

Equations of the Korteweg–de Vries type with non-trivial conserved quantities

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Abstract. We study a family of equations of the Korteweg–de Vries type. Different elements of this family are characterised by their linear dispersion relation $w(k)$. We prove that there are only three members of this family which possess a non-trivial polynomial conserved quantity and have a linear dispersion relation analytic in a neighbourhood of the origin. These correspond to the Korteweg–de Vries equation itself, the Benjamin–Ono equation and the intermediate long wave equation. As is well known, these equations have indeed infinitely many conserved quantities.

1. Introduction

One of the remarkable properties of the Korteweg–de Vries (κdV) equation is that it has infinitely many conserved quantities (see, e.g., Miura *et al* 1968). Typically, autonomous evolution equations with translation invariance have only three conserved quantities, namely the mass, the momentum and the energy. These are the ‘classical’ conserved quantities. The fact that the κdV equation has infinitely many conserved quantities is directly related to its complete integrability. So far two other equations of the κdV type with infinitely many conserved quantities have been found. These are the Benjamin–Ono equation (Benjamin 1967, Ono 1975), and the intermediate long wave (ILW) equation (Joseph 1977, Kubota *et al* 1978). A natural question we may ask is whether there exist other equations of the same form as the κdV equation which have infinitely many conserved quantities and are, therefore, likely to be completely integrable. This question was considered by Satsuma *et al* (1979) who, based on the existence of a Bäcklund transformation, gave a criterion (the so-called T conditions) for the existence of infinitely many conserved quantities (see also Ablowitz *et al* 1982). In this paper we consider a more modest question, namely we search for equations of the κdV type with one non-trivial (meaning non-classical) polynomial conserved quantity. By equations of the κdV type we mean evolution equations having the same non-linear structure as the κdV equation but different dispersion relations. We find a necessary condition on the dispersion relation (see equation (26) below) in order for the associated evolution equation to have a non-trivial conserved quantity. This equation is more general than the T conditions. Then we show that there are only three solutions to this equation (26) which are analytic in the neighbourhood of $k = 0$. These correspond to the dispersion relations for the κdV equation, for the Benjamin–Ono equation and for the ILW equation. We point out that there are other, non-analytic, solutions to equation (26).

The rest of this paper is organised as follows. In § 2 we introduce the equations of the κv type and we find the necessary condition on the dispersion relation in order to have non-trivial conserved quantities. Finally, in § 3 we discuss the existence and multiplicity of solutions to this necessary condition.

2. Equations of the κv type

By equations of the κv type we mean evolution equations of the form

$$u_t = \frac{\partial}{\partial x} (u^2 + G[u_x]) \quad (1)$$

defined on the whole real line and for $t > 0$. Here $u = u(x, t)$, and we use subscripts to denote partial derivatives. In (1), G is the pseudo-differential operator given by

$$G[f](x) \equiv \int_{-\infty}^{\infty} e^{ikx} \hat{g}(k) \hat{f}(k) dk \quad (2)$$

where $\hat{f}(k)$ denotes the Fourier transform of $f(x)$ and $\hat{g}(k)$ is the symbol of G . We call (1), (2), the family of equations of the κv type. The symbol $\hat{g}(k)$ determines the dispersion relation of the linearised equation obtained from (1). If G is antisymmetric with respect to the usual inner product $(f, g) = \int_{-\infty}^{\infty} fg dx$ (i.e. if \hat{g} is odd as a function of k) equations (1) and (2) have three 'classical' conserved quantities, namely

$$M = \int_{-\infty}^{\infty} u(x, t) dx \quad \text{the mass} \quad (3)$$

$$P = \int_{-\infty}^{\infty} u^2(x, t) dx \quad \text{the momentum} \quad (4)$$

$$E = \int_{-\infty}^{\infty} (\frac{1}{3}u^3 + \frac{1}{2}uG[u_x]) dx \quad \text{the energy.} \quad (5)$$

For the Korteweg-de Vries equation, the symbol is given by

$$\hat{g}(k) = ik \quad (6)$$

whereas for the Benjamin-Ono equation this is

$$\hat{g}(k) = ik/|k| \quad (7)$$

and for the intermediate long wave equation (ILW)

$$\hat{g}(k) = \frac{c_0}{2\pi} (1 - \frac{1}{2}kd \coth(kD))/k \quad (8)$$

where c_0 , d and D are free parameters. For the equations corresponding to these three symbols, (1) and (2) have non-classical conserved quantities; in fact they have infinitely many (see, e.g., Miura *et al* 1968, Case 1979, Satsuma *et al* 1979, respectively). Are there other symbols $\hat{g}(k)$ for which (1), (2) has at least one non-classical conserved quantity? Here we address a simpler problem: namely, are there other symbols for which (1), (2) has a polynomial conserved quantity of degree four, i.e. of the form

$$I_4 = \int_{-\infty}^{\infty} (\frac{1}{4}u^4 + \dots) dx. \quad (9)$$

As is well known (see, e.g., Gardner 1971, Faddeev *et al* 1971 or Lax 1975), equations of the form (1), (2) have a Hamiltonian structure. One can write (1) as

$$u_t = \{u, \mathcal{H}\} \tag{10}$$

where, as usual, the Poisson bracket between two functionals \mathcal{F}_1 and \mathcal{F}_2 of u is defined as

$$\{\mathcal{F}_1, \mathcal{F}_2\} = \int_{-\infty}^{\infty} \frac{\delta \mathcal{F}_1}{\delta u} \frac{d}{dx} \left(\frac{\delta \mathcal{F}_2}{\delta u} \right) dx \tag{11}$$

where $\delta \mathcal{F} / \delta u$ denotes the functional derivative of \mathcal{F} with respect to u . In equation (10), the Hamiltonian \mathcal{H} is just the energy, given by (5). Hence,

$$\frac{\delta \mathcal{H}}{\delta u} = u^2 + G[u_x] \tag{12}$$

and, therefore,

$$\{u, \mathcal{H}\} = \frac{\partial}{\partial x} (u^2 + G[u_x]). \tag{13}$$

Because of the Hamiltonian structure, I_4 is a conserved quantity if and only if

$$\{\mathcal{H}, I_4\} = 0. \tag{14}$$

Let

$$A_4 \equiv \frac{\delta I_4}{\delta u} = u^3 + \dots \tag{15}$$

denote the functional derivative of I_4 . Our problem reduces to determining whether there is an A_4 of the form (15) such that

$$\int_{-\infty}^{\infty} (u^2 + G[u_x])_x A_4 dx = 0. \tag{16}$$

Since $A_4 = \delta I_4 / \delta u$ we must require also that

$$\frac{\delta A_4(x)}{\delta u(y)} = \frac{\delta A_4(y)}{\delta u(x)}. \tag{17}$$

The above equation is just the integrability condition for A_4 .

The most general form A_4 can have, in order to satisfy (15), (17) and (16), is

$$A_4 = u^3 + \frac{3}{2}uG[u_x] + \frac{3}{2}G[uu_x] + K[u_x] \tag{18}$$

where K is an antisymmetric (i.e. with a symbol odd in k) pseudo-differential operator. In principle, we could allow the pseudo-differential operator G in (18) to be different than the G given in (2). However, one soon realises that the symbols of both operators have to be the same in order to satisfy (16). In fact, replacing A_4 , given by (18), in equation (16) we get,

$$\int_{-\infty}^{\infty} ((u^2)_x M[u_x] + \frac{3}{2}uG[u_x]G[u_{xx}]) dx = 0 \tag{19}$$

with

$$M[f] \equiv K[f] - \frac{3}{4}G[G[f_x]]. \tag{20}$$

Therefore, our original question can be rephrased in the following way. ‘For what functions G does there exist an M such that (19) holds for all u ?’ It is worth pointing out here that (19) is certainly satisfied if G satisfies the so-called T conditions (see Satsuma *et al* 1979, Ablowitz *et al* 1982).

Let $\hat{u}(k)$ denote the Fourier transform of u , and $\hat{g}(k)$, $\hat{h}(k)$ and $\hat{m}(k)$ the symbols of the pseudo-differential operators G , K , and M , respectively. In terms of \hat{u} , \hat{g} , \hat{h} and \hat{m} , equation (19) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{u}(k)\hat{u}(q)\hat{u}(-k-q)[\lambda kq(k+q)\hat{g}(k)\hat{g}(q) + \hat{m}(k)kq] dk dq = 0. \tag{21}$$

Here, $\lambda = 3i/8$ is a constant. A necessary and sufficient condition on \hat{g} and \hat{m} so that (21) holds for all \hat{u} is

$$\lambda kq(k+q)\hat{g}(k)\hat{g}(q) + \hat{m}(k)kq = b(k, q) + d(q, -k-q) + f(-k-q, k) + s(k, q, -k-q) \tag{22}$$

where $b(\cdot, \cdot)$, $d(\cdot, \cdot)$ and $f(\cdot, \cdot)$ are arbitrary antisymmetric functions of their arguments (i.e. $b(k, q) = -b(q, k)$, etc) and $s(\cdot, \cdot, \cdot)$ is an arbitrary completely antisymmetric function. We can easily get rid of s by changing b , d and f into \tilde{b} , \tilde{d} and \tilde{f} respectively, where $b(k, q) = \tilde{b}(k, q) + s(k, q, -k-q)$, $d(k, q) = \tilde{d}(k, q) - s(k, q, -k-q)$, and $f(k, q) = \tilde{f}(k, q) - s(k, q, -k-q)$. The form of equation (22) imposes strong conditions on $\hat{g}(k)$. In fact, as we shall see in the following, one can obtain an equation involving only $\hat{g}(k)$ (see equation (26), below) which is a necessary condition on $\hat{g}(k)$ so that (22) is satisfied. We proceed as follows.

Step 1. Consider the three equations obtained from (22) by setting respectively $k = l$, $q = -2l$; $k = -2l$, $q = l$; and $k = l$, $q = l$. By adding the three resulting equations we get

$$\hat{m}(q) - 2\hat{m}(2q) = 2\lambda q\hat{g}(q)(\hat{g}(q) - 2\hat{g}(2q)) \tag{23}$$

where we have substituted l by q as the independent variable. In deriving (23) we have used the antisymmetry of b , d , f and s .

Step 2. Consider now the six equations obtained from (22) by setting respectively $k = 2l$, $q = l$; $q = 2l$, $k = l$; $q = -3l$, $k = l$; $q = l$, $k = -3l$; $q = -3l$, $k = 2l$; and $q = 2l$, $k = -3l$. Then, adding the six resulting equations we get

$$9\hat{m}(3q) - 4\hat{m}(2q) - \hat{m}(q) = 12\lambda q(-\hat{g}(q)\hat{g}(2q) + \hat{g}(q)\hat{g}(3q) + \hat{g}(2q)\hat{g}(3q)) \tag{24}$$

where we have substituted l by q as the independent variable.

Step 3. Finally consider the six equations obtained from (22) by setting respectively $k = 3l$, $q = l$; $k = l$, $q = 3l$; $k = -4l$, $q = l$; $q = -4l$, $k = l$; $k = -4l$, $q = 3l$; and $k = 3l$, $q = -4l$. Then adding the six resulting equations we get

$$16\hat{m}(4q) - 9\hat{m}(3q) - \hat{m}(q) = 24\lambda q(\hat{g}(4q)\hat{g}(q) - \hat{g}(3q)\hat{g}(q) + \hat{g}(4q)\hat{g}(3q)) \tag{25}$$

where we have substituted again l by q as the independent variable.

The left-hand sides of equations (23), (24) and (25) are all linear in \hat{m} , and it is rather simple to eliminate \hat{m} from these three equations. By doing so we finally get one equation for the symbol \hat{g} of G , which is a necessary condition on equation (1) for it to have a non-classical polynomial conserved quantity. The equation we get for \hat{g} is

$$(3\hat{g}(3q) - \hat{g}(q))(\theta(2q) + \theta(q)) = 4\theta(2q)\theta(q) \tag{26}$$

where we have introduced

$$\theta(q) \equiv 2\hat{g}(2q) - \hat{g}(q). \tag{27}$$

Properties of equation (26). It is rather straightforward to verify that the equation we have found for $\hat{g}(q)$ (i.e. equation (26)) has the following properties.

- (i) If $\hat{g}(q)$ solves (26), so does $\alpha\hat{g}(\gamma q) + \beta/q$ for any α, β and γ .
- (ii) $\hat{g}(q) = q$ (the symbol corresponding to the κ av equation), $\hat{g}(q) = c_0$, for $q \geq 0$ (the symbol corresponding to the Benjamin-Ono equation), and $\hat{g}(q) = \coth(q) - 1/q$, for $q \geq 0$ (the symbol corresponding to the ILW equation) are all analytic solutions of (26) in a neighbourhood of the origin. (Note that here it is enough to consider $\hat{g}(q)$ defined in $[0, \infty)$. Recall that $\hat{g}(-q) = -\hat{g}(q)$.)
- (iii) There are many other solutions of (26) which are not analytic near 0. For example

$$\hat{g}(q) = \frac{1}{q} \sin\left(2\pi \frac{\ln(q)}{\ln(2)}\right)$$

solves (26), and so does any other function satisfying $\hat{g}(2q) = \frac{1}{2}\hat{g}(q)$ i.e. $\theta(q) = 0$. At this point it is worth recalling that (26) is only a necessary condition on \hat{g} for equation (1) to have a non-classical conserved quantity.

3. Analytic solutions of the equation for \hat{g} .

In this section we find all the solutions of equation (26) which are analytic in a neighbourhood of $q = 0$. Let

$$\hat{g}(q) = \sum_{n=0}^{\infty} c_n q^n \tag{28}$$

be the power series for $\hat{g}(q)$ around $q = 0$. Introducing this expression for $\hat{g}(q)$ in equation (26) we get

$$\sum_{n=0}^{\infty} \omega_n q^n = 0 \tag{29}$$

for all $q > 0$ in a neighbourhood of the origin and, therefore,

$$\omega_n = 0 \quad \text{for } n = 0, 1, 2, \dots \tag{30}$$

where

$$\omega_n = \begin{cases} d_n c_{n/2}^2 + \sum_{l=0}^{(n/2)-1} c_l c_{n-l} A_{ln} & \text{for } n = 2, 4, \dots \\ \sum_{l=0}^{(n-1)/2} c_l c_{n-l} A_{ln} & \text{for } n = 1, 3, 5, \dots \end{cases} \tag{31}$$

and $\omega_0 \equiv 0$. Here,

$$A_{ln} = B_{ln} + B_{n-ln} \quad \text{for } l < n/2 \tag{33}$$

with

$$B_{ln} = 6 \times 4^l - 3^{l+1} + 6 \times 4^l \times 3^{n-l} + 3^{l+1} 2^{n-l} - 5 \times 2^l - 16 \times 4^l \times 2^{n-l} + 8 \times 2^n + 1 \tag{34}$$

and

$$d_n = 14 \times 2^n + 3^{n/2}(6 \times 2^n + 3 \times 2^{n/2} - 3) - 5 \times 2^{n/2} + 1 - 16 \times 2^{3n/2}. \quad (35)$$

Equation (30) defines a recurrence relation for the coefficients c_n which assumes the general form

$$c_0 c_n A_{0n} = f_n(c_1, c_2, \dots, c_{n-1}). \quad (36)$$

Denote $b_n = A_{0n} = 2[3^{n+1} - 2 \times 4^n - 2^n]$. Clearly $b_0 = 0$ and $b_{n+1} = 4b_n + 2(2^{n+1} - 3^{n+1})$. Therefore, $b_n < 0$ for all $n \geq 1$. Now, we divide the study of the recurrence relation (36) (or (30)) into three separate cases.

Case 1, $c_0 \neq 0$. We will show that in this case $c_n = 0$ for all $n \geq 1$. If $n = 1$, (36) is

$$c_0 c_1 b_1 = 0 \quad (37)$$

whereas, for $n \geq 2$, (36) is given by

$$c_0 c_n b_n = f_n(c_1, c_2, \dots, c_{n-1}) \quad (38)$$

where f_n is a homogeneous function of degree 2. Since $b_n \neq 0$ for all $n \geq 1$ (in fact $b_n < 0$), (37) implies $c_1 = 0$ and, therefore, from (38) we conclude inductively that $c_n = 0$ for all $n \geq 2$. Therefore, in this case the only analytic solution to equation (26) is

$$\hat{g}(q) = c_0 \quad (39)$$

which is the symbol associated with the Benjamin-Ono equation in equation (1).

Case 2, $c_0 = c_1 = 0$. We will show that in this case $c_n = 0$ for all $n \geq 2$ and, therefore, $\hat{g} \equiv 0$. From (31) we have that $\omega_4 = 0$ implies

$$d_4 c_2^2 = 0 \quad (40)$$

whereas for n even, $n \geq 6$, we have from (31) that

$$\sum_{l=2}^{(n/2)-1} c_l c_{n-l} A_{ln} + d_n c_{n/2}^2 = 0. \quad (41)$$

From (35) we have that $d_4 = 126$ and therefore (40) implies $c_2 = 0$. Moreover, $d_n > 0$ for all $n \geq 6$. In fact, after some regrouping we get from (35) that

$$d_n = 2^{n/2} \sigma_n + 3^{(n/2)+1} \tau_n + 2^{3n/2} \theta_n \quad (42)$$

with

$$\sigma_n = 14 \times 2^{n/2} - 5 \quad \tau_n = 2^{n/2} - 1 \quad \theta_n = (3/2)^{n/2} 6 - 16.$$

It is clear that $\sigma_n > 0$, $\tau_n > 0$ for all $n > 0$. Moreover, $\theta_n > 0$ for $n \geq 6$. Thus, $d_n > 0$ for all $n \geq 6$. (In fact, $d_n > 0$ for $n = 2, 4, \dots$) Since $d_n \neq 0$ for all $n \geq 6$, n even, and $c_2 = 0$, it follows inductively from (41) that $c_n = 0$ for all n . This also ensures that $\omega_n = 0$ is satisfied when n is odd.

Case 3, $c_0 = 0$, $c_1 \neq 0$. If $c_0 = 0$ we get from (31) and (32) that the equation $\omega_n = 0$ reduces to

$$\sum_{l=1}^{(n/2)-1} c_l c_{n-l} A_{ln} + d_n c_{n/2}^2 = 0 \quad \text{for } n = 4, 6, 8, \dots \quad (43)$$

and

$$\sum_{l=1}^{(n-1)/2} c_l c_{n-l} A_{ln} = 0 \quad \text{for } n = 3, 5, 7, \dots \tag{44}$$

Denote by $e_n = A_{1n} = 7 + 9 \times 3^n - 14 \times 2^n - 2 \times 4^n$. e_n satisfies the relation

$$e_{n+1} = -14 + 3e_n + 2^{n+1}(7 - 2^n). \tag{45}$$

Since $e_0 = 0$, we can compute the first few terms of this sequence. This yields $e_1 = -2$, $e_2 = 0$, $e_3 = 10$ and $e_4 = 0$. Since $7 - 2^n \leq 0$ for $n \geq 3$, and $e_4 = 0$, (45) implies $e_n < 0$ for all $n \geq 5$.

For $n = 3$, equation (44) is $c_1 c_2 e_3 = 0$. Since $e_3 = 10 \neq 0$, and $c_1 \neq 0$, $c_2 = 0$. Since $e_n \neq 0$ for $n \geq 5$ we conclude inductively from (44) that $c_n = 0$ for all even n . For $n = 4$, equation (43) is satisfied identically, no matter the value of c_3 , since $c_2 = e_4 = 0$. Thus, the value of c_3 is not determined by the recursion relation (43). We will, finally, consider the two possibilities, $c_3 = 0$ or $c_3 \neq 0$, separately. We need only consider equation (43), which takes the generic form

$$c_1 s_k e_{2k} = g_k(s_2, s_3, \dots, s_{k-1}) \quad \text{for } k = 3, 4, 5, \dots \tag{46}$$

Here, we have introduced the notation $s_k \equiv c_{2k-1}$ for $k = 2, 3, \dots$. The functions g_k , on the right-hand side of (46), are homogeneous of degree 2 with respect to their arguments. If $s_2 \equiv c_3 = 0$, we conclude inductively from (46) that $s_k = 0$ for all $k \geq 2$, since $c_1 \neq 0$ and $e_{2k} \neq 0$ for $k \geq 3$. Therefore, if $c_3 = 0$, the analytic solution $\hat{g}(q)$ to equation (26) takes the form

$$\hat{g}(q) = c_1 q \tag{47}$$

which is the symbol associated with the Korteweg-de Vries equation in equation (1).

To conclude our analysis, given $s_2 \equiv c_3 \neq 0$, equation (46) yields a well defined sequence c_3, c_5, \dots (since $c_1 \neq 0$ and $e_{2k} \neq 0$ for $k \geq 3$), where c_5, c_7, \dots , are uniquely determined in terms of c_3 . Since the solution to the recursion relation (given by equation (46)) is unique, and since we already know that

$$\hat{g}(q) = \beta \left(\coth(\alpha q) - \frac{1}{\alpha q} \right) \tag{48}$$

is an analytic solution to equation (26) with $c_1 = \frac{1}{3}\alpha\beta \neq 0$ and $c_3 = -\frac{1}{3}\alpha^3\beta \neq 0$, this is the only analytic solution (having $c_1 \neq 0 \neq c_3$). The solution $\hat{g}(q)$ given by equation (48) is the symbol associated with the ILW equation in equation (1).

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