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Finite mode analysis of the generalized Kuramoto–Sivashinsky equation

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We present numerical results concerning a five mode truncation of the equation $u_t + uu_x + \delta u_{xxx} + u_{xx} + u_{xxxx} = 0$ subject to periodic boundary conditions. We find that for large δ the system evolves from most initial conditions into a final state consisting of one or two traveling pulses, depending on the initial condition and horizontal periodicity. This is due to a region of simultaneous stability of the first two branches that bifurcate from the trivial solution. An additional two pulse traveling wave which does not bifurcate from $u = 0$ is also present.

1. Introduction

The equation $u_t + uu_x + \delta u_{xxx} + u_{xx} + u_{xxxx} = 0$, is one of the simplest which includes the combined action of dispersion, nonlinearity, dissipation and instability. In the absence of dispersion, that is, for $\delta = 0$, it is the well studied Kuramoto–Sivashinsky (KS) equation. If sufficiently large dispersion is included, the time evolution of the system is significantly changed. A series of numerical studies [1–3] have shown that, in a periodic domain, most initial conditions evolve into a final state consisting of a row of traveling solitary like pulses of the same amplitude. The interpulse distances have been observed to adopt distinct fixed values or, in other cases, a single interpulse distance is obtained giving rise to a periodic arrangement of pulses. Different equilibrium states arise depending on the initial conditions and on the periodicity L of the spatial domain. These final states differ in the number of pulses and in their amplitude. For lower values of dispersion the evolution becomes chaotic and creation or annihilation of pulses occurs [4]. Analytical studies of this equation include perturbation theory around the Korteweg–de Vries equation [2] and the study of pulse interactions [5,6]. These two approaches do not address the problem of the time evolution of an initial condition, a final state consisting of a stable array of N pulses is assumed. Exact traveling wave solutions have been obtained for special values of δ [7].

Our interest in this paper is to provide some understanding of the time evolution of the system, for large δ, from an arbitrary initial condition to a final state consisting of a stable array of traveling pulses. In the simple case of small periodicity interval this can be done including few Fourier modes, since as numerical results show, the linearly stable modes decay rapidly [2]. In previous work [8] we performed a three mode truncation which enabled us to understand analytically the role of dispersion in the evolution of the system. We showed that in a small box a single traveling solitary hump arises with its amplitude proportional to δ when δ is large. When dispersion is low, a sharp feature is not developed. Due to the smallness of the box only one pulse appears in agreement with the numerical integrations of the complete equation. In this article we describe numerical results for a five mode truncation which exhibits the appearance of one or two solitary pulses depending on L and in some cases on initial conditions. A similar truncated system has been used to study the solutions

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of the Kuramoto–Sivashinsky equation [9] near the second bifurcation from $u = 0$. A center manifold analysis near this bifurcation predicts accurately the transitions to different types of solutions near this point. The bifurcation of traveling waves from the steady state, attracting heteroclinic cycles and unstable modulated traveling waves is explained. A crucial fact in their existence is the O(2) symmetry of the KS equation in the space of Fourier coefficients [10,11]. The inclusion of dispersion breaks this symmetry, as a result of this we find that the solutions are significantly changed, the first solution that bifurcates from $u = 0$ is stabilized, its region of stability extends beyond the value of L at which the second branch that bifurcates from $u = 0$ gains stability leading to regions where the final state is dependent of initial conditions. An additional branch leading to two pulse traveling waves is also present at larger L.

The results obtained when dispersion is included can be summarized as follows, for small L all initial conditions lead to the formation of a single solitary pulse, as L increases there is a region where one or two pulses are formed depending on initial conditions. Increasing L further we find that some initial conditions lead to the formation of two traveling pulses, two different two pulse states exist, other initial conditions lead to modulated traveling waves.

2. Mathematical formulation

We consider the equation

$$u_t + uu_x + \delta u_{xxx} + u_{xx} + u_{xxxx} = 0 \qquad (1)$$

subject to the periodic boundary conditions $u(0,t) = u(L,t)$ with initial condition $u(x,0) = u_0(x)$.

We expand the solution for u in the Fourier series $u(x,t) = \sum_{n=-\infty}^{\infty} a_n(t)\, \mathrm{e}^{\mathrm{i}k_n x}$ where $k_n = 2n\pi/L$ and the coefficients satisfy $a_{-n}(t) = \overline{a}_n$. Here $\overline{a}$ denotes the complex conjugate of a. Since the spatial average of u is conserved we fix it to be zero by letting $a_0 = 0$. Replacing the series expansion in the equation we obtain the following system for the time evolution of the Fourier amplitudes:

$$\dot{a}_n + (k_n^4 - k_n^2 - \mathrm{i}\delta k_n^3)a_n + \tfrac{1}{2}\mathrm{i}k_n \sum_{t=0}^{\infty} (a_t a_{n-t} + \overline{a}_t a_{n+t}) = 0. \qquad (2)$$

It is this system which we truncate at five modes. This means that we are restricted to a small box size L. Here, as in the KS equation, since the linearly stable modes are strongly damped [2], keeping one or two linearly stable modes will provide a qualitative correct solution for the time evolution of the system. The five mode truncated system is given by

$$\dot{a}_1 + (\mu_1 - \mathrm{i}\delta k^3)a_1 + \mathrm{i}k(\overline{a}_1 a_2 + \overline{a}_2 a_3 + \overline{a}_3 a_4 + \overline{a}_4 a_5) = 0, \qquad (3)$$

$$\dot{a}_2 + (\mu_2 - 8\mathrm{i}\delta k^3)a_2 + \mathrm{i}k(a_1^2 + 2\overline{a}_1 a_3 + 2\overline{a}_2 a_4 + 2\overline{a}_3 a_5) = 0, \qquad (4)$$

$$\dot{a}_3 + (\mu_3 - 27\mathrm{i}\delta k^3)a_3 + 3\mathrm{i}k(a_1 a_2 + \overline{a}_1 a_4 + \overline{a}_2 a_5) = 0, \qquad (5)$$

$$\dot{a}_4 + (\mu_4 - 64\mathrm{i}\delta k^3)a_4 + 2\mathrm{i}k(a_2^2 + 2a_1 a_3 + 2\overline{a}_1 a_5) = 0, \qquad (6)$$

$$\dot{a}_5 + (\mu_5 - 125\mathrm{i}\delta k^3)a_5 + 5\mathrm{i}k(a_1 a_4 + a_2 a_3) = 0, \qquad (7)$$

where $k = 2\pi/L$ and $\mu_n = k_n^4 - k_n^2$. All numerical integrations are performed on this system of five complex differential equations. We find that the system evolves into traveling waves consisting of one or two solitary like pulses. Using the polar representation for the amplitudes $a_n = \rho_n\, \mathrm{e}^{\mathrm{i}\theta_n}$ the expression for u may be written after some rearrangement as

$$u(x,t) = \sum_{n=n_0}^{\infty} \rho_n \exp\left[\mathrm{i}\left(\theta_n - \frac{n}{n_0}\theta_{n_0}\right)\right] \times \exp\left[\mathrm{i}nk\left(x + \frac{\theta_{n_0}}{n_0 k}\right)\right] + \text{c.c.}, \qquad (8)$$

where n_0 is the first positive nonvanishing Fourier component of the solution. We see then that traveling waves $u(x-ct)$ correspond to fixed points of ρ_n, and of $\theta_n - (n/n_0)\theta_{n_0}$ with constant time derivative of θ_{n_0}. The speed of the wave is then $c = -\dot{\theta}_{n_0}/kn_0$. In terms of the amplitudes, the traveling waves satisfy $\dot{a}_n = \mathrm{i}(n/n_0)\dot{\theta}_{n_0}a_n$. Two types of traveling waves $u(x-ct)$ are present in this five mode truncated system, that in which none of the amplitudes vanish and satisfies $\dot{a}_n = \mathrm{i}n\dot{\theta}_1 a_n$ which propagates with speed $c = -\dot{\theta}_1/k = -\delta k^2 + \mathrm{Re}[(\bar{a}_1 a_2 + \bar{a}_2 a_3 + \bar{a}_3 a_4 + \bar{a}_4 a_5)/a_1]$, and a traveling wave with $a_1 = a_3 = a_5 = 0$ for which $\dot{a}_4 = 2\mathrm{i}\dot{\theta}_2 a_4$ and whose speed of propagation is $c = -\dot{\theta}_2/2k = -4\delta k^2 + \mathrm{Re}(\bar{a}_2 a_4/a_2)$. The first type of traveling wave leads to a one pulse solution that bifurcates from $u = 0$ and to a secondary two pulse branch. The second type leads to the two pulse solution that bifurcates from $u = 0$. Due to the translational invariance of the equation the speed can be expressed as a function of the values of the fixed points ρ_n and $\theta_n - (n/n_0)\theta_{n_0}$ [12].

3. Numerical results

We have performed a series of numerical integrations on system of equations (3)–(7) for different initial conditions, high dispersion, and for values of L in the range $2\pi < L < 6.4\pi$. The solution that bifurcates at $L = 6\pi$ and whose main components are a_3 and a_6 is not included in this system and we cannot expect the truncated system to approximate the solution beyond this value of L. All the initial conditions explored up to $L = 5.84\pi$ lead to traveling waves, for $5.84\pi < L < 6.22\pi$ some initial conditions lead to modulated traveling waves, others to two pulse solutions. In the region where a traveling wave is always reached, depending on L and initial condition a final state with one or two pulses is reached. We have calculated tha stability of the traveling wave solutions of the system. In fig. 1a we have plotted the norm of u versus $(L/\pi)^2$ for $\delta = 3$ for the traveling waves. The solid lines correspond to the stable part of the branches. The three branches shown lose stability through a Hopf bifurcation leading to modulated traveling waves. The branch that bifurcates from $u = 0$ at $L = 2\pi$ is stable up to $L = 5.84\pi$. The branch which bifurcates at $L = 4\pi$ from $u = 0$ gains stability at $L = 4.3\pi$ and remains stable until $L = 6.103\pi$. A different two pulse branch appears at $L = 5.66\pi$ which is stable up to $L = 6.22\pi$, this branch extends unstably into the larger L region. More modes

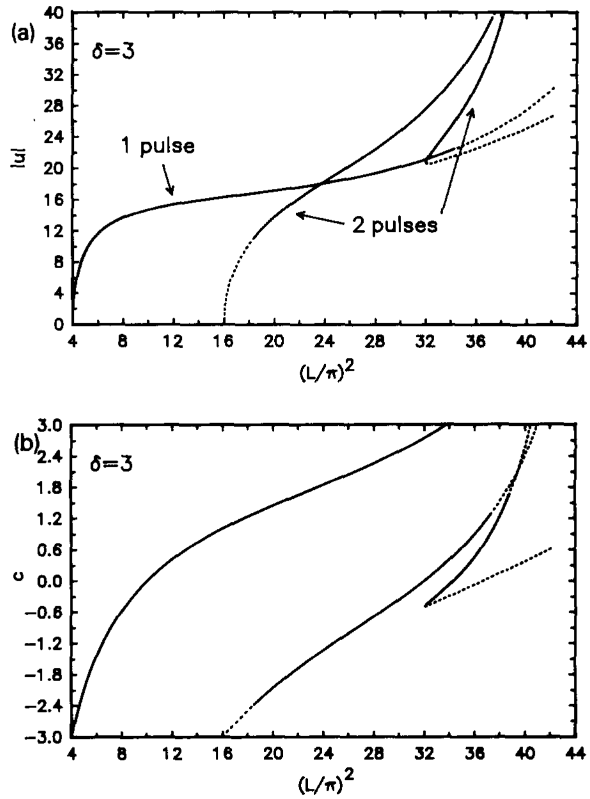

Fig. 1. The norm of u as a function of $(L/\pi)^2$ for the traveling wave solutions. The solid line indicates the stable regions. (b) The speed c as a function of $(L/\pi)^2$ for the traveling waves shown in (a).

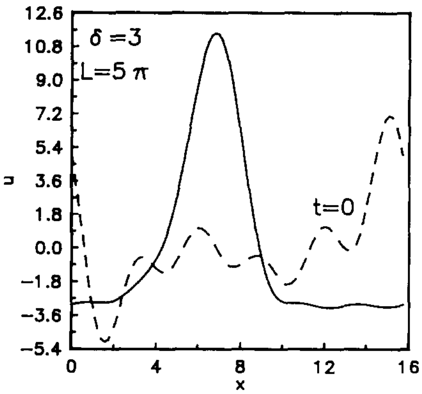

Fig. 2. At $L = 5\pi$ the initial condition $a_1 = (0.5, 0.5)$, $a_2 = (0.5, 1)$, $a_3 = (0.5, 0.5)$, $a_4 = (0.5, 0.5)$, $a_5 = (0.5, 0.5)$ evolves into a single pulse traveling wave. In this and the following figures the dashed line shows the initial condition while the solid line represents the final state.

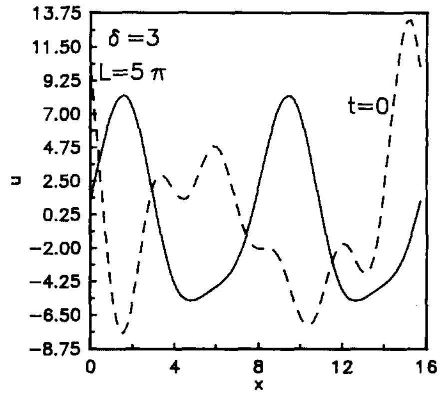

Fig. 3. A different initial condition as that of fig. 2 gives rise to a two pulse traveling solution.

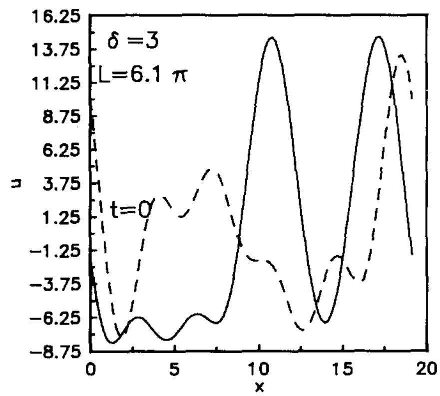

Fig. 4. One of the two pulse solutions found at larger L. Notice the spacing between the pulses. This state lies on the secondary two pulse branch.

should be taken into account in this region. The speed of the waves is shown in fig. 1b. Along all the branches the speed is reversed as L increases; in each branch, for small L waves are traveling to the left. We see then that for $2\pi < L < 4.3\pi$ the single pulse traveling wave is stable. For $4.3\pi < L < 5.84\pi$ the final state may be a one or two pulse traveling wave depending on initial conditions. For $5.84\pi < L < 6.22\pi$ the system evolves into a two pulse traveling wave or into a modulated traveling wave. This region however may be significantly altered if more modes are included. In fig. 2 we show the initial condition $a_1 = (0.5, 0.5)$, $a_2 = (0.5, 1)$, $a_3 = (0.5, 0.5)$, $a_4 = (0.5, 0.5)$, $a_5 = (0.5, 0.5)$ which at $L = 5\pi$ and $\delta = 3$ leads to a fixed point on the branch that bifurcates from $u = 0$ at $L = 2\pi$ given by $\rho_1 = 2.623$, $\rho_2 = 1.695$, $\rho_3 = 0.905$, $\rho_4 = 0.421$, $\rho_5 = 0.173$ which travels with speed 1.945. The initial condition IC1, given by $a(1) = (1, -1), a(2) = (1, 2), a(3) = a(4) = a(5) = (1, 1)$ shown in fig. 3, leads instead to the fixed point $a_1 = a_3 = a_5 = 0$, $\rho_2 = 3.350, \rho_4 = 0.805$ which corresponds to two pulses traveling to the left with speed 1.168. At larger L the system may evolve into different two pulse solutions depending again on the initial condition. In fig. 4 we show the initial condition IC1 which for $L = 6.1\pi$ leads to the fixed point $\rho_1 = 3.335, \rho_2 = 2.510, \rho_3 = 3.572, \rho_4 = 1.148, \rho_5 = 0.555$ which corresponds to two traveling pulses with speed 0.873. In fig. 5 we show the evolution of the system from initial condition $a(1) = (1, -1)$, $a(2) = (5, 2)$, $a(3) = (0, 1)$, $a(4) = (5, 1)$, $a(5) = (1, 1)$ to the bimodal state $a_1 = a_3 = a_5 = 0$, $\rho_2 = 5.761$, $\rho_4 = 2.563$ which travels with speed 1.246. There is a short interval $6.1\pi < L < 6.22\pi$ in which the system evolves into the new fixed point which then undergoes a Hopf bifurcation.

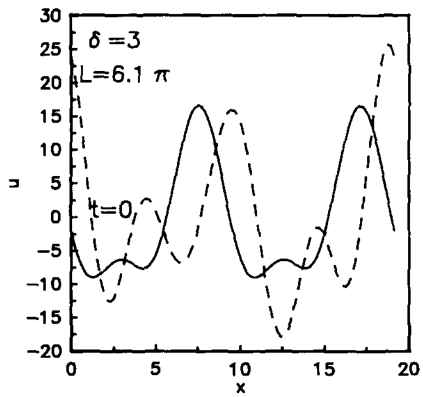

Fig. 5. A different initial condition leads to a different two pulse state. This state lies in the same branch as the one shown in fig. 3.

4. Summary

We performed a series of numerical integrations of a five mode truncation of the generalized Kuramoto–Sivashinsky equation for large dispersion δ. For most initial conditions, for a large range of box sizes the system evolves into single or two pulse traveling waves depending on initial conditions and box size. The role of dispersion is to stabilize the traveling wave solutions that bifurcate from the trivial solution $u = 0$ leading to a region where the first two branches that bifurcate from $u = 0$ are simultaneously stable. This simple five mode truncation exhibits the main features found in numerical integrations of the full equation, namely, evolution of most initial conditions into traveling pulses of amplitude increasing with δ, different number of pulses arise depending on the box size L and for a given L on initial conditions. For small δ the situation is not as simple and ressembles the behavior found in the KS equation [6]. A normal form analysis of the truncated equations near $L = 4\pi$ as in [9] will provide information on the transition from small to large δ behavior.

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