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A five-mode bifurcation analysis of a Kuramoto–Sivashinsky equation with dispersion

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We study the bifurcation structure of the equation $u_t + uu_x + \delta u_{xxx} + u_{xx} + u_{xxxx} = 0$ in a small domain L by using a five-mode truncation. As dispersion increases quantitative and qualitative changes occur. A characteristic feature due to dispersion is the coexistence of stable travelling waves of different basic wavenumber and speed.

1. Introduction

In several problems where a long wavelength oscillatory instability is found, the nonlinear evolution of the perturbations near criticality is governed by the dispersion modified Kuramoto–Sivashinsky equation (KS–KdV). It appears in problems of fluid flow along an inclined plane [1–3], convection in fluids with a free surface [4], drift waves in plasmas [5], vertically falling liquid films in the presence of interfacial viscosities [6], etc. While in extended systems in the nondispersive case $\delta = 0$, i.e., in the KS equation, disordered behavior predominates, numerical studies have shown that for large dispersion the system evolves into rows of solitary like pulses of equal amplitude which travel as a whole [7–9]. The number of pulses that appear depends on the initial conditions, the horizontal extension of the domain and on dispersion. Fourier analysis of the solutions obtained from time integrations of the partial differential equation show that the final state is composed of few modes and that stable modes decay rapidly [8]. An equilibrium between two modes, k and $2k$, gives approximate agreement with some numerical results [5]. A three-mode system shows the secondary bifurcation from a steady state to rotating waves in the KS equation and the imperfect bifurcation when dispersion is included. Asymptotic results for large dispersion in the three-mode equilibria show that the solution evolves into a localized travelling pulse of amplitude which increases linearly with δ [10]. Our aim in this Letter is to extend the work mentioned above to include the competition of two unstable modes, which is not allowed for unless four or more modes are included. A key feature in the study of the KS equation is the O(2) symmetry of the equation [11–14]. The inclusion of dispersion breaks this symmetry, the equation is no longer symmetric under reflections, which leads to qualitatively different behavior. In the absence of dispersion the primary bifurcations from the steady state $u = 0$ are to a new steady state. If the reflection symmetry is broken the primary bifurcations are to travelling waves of speed proportional to the amount of symmetry breaking. This effect has been studied theoretically and experimentally in rotating Bénard convection in a cylinder [15]. It has been found in other problems that even a small symmetry breaking can significantly change both the stability properties and type of solutions [16–18].

In this work we present numerical results on the bifurcation structure of the first two primary bifurcation branches. The results are obtained using a five-mode Galerkin truncation of the KS–KdV equation which is expected to be accurate qualitatively and quantitatively for small values of the periodicity interval. For several dissipative partial differential equations, and in particular for the KS equation, based on the existence of an inertial manifold, several schemes of approximation of this manifold have been devised. These methods enable a description of the bifurca-

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tion structure of the solution in terms of fewer modes than if a traditional Galerkin approximation is employed [19]. We have not used such an approach in the present case since our interest is the effect of dispersion on the first two bifurcating branches. For the KS equation these branches are described accurately with a five-mode Galerkin truncation, or more efficiently with a three-mode approximate inertial form. We expect an analogous situation here; the use of a more efficient scheme is not indispensable in the present situation but should certainly be employed if the domain of periodicity is extended.

In addition to the numerical studies of the bifurcation structure of the KS equation the bifurcating structure of the first two branches has been studied analytically by means of a reduction to a center unstable manifold around the point where the second mode becomes unstable [12]. We have not carried out such a reduction, we present no analytical results on the role of dispersion, we refer to ref. [16] for some results. A complete analysis of such a reduction remains a task to be undertaken.

The stability of the first bifurcating branch of the regularized Kuramoto–Sivashinsky equation has been studied as well [20].

2. Formulation of the problem

Our starting point is the equation

$$u_t + uu_x + \delta u_{xxx} + u_{xx} + u_{xxxx} = 0 \tag{1}$$

subject to periodic boundary conditions in the interval $[0, L]$, with initial conditions $u(x,0) = u_0(x)$. We only consider solutions with zero spatial average. We recall that for $L \leqslant 2\pi$ all initial conditions evolve into $u(x,t) = 0$.

We expand the solution for u in the Fourier series

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n(t)\exp(\mathrm{i}k_n x),$$

where $k_n = 2n\pi/L$ and the coefficients satisfy

$$a_{-n}(t) = \overline{a}_n.$$

Here $\overline{a}$ denotes the complex conjugate of a, and since we have chosen solutions with zero average $a_0 = 0$. Replacing the series expansion in the equation we obtain the following system for the time evolution of the Fourier amplitudes,

$$\dot{a}_n + (k_n^4 - k_n^2 - \mathrm{i}\delta k_n^3)a_n + \tfrac{1}{2}\mathrm{i}k_n \sum_{t=0}^{\infty} (a_t a_{n-t} + \overline{a}_t a_{n+t}) = 0. \tag{2}$$

Keeping only the first five modes we obtain the system

$$\dot{a}_1 + (\mu_1 - \mathrm{i}\delta k^3)a_1 + \mathrm{i}k(\overline{a}_1 a_2 + \overline{a}_2 a_3 + \overline{a}_3 a_4 + \overline{a}_4 a_5) = 0, \tag{3}$$

$$\dot{a}_2 + (\mu_2 - 8\mathrm{i}\delta k^3)a_2 + \mathrm{i}k(a_1^2 + 2\overline{a}_1 a_3 + 2\overline{a}_2 a_4 + 2\overline{a}_3 a_5) = 0, \tag{4}$$

$$\dot{a}_3 + (\mu_3 - 27\mathrm{i}\delta k^3)a_3 + 3\mathrm{i}k(a_1 a_2 + \overline{a}_1 a_4 + \overline{a}_2 a_5) = 0, \tag{5}$$

$$\dot{a}_4 + (\mu_4 - 64\mathrm{i}\delta k^3)a_4 + 2\mathrm{i}k(a_2^2 + 2a_1 a_3 + 2\overline{a}_1 a_5) = 0, \tag{6}$$

$$\dot{a}_5 + (\mu_5 - 125\mathrm{i}\delta k^3)a_5 + 5\mathrm{i}k(a_1 a_4 + a_2 a_3) = 0, \tag{7}$$

where $k = 2\pi/L$ and $\mu_n = k_n^4 - k_n^2$. Notice that the Fourier series expansion for $u(x,t)$ can be rewritten as

$$u(x,t) = \sum_{n=n_0}^{\infty} \rho_n \exp\{\mathrm{i}[\theta_n - (n/n_0)\theta_{n_0}]\} \times \exp[\mathrm{i}nk(x + \theta_{n_0}/n_0 k)] + \text{c.c.}, \tag{8}$$

where n_0 is the first positive nonvanishing Fourier component of the solution. We see then that travelling waves $u(x-ct)$ correspond to fixed points of ρ_n, and of $\theta_n - (n/n_0)\theta_{n_0}$ with constant time derivative of θ_{n_0}. The speed of the wave is then $c = -\dot{\theta}_{n_0}/kn_0$. Two types of travelling waves $u(x-ct)$ are present in this truncated five-mode system, those in which none of the amplitudes vanish and satisfy $\dot{a}_n = \mathrm{i}n\dot{\theta}_1 a_n$ which propagate with speed $c = -\dot{\theta}_1/k$ and a travelling wave with $a_1 = a_3 = a_5 = 0$ for which $\dot{a}_4 = 2\mathrm{i}\dot{\theta}_2 a_4$ whose speed of propagation is $c = -\dot{\theta}_2/2k$. The first type of travelling wave (mixed mode) is found in primary and secondary branches, and may lead to single or multipulsed solutions. A preliminary study of this system with large dispersion showed the simultaneous stability of the two travelling waves that bifurcate from the trivial solution [21] and an additional secondary branch. In this work we present results that show the evolution of the bifurcation structure from small to large dispersion. We refer to ref. [14] for the corresponding results for the KS equation. As we describe in the next section quantitative and qualitative changes arise as dispersion increases. For large dispersion, the structure of the bifurcation diagram is considerably simpler.

3. Numerical results

Travelling waves were calculated by obtaining reduced equations for the fixed points and their speed as in ref. [10]. These were verified with the package Auto [22,23]. The stability of the modulated travelling waves was determined using Auto.

We have defined the norm

$$|u| = \left(\frac{1}{2L} \int_0^L u^2(x,t)\, \mathrm{d}x \right)^{1/2},$$

which is time independent for travelling waves; for modulated waves, which correspond to time periodic solutions of ρ_n and $\theta_n - (n/n_0)\theta_{n_0}$, we define the norm as the time average

$$|u| = \frac{1}{T} \int_0^T |u(t)|\, \mathrm{d}t,$$

where T is the period of the solution for ρ_n and $\theta_n - (n/n_0)\theta_{n_0}$. In fig. 1 we show the norm as a function of $(L/\pi)^2$ for $\delta = 0.1$ for the travelling wave solutions. The first branch of travelling waves (TW1) that bifurcates from $u = 0$ at $L = 2\pi$ is stable, it loses stability through a Hopf bifurcation to a stable modulated travelling wave (MTW1). The branch that bifurcates from $L = 4\pi$ (TW2) begins as an unstable TW, and has a branch point where a secondary unstable TW branch is born, which we have labelled TW3. TW3 is stabilized at a limit point and subsequently loses stability through a Hopf bifurcation. TW2 gains stability at a second branch point and at larger L loses stability through a Hopf bifurcation. The two branch points along TW2 correspond to a single branch TW3 as we can see in fig. 2. The MTW (MTW3) that bifurcates from TW3 is unstable and terminates in a homoclinic orbit. There is an interval at larger L where TW2 loses stability and then regains it. From direct time integrations we have found period doubling and chaotic solutions in this interval. Its proximity to $L = 6\pi$, the point where a new branch bifurcates from $u = 0$, indicates that more modes need to be included to study this region so we have not pursued it further. We observe the simultaneous stability of different solutions. There is a region where TW1 is the only stable solu-

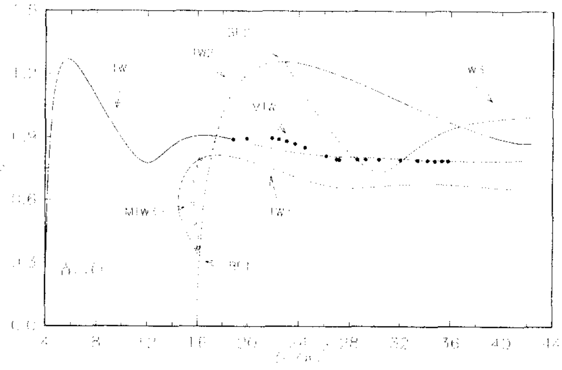

Fig. 1. Bifurcation diagram for $\delta = 0.1$. The solid lines correspond to stable travelling waves, the dashed lines to unstable travelling waves. Full circles correspond to stable modulated travelling waves, open circles to unstable ones. BP1 and BP2 correspond to the two branch points on TW2.

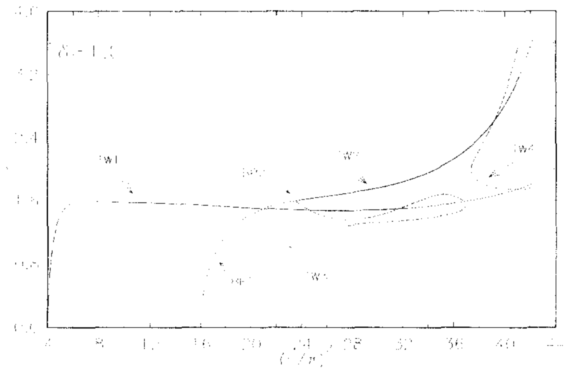

Fig. 2. Travelling waves for $\delta = 1$. TW1 is stable over a large region, TW3 is unstable. A new two-pulse branch, TW4, has appeared at large L.

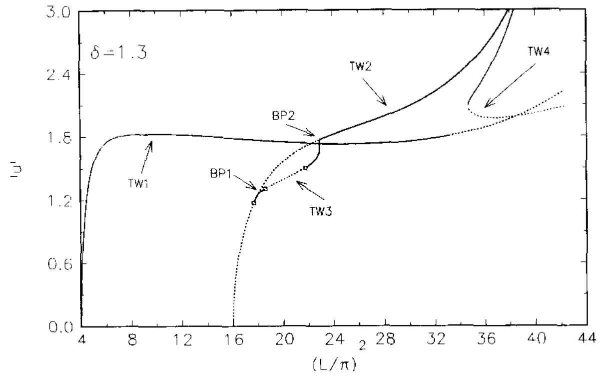

Fig. 3. At larger δ qualitative changes arise in the stability of TW2 and TW3. TW2 gains stability at a Hopf bifurcation, loses it at a branch point and regains stability at a second branch point. Hopf bifurcation points on TW2 and TW3 are indicated by open squares. They correspond to HB1, HB2 and HB3 in increasing order of L in the following figures.

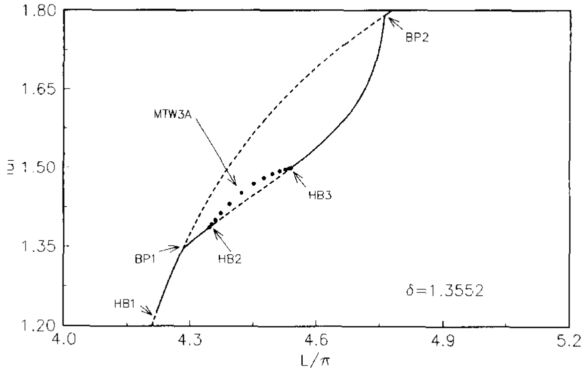

Fig. 4. Enlargement of the TW3 shown in fig. 3. The modulated travelling wave, MTW3A, that bifurcates from TW3 is stable.

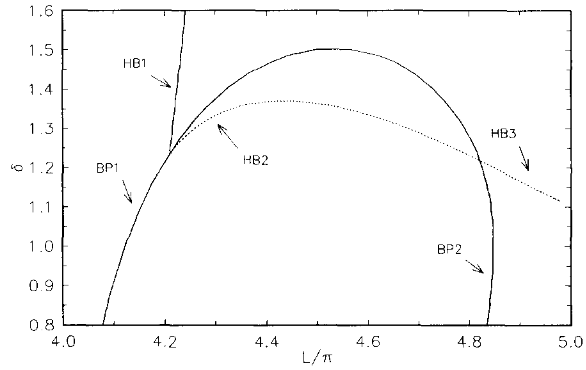

Fig. 5. Locus of the bifurcation points shown in fig. 4.

tion, another region where TW1 and TW3 are stable, and characteristic of low dispersion, a region where no travelling wave is stable. Then a region where TW2 and MTW1 are stable. In fig. 2, we show the travelling waves for $\delta = 1$. We can now see that TW3 connects the two branch points on TW2. For $\delta = 1$ TW3 is no longer stable. We observe that the region of stability of TW1 is greatly increased, and TW1 and TW2 are now simultaneously stable in a region of L. Forward time integrations may lead to a single or double pulse travelling wave solution of different speed, depending on the initial conditions. A new branch TW4 has appeared, leading to a different two-pulse state [21], but more modes should be included to verify its relation to the original PDE. The MTW1 that bifurcates from TW1 is stable, in the region where it exists, and coexists whit the stable TW2. Hence some initial conditions lead to MTW1, others to the two-pulse TW2. In fig. 3 we show the corresponding graph for $\delta = 1.3$. Qualitative changes are found in the stability of TW2. As L is increased the TW2 first gains stability at a Hopf bifurcation (labeled HB1) producing an unstable MTW2, but then loses it at the branch point BP1 where TW3 appears. At the next branch point BP2 on TW2, the TW3 reconnects to TW2; this bifurcation is subcritical. Both ends of TW3 bifurcate stably from TW2, later losing stability to an interval of MTW3A. As δ is increased the branch points BP1 and BP2 on TW2 and the points HB2 and HB3 where the Hopf bifurcation on TW3 occurs, come together until finally the TW3 entirely disappears. In fig. 4 we show a detail of this region for $\delta = 1.355$. The locus of the bifurcation points is shown in fig. 5. For $\delta > 1.24$, TW2 gains stability at a Hopf bifurcation. At $\delta = 1.502$ and $L = 4.564$ BP1 merges with BP2, and for larger values of δ TW3 no longer exists. Prior to that, at $\delta = 1.369$ and $L = 4.419$ HB2 merges with HB3, so that between this value of δ and 1.502 TW3 is stable. After δ exceeds the value 1.502 we find no further qualitative changes in the stability of TW1 and TW2. Finally in fig. 6 we show the travelling waves for $\delta = 3$. This diagram is representative for δ above 1.502. All three, TW1, TW2 and TW4, lose stability through Hopf bifurcation at large values of L. We see that for large δ, for any L, we find stable travelling waves. Moreover, depending on initial con-

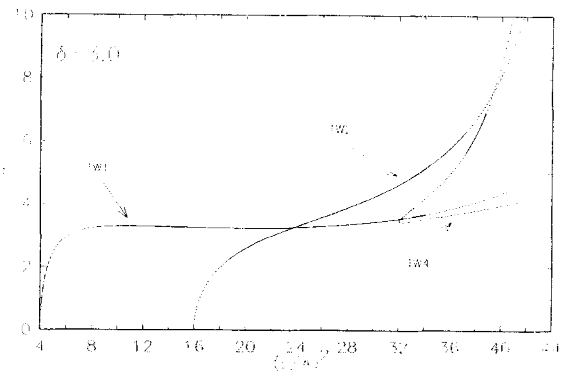

Fig. 6. At larger δ TW3 is no longer present, the travelling waves that bifurcate from $u = 0$ are stable for large ranges of L.

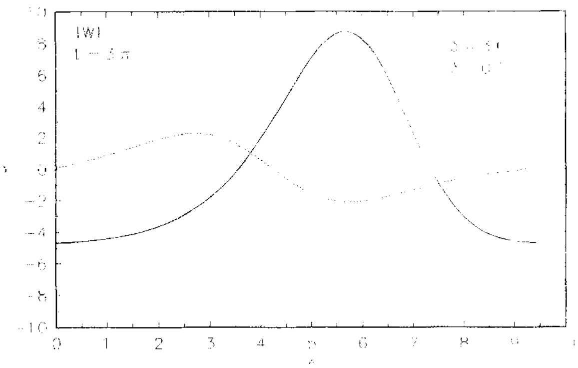

Fig. 7. Waveform $u(x)$ on TW1 for different values of dispersion. Dispersion increases the amplitude and localization.

ditions the system will evolve into one or two-pulse travelling waves. The secondary branch TW4 should be studied including more modes. In the following figures we show the waveform $u(x,t)$ for different values of δ. In fig. 7 we show the waveform on TW1 for two different values of dispersion. The main feature is the increased localization as dispersion increases. The two-pulse branch TW2 which is not shown corresponds to the periodic repetition of TW1. The solution for $u(x,t)$ when the system has evolved into the modulated travelling wave MTW1 is shown for $\delta = 0.1$ and $L = 4.5\pi$ in fig. 8. A clear identification of pulses is not possible now. The modulus of the Fourier amplitudes is periodic as is shown in fig. 9.

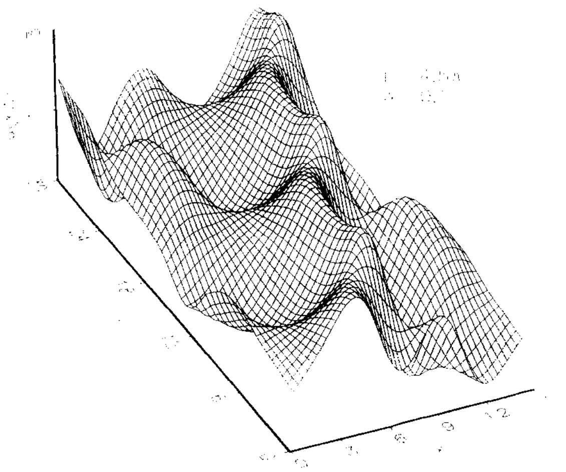

Fig. 8. Time evolution of the solution once the system has evolved into the modulated travelling wave MTW1. A complete period has been plotted.

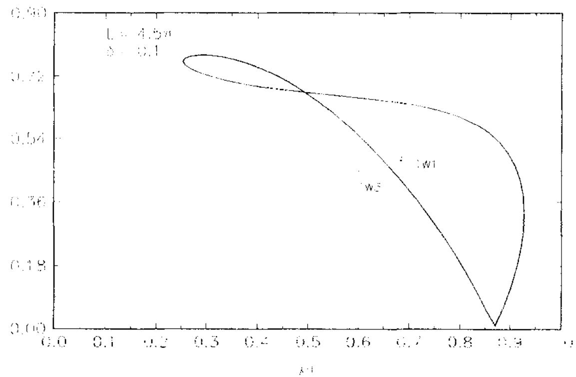

Fig. 9. The periodic solution for the amplitudes of the first two Fourier components of the modulated travelling wave MTW1 shown in fig. 8. The fixed points corresponding to the travelling waves TW1 and TW3 are indicated by a cross. The fixed point for TW2, not shown here, lies on the y-axis.

4. Summary

We have studied the bifurcation diagram of a five-mode truncation of the equation $u_t + uu_x + \delta u_{xxx} + u_{xx} + u_{xxxx} = 0$. As numerical integrations of the full equation show, for large dispersion, almost all initial conditions evolve into rows of solitary like travelling pulses. The number of pulses depends on the size of the interval and on the initial conditions. Our interest has been to gain some understanding of the role of increasing dispersion on the structure of the bifurcations on a small domain which may explain in part the behavior observed in integrations of the PDE. For small dispersion, we have found that initial conditions may evolve into a varied assortment of states, ranging from travelling waves, which can be steady at some precise values of L, to chaotic solutions. As dispersion increases, the bifurcation diagram becomes gradually simpler. The main effect of dispersion is to extend the region of existence and stability of the travelling waves that bifurcate from $u = 0$ and to increase the localization of the pulses. This stabilization implies that, for large dispersion, anywhere in the region studied, stable travelling waves exist. Furthermore, different types of travelling waves coexist stably in wide regions so that initial conditions select the wavelength for fixed parameters of the problem.

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