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On the principal bifurcation branch of a third-order nonlinear long-wave equation

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Abstract

We study the principal bifurcation curve of a third-order equation which describes the nonlinear evolution of several systems with a long-wavelength instability. We show that the main bifurcation branch can be derived from a variational principle. This allows us to obtain a close estimate of the complete branch. In particular, when the bifurcation is subcritical, the large amplitude stable branch can be found in a simple manner.

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1. Introduction

Long-wavelength instabilities have been found in diverse physical systems, such as Rayleigh–Bénard convection between insulating boundaries [1], Marangoni convection in the presence of a free surface [2–4], directional solidification [5, 6], Langmuir circulation in the ocean [7] and others [8]. Evolution equations that describe the nonlinear evolution of the instability can be derived by means of an expansion procedure based on the long scale of the instability [9]. Unlike the case of instabilities with non-zero wavelength where the weakly nonlinear evolution of the instability is governed by the Ginzburg–Landau equation, for long-wavelength instabilities different equations arise according to the problem considered [10]. In the absence of a unique equation to describe these instabilities each equation needs to be examined to determine whether it provides an adequate description of the problem under study. It is often found that the bifurcation from the basic steady state is subcritical, thus unstable. In such cases it is important to determine whether large amplitude stable solutions exist. The usual procedure to capture the turning back of a subcritical branch is to perform an asymptotic expansion close to the bifurcation point. Suitable scaling of parameters is necessary in order to capture stable solutions. If this procedure fails one may search for approximate large amplitude solutions of the equation which is not always feasible. The alternative is to obtain the bifurcation curve numerically.

The purpose of this work is to study a prototype long-wave evolution equation which, in a given parameter range, bifurcates subcritically from the basic state [11, 12], and show that the principal bifurcation branch derives from an integral variational principle. It is then possible to obtain lower bounds on the complete branch that are as close as desired to the exact solution. While we treat a specific equation, the method can be applied to other equations as well. In section 2 the main result is derived. In section 3 a particular exactly solvable case is discussed. Analytical and numerical results for the general case are given in section 4.

2. Formulation of the problem

We consider the long-wavelength equation

$$f_t + \lambda f_{\xi\xi} + \kappa f_{\xi\xi\xi\xi} - \nu\left(f_\xi^3\right)_\xi + \mu(f_\xi f_{\xi\xi})_\xi + \delta(f f_\xi)_\xi = 0, \tag{1}$$

in a periodic box of length $2L$. The boundary conditions $f_\xi(\pm L) = f_{\xi\xi\xi}(\pm L) = 0$ are chosen to satisfy physical requirements. With these boundary conditions the space average $\overline{f}$ is conserved.

This equation describes the weakly nonlinear evolution of Rayleigh–Bénard and Marangoni convection between insulating boundaries [11–15]. Particular cases of steady state solutions to this equation have been studied in connection with directional solidification. In this context the case $\nu = 0$ is known as the (steady state) Riley–Davis equation with vanishing segregation coefficient. The case $\nu = 0$ and $\mu = 0$ is known as the Sivashinsky equation. The bifurcation structure of these two equations has been studied extensively [5, 6, 16, 17]. In the context of Rayleigh–Bénard convection, the term proportional to μ arises due to the asymmetry between upper and lower mechanical boundary conditions [9] and the term proportional to δ arises when non-Boussinesq effects are included [11, 12, 14]. The purpose of this work is to study the *principal* (monotonic) branch that bifurcates from the first eigenvalue. These solutions are sometimes called *simple* [16].

Introducing a new length scale $x = \xi/L$, scaling the amplitude as $f = \sqrt{\kappa/\nu}\phi$ and integrating once, the equation can be written as

$$r\phi_x + \phi_{xxx} - \phi_x^3 + A\phi_x\phi_{xx} + B\phi\phi_x = 0, \tag{2}$$

for appropriate constants A and B. The constant of integration in the equation above vanishes due to the boundary conditions. Due to the symmetry $x \to -x$ we may consider the solution on half the interval and impose

$$\phi_x(0) = \phi_x(1) = 0. \tag{3}$$

In addition, for the principal branch we impose

$$\phi(0) = \phi_m, \qquad \phi(1) = 0, \qquad \phi_x < 0. \tag{4}$$

Previous studies of this equation show that the quadratic term $B\phi\phi_x$ is destabilizing, it gives rise to a subcritical instability ([5, 11, 12]). If $B < 0$ the bifurcation is supercritical. The bending back of the unstable branch can be described close to the transition point $B = 0$ by means of small amplitude expansions. Expanding the solution in a small parameter ϵ, as $\phi(x) = \epsilon(\phi_0 + \epsilon\phi_1 + \cdots)$ and the eigenvalue as $r = r_0 + \epsilon r_1 + \cdots$, we obtain for the eigenfunction in leading order

$$\phi_0(x) = \phi_m \frac{1 + \cos[\pi x]}{2},$$

and at order ϵ

$$\phi_1(x) = \frac{1}{48\pi^2}(B - A\pi^2)(\cos[2\pi x] - 1).$$

The eigenvalue up to order ϵ^2 is given by

$$r = \pi^2 - \epsilon B\frac{\phi_m}{2} + \epsilon^2\frac{\phi_m^2}{96\pi^2}[18\pi^4 + (B - \sqrt{2}\pi^2 A)^2 - (3 - 2\sqrt{2})\pi^2 AB].$$

For positive B the bifurcation is subcritical, but the order ϵ^2 term shows that it regains stability for small enough B. This result is valid only for small amplitude and sufficiently low values of B.

Our purpose is to show that the principal branch $r(\phi_m)$ can be obtained from a variational principle from which lower bounds on the bifurcation curve are obtained. This enables us to obtain non-perturbative results valid at arbitrary large amplitude and for all parameter values. The method used to derive this result is related to that developed to study a class of reaction diffusion problems [18–20] and others [21].

The first step is to rewrite the equation in phase space, therefore we define

$$p(\phi) = -\frac{\mathrm{d}\phi}{\mathrm{d}x}$$

and write the problem to be solved as [12]

$$rp + p\frac{\mathrm{d}}{\mathrm{d}\phi}\left(p\frac{\mathrm{d}p}{\mathrm{d}\phi}\right) - p^3 + Ap^2\frac{\mathrm{d}p}{\mathrm{d}\phi} + B\phi p = 0, \tag{5}$$

with

$$p(0) = p(\phi_m) = 0, \qquad p > 0. \tag{6}$$

Even though the equation above can be reduced to quadratures, an explicit expression for the eigenvalue $r(\phi_m)$ cannot be obtained except in the case $B = 0$ (appendix A). From the variational principle that we construct below, the exact solution for the case $B = 0$ can be recovered, and arbitrarily accurate bounds can be obtained when $B \neq 0$.

Let $g(\phi)$ be an arbitrary positive continuous function, with continuous second derivatives, such that $g(0) = g(\phi_m) = 0$. Multiplying equation (5) by g/p and integrating by parts we obtain, after regrouping terms,

$$r\int_0^{\phi_m} g(\phi)\,\mathrm{d}\phi + B\int_0^{\phi_m} \phi g(\phi)\,\mathrm{d}\phi = \int_0^{\phi_m} p^2(\phi)h(\phi)\,\mathrm{d}\phi, \tag{7}$$

where we have defined

$$h(\phi) = -\frac{1}{2}g'' + g + \frac{A}{2}g'. \tag{8}$$

Boundary terms which appear when integrating by parts vanish because of (6) and the conditions on g. In addition we require that the function g be such that $h > 0$ in $(0, \phi_m)$, a condition which can always be met (see appendix B).

We now use Hölder's inequality,

$$\left(\int_0^{\phi_m} |s(\phi)|^\ell\,\mathrm{d}\phi\right)^{1/\ell}\left(\int_0^{\phi_m} |q(\phi)|^k\,\mathrm{d}\phi\right)^{1/k} \geqslant \int_0^{\phi_m} s(\phi)q(\phi)\,\mathrm{d}\phi,$$

with

$$\frac{1}{\ell} + \frac{1}{k} = 1$$

to bound the right-hand side of equality (7). Choosing $s = (p^2h)^{1/3}$, $q = p^{-2/3}$, $\ell = 3$ and $k = 3/2$ we obtain

$$\left(\int_0^{\phi_m} p^2h\,\mathrm{d}\phi\right)^{1/3}\left(\int_0^{\phi_m} \frac{1}{p}\,\mathrm{d}\phi\right)^{2/3} \geqslant \int_0^{\phi_m} h^{1/3}\,\mathrm{d}\phi.$$

Using

$$1 = \int_0^1 \mathrm{d}x = \int_0^{\phi_m} \frac{\mathrm{d}\phi}{p},$$

the above inequality becomes

$$\int_0^{\phi_m} p^2 h \, \mathrm{d}\phi \geqslant \left[\int_0^{\phi_m} h^{1/3} \, \mathrm{d}\phi \right]^3. \tag{9}$$

Inserting (9) in (7) we obtain the desired lower bound,

$$r \geqslant \frac{\left[\int_0^{\phi_m} h^{1/3}(\phi) \, \mathrm{d}\phi\right]^3 - B \int_0^{\phi_m} \phi g(\phi) \, \mathrm{d}\phi}{\int_0^{\phi_m} g(\phi) \, \mathrm{d}\phi}. \tag{10}$$

To show that this is a variational principle we must prove that there exists a function $g = \hat{g}$ for which equality holds in (10) [18]. In Hölder's inequality, the case of equality corresponds to $s^\ell = q^k$. In the present case we have that equality in (9) holds when

$$h = \hat{h} = \frac{1}{p^3}. \tag{11}$$

From the definition of h we have equality in (10) when g is a solution of

$$-\frac{1}{2} g'' + g + \frac{A}{2} g' = \frac{1}{p^3}, \tag{12}$$

satisfying $g(0) = g(\phi_m) = 0$. To prove that the solution to this boundary value problem, which we call $\hat{g}$, exists and that it yields finite values for the appropriate integrals in (10) is straightforward (see appendix B for details).

Since equality holds for a special function of g we conclude that the principal bifurcation curve of (2) is given by

$$r = \max_g \frac{\left[\int_0^{\phi_m} h^{1/3}(\phi) \, \mathrm{d}\phi\right]^3 - B \int_0^{\phi_m} \phi g(\phi) \, \mathrm{d}\phi}{\int_0^{\phi_m} g(\phi) \, \mathrm{d}\phi}, \tag{13}$$

where the maximum is taken among all positive continuous functions g (with continuous second derivatives and such that $g(0) = g(\phi_m) = 0$) for which $h > 0$ and integrals exist. The maximum is achieved for $g = \hat{g}$ up to a multiplicative constant.

It is worth pointing out that the Euler–Lagrange equation for (13) can be obtained easily (appendix C) and is, effectively, equation (5) when $g = \hat{g}$.

Expressions (10) and (13) constitute our main result. Using adequate trial functions g in (10) we obtain lower bounds on the bifurcation branch $r(\phi_m)$. Moreover we know there exists a trial function for which equality can be attained.

In the following section we study the case $B = 0$ and show that the maximizing $\hat{g}$ can be constructed explicitly in this case.

3. The case $B = 0$

In this section we show that the exact formula for the eigenvalue can be obtained from the variational principle when $B = 0$. Furthermore, the optimal g obtained for this case will provide a good trial function for $B \neq 0$.

To obtain the solution for the eigenvalue, instead of solving the maximizing problem (13), we address the equivalent problem of minimizing its inverse. Let

$$J_m = \min_g J[g] = \min_g \frac{\int_0^{\phi_m} g(\phi) \, \mathrm{d}\phi}{\left(\int_0^{\phi_m} h^{1/3}(\phi) \, \mathrm{d}\phi\right)^3}, \tag{14}$$

with $g > 0$, and $h > 0$ as defined in (8). Given h, g can be found in terms of h by means of Green's function $G(\phi, \phi')$ for (8). Defining

$$s(\phi) = h^{1/3}(\phi) \tag{15}$$

we have that

$$g(\phi) = \int_0^{\phi_m} G(\phi, \phi') s^3(\phi')\,\mathrm{d}\phi', \tag{16}$$

where Green's function $G(\phi, \phi')$ corresponding to equation (8) is given by

$$G(\phi, \phi') = \begin{cases} \dfrac{2}{b\sinh(b\phi_m)}\,\mathrm{e}^{A(\phi-\phi')/2}\sinh(b\phi)\sinh(b(\phi_m-\phi')) & \text{for} \quad \phi < \phi' \\ \dfrac{2}{b\sinh(b\phi_m)}\,\mathrm{e}^{A(\phi-\phi')/2}\sinh(b\phi')\sinh(b(\phi_m-\phi)) & \text{for} \quad \phi > \phi', \end{cases} \tag{17}$$

with

$$b = \tfrac{1}{2}\sqrt{A^2+8}.$$

Then,

$$\int_0^{\phi_m} g(\phi)\,\mathrm{d}\phi = \int_0^{\phi_m} \mathrm{d}\phi'\, F(\phi') s^3(\phi'), \tag{18}$$

where

$$\begin{aligned} F(\phi) = \int_0^{\phi_m} G(\xi, \phi)\,\mathrm{d}\xi = 1 - \mathrm{e}^{-A\phi/2}[&\cosh(b\phi) \\ &- \coth(b\phi_m)\sinh(b\phi) + \mathrm{e}^{A\phi_m/2}\operatorname{cosech}(b\phi_m)\sinh(b\phi)]. \end{aligned} \tag{19}$$

Finally, (14) can be written as

$$J_m = \min_s J[g] = \min_s \frac{\int_0^{\phi_m} F(\phi)s^3(\phi)\,\mathrm{d}\phi}{\left(\int_0^{\phi_m} s(\phi)\,\mathrm{d}\phi\right)^3}, \tag{20}$$

which is a simple variational problem for which it is straightforward to find the minimizing function. The minimizing s is given by

$$s_{\min}(\phi) = \frac{1}{N}\frac{1}{\sqrt{F(\phi)}}, \tag{21}$$

where

$$N = \int_0^{\phi_m} \frac{\mathrm{d}\phi}{\sqrt{F(\phi)}}. \tag{22}$$

Using this function in (20), we find

$$J_{\min} = J(s_{\min}) = \frac{1}{N^2}$$

and the exact value of r for $B = 0$ is $1/J(s_{\min})$. Spelled out explicitly,

$$r(\phi_m) = \left(\int_0^{\phi_m} \frac{\mathrm{d}\phi}{\sqrt{F(\phi)}}\right)^2,$$

which is the exact expression for the bifurcating branch when $B = 0$ (appendix A). Note that we have also obtained implicitly the solution to the equation itself, since equality in (13) holds when $\hat{h} = s_{\min}^3 = 1/p^3$. The solution of the equation is obtained by integrating $p = -\mathrm{d}\phi/\mathrm{d}x = 1/s_{\min}$.

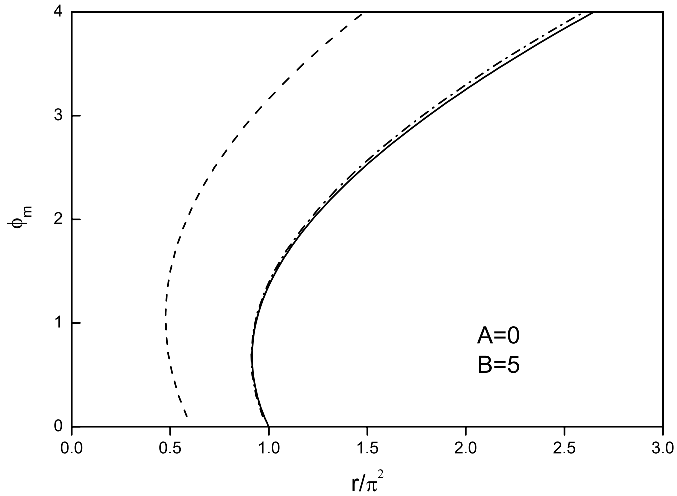

Figure 1. Exact bifurcation curve and bounds with different trial functions.

4. Bounds for $B \neq 0$

The exact solution for the principal branch cannot be obtained in closed form when $B \neq 0$. Instead, upper bounds can be obtained using different trial functions in (10). In this section accurate bounds will be obtained numerically. A simple analytical bound will be obtained in the special case $A = 0$.

4.1. Numerical bounds

Guided by the results of the previous section, we choose as a trial function the exact solution for the case $B = 0$, that is,

$$g_1(\phi) = \int_0^{\phi_m} G(\phi, \phi') s_{\mathrm{min}}^3(\phi')\, \mathrm{d}\phi'. \tag{23}$$

Replacing this choice for g in (10) we obtain

$$r \geqslant \left(\int_0^{\phi_m} \frac{\mathrm{d}\phi}{\sqrt{F(\phi)}} \right)^2 - \frac{B}{N} \int_0^{\phi_m} \frac{1}{F^{3/2}(\phi')} \left(\int_0^{\phi_m} \phi G(\phi, \phi')\, \mathrm{d}\phi \right) \mathrm{d}\phi' \tag{24}$$

where N is defined in (22).

A different, simpler trial function, is obtained choosing $h(\phi) = 1$, which gives

$$g_2(\phi) = \int_0^{\phi_m} G(\phi, \phi')\, \mathrm{d}\phi'. \tag{25}$$

In figure 1 below we show the bifurcation curve for $A = 0$, $B = 5$. The solid line is the exact numerical solution obtained using the software Auto [22]. The dot-dashed line is the lower bound obtained using the trial function (23), and the dashed line is the bound obtained with the trial function (25). The bound obtained using g_1 as a trial function provides a very close estimation of the bifurcating branch, especially at low amplitudes. Already the simple trial function g_2 which does not give a close estimate is enough to prove that the subcritical bifurcation is stabilized at larger amplitudes. In figure 2 we show the numerical results for $A = 0$, $B = 10$. The curves shown are as in figure 1. At small amplitudes g_1

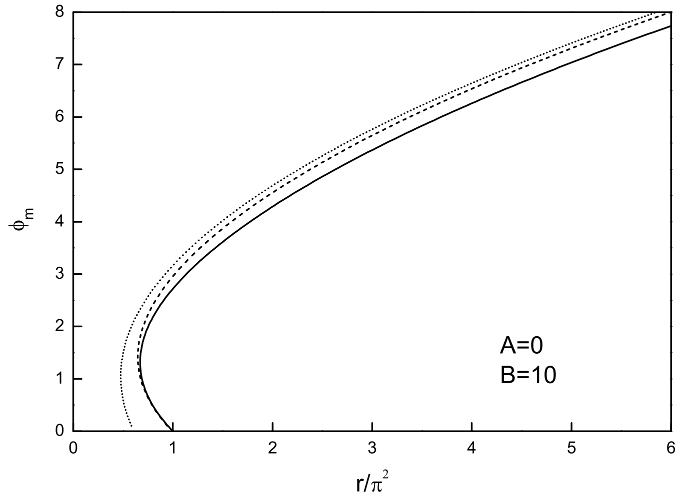

Figure 2. As in figure 1 for different parameter values.

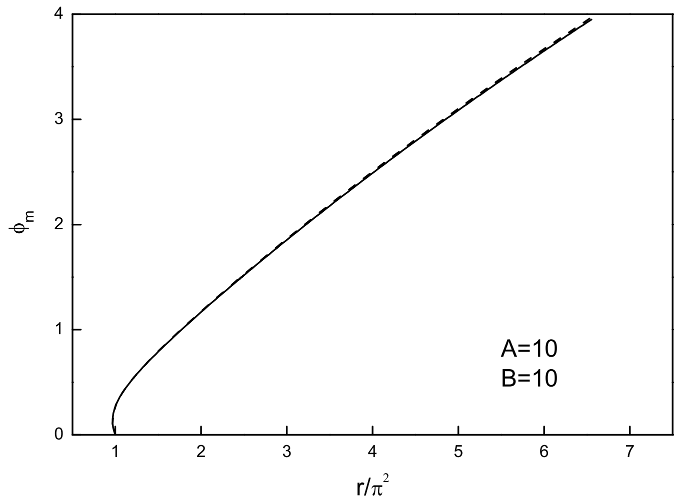

Figure 3. Exact bifurcation curve and bound with the $B = 0$ exact solution.

gives a close estimate, at large amplitudes it captures the qualitative behaviour, still providing a good bound. Here too the simple trial function g_2 suffices to capture the stabilization of the subcritical branch at larger amplitudes.

In figure 3 we show the numerical results for $A = 10$, $B = 10$. The solid line is the numerical solution of the bifurcation problem using Auto, the dashed line is the bound obtained with g_1.

4.2. Analytical bounds

In the case $A = 0$ a simple admissible trial function is $g_3(\phi) = \phi(\phi_m - \phi)$. Then,

$$h(\phi) = 1 + \phi(\phi_m - \phi) > \phi(\phi_m - \phi).$$

Using $g_3(\phi)$ and this lower bound for the corresponding $h(\phi)$ in (10) we obtain

$$\begin{aligned} r &\geqslant \frac{\int_0^{\phi_m} [\phi(\phi_m - \phi)]^{1/3}\,\mathrm{d}\phi - B\int_0^{\phi_m} \phi^2(\phi_m - \phi)\,\mathrm{d}\phi}{\int_0^{\phi_m} \phi(\phi_m - \phi)\,\mathrm{d}\phi} \\ &\geqslant 6B(4/3, 4/3)\phi_m^2 - \frac{B}{2}\phi_m \\ &\geqslant 3, 18\phi_m^2 - \frac{B}{2}\phi_m, \end{aligned} \tag{26}$$

where $B(x, y)$ is the standard Beta function. This bound, valid for all amplitudes and all values of B shows that the subcritical branch always turns around and becomes stable at sufficiently large amplitude.

4.3. Bounds for solutions of zero average

Here we address the problem of solutions to (2) of zero average. The bifurcating branch $r(\phi_m)$ obtained above does not correspond to a solution of fixed average. Each point along the branch has different average, $\overline{\phi} = \overline{\phi}(\phi_m)$. It is possible to obtain bounds for solutions of zero average as follows. Define $u = \phi - \overline{\phi}$. In terms of u, equation (2) becomes

$$(r + B\overline{\phi})u_x + u_{xxx} - u_x^3 + Au_x u_{xx} + Buu_x = 0,$$

with

$$u_x(0) = u_x(1) = 0, \qquad u(0) = \phi_m - \overline{\phi}, \qquad u(1) = -\overline{\phi}.$$

The amplitude of the function u is ϕ_m, its average is zero and the bifurcating branch $r_{\overline{u}=0}$ is given by

$$r_{\overline{u}=0}(\phi_m) = r(\phi_m) + B\overline{\phi}.$$

Since $0 \leqslant \overline{\phi} \leqslant \phi_m$, we know that for $B > 0$, $B\overline{\phi} > 0$, whereas for $B < 0$, $B\overline{\phi} > B\phi_m$. Then we know that for a solution of zero average

$$r_{\overline{u}=0} \geqslant \begin{cases} r(\phi_m) & \text{for} \quad B > 0 \\ r(\phi_m) + B\phi_m & \text{for} \quad B < 0. \end{cases} \tag{27}$$

The analytical bound (26) for $A = 0$ becomes for solutions of zero average,

$$r_{\overline{u}=0} \geqslant \begin{cases} 3.18\phi_m^2 - B\phi_m/2 & \text{for} \quad B > 0 \\ 3.18\phi_m^2 + B\phi_m/2 & \text{for} \quad B < 0. \end{cases} \tag{28}$$

This shows that also for zero average solutions, all subcritical branches turn back at sufficiently large amplitude.

5. Summary

We have studied the principal bifurcation branch of a third-order equation which exhibits a subcritical bifurcation for certain parameter values. The bifurcation problem can be characterized by an integral variational principle from which lower bounds on the complete branch can be found. With the use of appropriate trial functions it is possible to obtain an accurate estimation of the complete branch, without resource to perturbation theory. This permits us to capture the stabilization of the subcritical branch even when the turning point occurs at large amplitude.

While we focused on a specific equation, the method can be applied to other nonlinear equations [18–21].

Acknowledgments

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Appendix A

In this appendix we show that even though the problem can be reduced to quadratures, an explicit expression for the eigenvalue can be obtained only when $B = 0$.

Equation (5) can be reduced to a linear equation for $u = p^2$,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + A\frac{\mathrm{d}u}{\mathrm{d}\phi} - 2u + 2(r + B\phi) = 0. \tag{A.1}$$

The solution to this equation which satisfies the boundary conditions $u(0) = u(\phi_m) = 0$ is given by

$$u = \left(r + \frac{AB}{2}\right) F(\phi) + BZ(\phi), \tag{A.2}$$

with $F(\phi)$ as defined in (19) and

$$Z(\phi) = \phi - \phi_m\, \mathrm{e}^{-A(\phi-\phi_m)/2} \operatorname{cosech}(b\phi_m) \sinh(b\phi). \tag{A.3}$$

The eigenvalue r follows from the condition:

$$\int_0^{\phi_m} \frac{\mathrm{d}\phi}{p(\phi)} = 1, \tag{A.4}$$

which may be written as

$$\int_0^{\phi_m} \frac{\mathrm{d}\phi}{\sqrt{(r + AB/2)F(\phi) + BZ(\phi)}} = 1. \tag{A.5}$$

Since this integral cannot be performed analytically, only in the case $B = 0$ an explicit expression for r is found.

Appendix B

In this appendix we analyse the properties of the solution of (12), i.e., the solution of the two point boundary value problem

$$-\frac{1}{2}g'' + \frac{A}{2}g' + g = \frac{1}{p^3} \tag{B.1}$$

with $g(0) = g(\phi_m) = 0$, where $p(\phi)$ is a solution of (5) with $p(0) = p(\phi_m) = 0$. One can conveniently write (B.1) as

$$L(g) = \frac{2}{p^3}\, \mathrm{e}^{-A\phi}, \tag{B.2}$$

where the linear operator L is given by

$$L = -\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\mathrm{e}^{-A\phi}\frac{\mathrm{d}}{\mathrm{d}\phi}\right) + \mathrm{e}^{-A\phi}. \tag{B.3}$$

The operator L acting on functions g that vanish in 0 and ϕ_m is self-adjoint (with respect to the usual inner product $(f, g) = \int_0^{\phi_m} f(\phi)g(\phi)\,\mathrm{d}\phi$) and positive definite (to be precise L is acting on $H_0^1(0, \phi_m)$). In fact Green's function associated with the operator L (i.e., the kernel of L^{-1}) is explicitly given by $\mathrm{e}^{A\phi'}G(\phi, \phi')$ where G is given by (17). Note that $\mathrm{e}^{A\phi'}G(\phi, \phi')$ is symmetric under the exchange of ϕ and ϕ', and pointwise positive in the interval $(0, \phi_m)$.

Since the right-hand side of (B.2) is non-negative for the solutions to (2) that we are considering, i.e., for the *principal branch*, and since L is positive definite, the corresponding solution g to the boundary value problem (B.1) is positive in $(0, \phi_m)$. By standard regularity properties of differential equations, the solution g is continuous in $(0, \phi_m)$. In the neighbourhood of the end points, the solution g of (B.1) behaves as follows: $g(\phi)$ vanishes like $\sqrt{\phi}$ near 0, whereas $g(\phi)$ vanishes like $\sqrt{\phi_m - \phi}$ near ϕ_m. To determine this behaviour, note that the solutions of (2) corresponding to the *principal branch* satisfying the boundary conditions (3), (4) are real analytic. Thus, in the neighborhood of $x = 0$ (because of the boundary conditions), $\phi(x) \approx \phi_m - cx^2$, for some $c > 0$, hence, $p = -\phi'(x) \approx 2cx$ near $x = 0$, and inverting (i.e., expressing x in terms of ϕ we finally have $p(\phi) \approx \sqrt{(\phi_m - \phi)}$ in the neighborhood of $\phi = \phi_m$. Analogously, $p(\phi) \approx \sqrt{\phi}$ in the neighborhood of $\phi = 0$. From the properties of Green's function for the operator L (note that G vanishes linearly near each of the endpoints) we see that the solution of (B.1) also behaves as $\sqrt{\phi}$ near $\phi = 0$ and as $\sqrt{(\phi_m - \phi)}$ near $\phi = \phi_m$.

Concerning the existence of the different integrals that appear on the right-hand side of (10) when g is the solution of (B.1), note that h is continuous in $(0, \phi_m)$, and although h is singular near the end points, $h^{1/3}$ behaves like $1/\sqrt{\phi}$ (respectively like $1/\sqrt{\phi_m - \phi}$) in the neighborhood of 0 (respectively in the neighborhood of ϕ_m), so $h^{1/3}$ is locally integrable near the end points. The other two integrals also exist, because g is continuous in $[0, \phi_m]$.

Appendix C

In this appendix we derive the Euler–Lagrange equation for (13). Consider the maximization of the functional

$$K[g] = \left[\int_0^{\phi_m} h^{1/3}(\phi)\,\mathrm{d}\phi\right]^3 - B\int_0^{\phi_m} \phi g(\phi)\,\mathrm{d}\phi, \tag{C.1}$$

where

$$h(\phi) = -\frac{1}{2}g'' + g + \frac{A}{2}g' \tag{C.2}$$

subject to

$$\int_0^{\phi_m} g(\phi)\,\mathrm{d}\phi = 1.$$

Introducing the constraint with a Lagrange multiplier λ, we obtain the Euler–Lagrange equation

$$\left(\int_0^{\phi_m} h^{1/3}\,\mathrm{d}\phi\right)^2 \left[\frac{1}{h^{2/3}} - \frac{A}{2}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{h^{2/3}}\right) - \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left(\frac{1}{h^{2/3}}\right)\right] - B\phi + \lambda = 0. \tag{C.3}$$

At the extrema $h = 1/p^3$ and the above equation becomes

$$\left(\int_0^{\phi_m} \frac{\mathrm{d}\phi}{p}\right)^2 \left[p^2 - \frac{A}{2}\frac{\mathrm{d}p^2}{\mathrm{d}y} - \frac{1}{2}\frac{\mathrm{d}^2p^2}{\mathrm{d}y^2}\right] - B\phi + \lambda = 0. \tag{C.4}$$

Identifying the Lagrange multiplier λ with the eigenvalue r and observing that the integral in the equation above is equal to 1, we verify that the Euler–Lagrange equation for extrema of $K[g]$ is the original equation (5) whose solution we seek.

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