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On the transition from pulled to pushed monotonic fronts of the extended Fisher–Kolmogorov equation

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Abstract

The extended Fisher–Kolmogorov equation $u_t = u_{xx} - \gamma u_{xxxx} + f(u)$ with arbitrary positive $f(u)$, satisfying $f(0) = f(1) = 0$, has monotonic traveling fronts for $\gamma < \frac{1}{12}$. We find a simple lower bound on the speed of the fronts which allows to determine, for a given reaction term, when will the front of minimal speed be pushed.

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1. Introduction

The extended Fisher–Kolmogorov equation (EFK),

$$u_t = u_{xx} - \gamma u_{xxxx} + f(u) , \tag{1}$$

with $f(u) = u - u^3$ arises in the description of different systems. It appears, for example, in the study of phase transitions near a Lifshitz point [1,2]. It has been derived as an amplitude equation at the onset of instabilities near certain degenerate points [3]. It has also been proposed as a model for the onset of spatiotemporal chaos

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in bistable systems [4] and as a natural extension to the reaction diffusion equation ($\gamma = 0$) on which to study the dynamics of front propagation and pattern formation [5,6], etc. Its steady version with different functions $f(u)$,

$$u_{xxxx} + qu_{xx} + f(u) = 0 \, ,$$

is of interest in different fields and much work has been devoted to it. A very complete account of its solutions can be found in Ref. [7].

For $\gamma < \frac{1}{12}$ numerical results indicate that sufficiently localized conditions evolve into a uniform translating front joining the stable point $u = 1$ to the unstable $u = 0$ point [8]. Similarly, to what is found in the reaction diffusion equation, for the Fisher case [9,10] $f(u) = u - u^2$ and for $f(u) = u - u^3$ the front propagates with the linear speed which now is [6]

$$c_L = \frac{2}{\sqrt{54\gamma}}[1 + 36\gamma - (1 - 12\gamma)^{3/2}]^{1/2} \, , \tag{2}$$

obtained from linear analysis near $u = 0$. If $\gamma > \frac{1}{12}$ monotonic fronts do not exist, and a pattern may appear.

Numerical results of the integrations of the EFK equation with arbitrary $f(u)$ show that, as it occurs in the reaction diffusion equation, transition from pulled fronts (propagating with speed c_L) to pushed fronts (propagating at a speed greater than c_L) will occur as parameters in $f(u)$ are varied [8].

In recent work, a sufficient criterion on $f(u)$ for the existence of pulled fronts analogous to the KPP [10] and Aronson–Weinberger [11] criteria for the reaction diffusion equation has been established [12]. As for the reaction diffusion equation, this criterion gives sufficient but not necessary conditions on the reaction term for the appearance of the front propagating at the linear speed. Numerical results indicate that for small γ fronts of the EFK equation have similar properties to fronts of the reaction diffusion equation. Rigorous existence results of fronts of the EFK equation have been given for general functions $f(u)$ [15]. For $\gamma \to 0$ it, was proved that there is a minimal speed c^* for the existence of monotonic fronts, and that the fronts are stable. For $\gamma = \varepsilon^2$, the minimal speed is given by $c^* \geqslant 2 - \varepsilon^2 + \cdots$ [16].

The purpose of the present work is to establish a simple lower bound on the speed c^* for which monotonic fronts exist. This enables to test whether for a given function $f(u)$ the minimal speed is the linear value c_L obtained from the linear analysis at the edge of the front. The bound given in this work is not sharp, but the derivation suggests that it is possible to obtain a variational formulation for the minimal speed analogous to that given for the reaction diffusion equation [13,14]. Future work will address this aspect.

2. Monotonic fronts

A traveling monotonic front $u = q(x - ct)$ joining the stable state $u = 1$ to $u = 0$ satisfies the ordinary differential equation

$$q_{zz} + cq_z - \gamma q_{zzzz} + f(q) = 0 \, ,$$

with

$$\lim_{z\to-\infty} q = 1, \quad \lim_{z\to\infty} q = 0, \quad q_z < 0\,,$$

where $z = x - ct$ and subscripts denote derivatives.

Following the usual procedure, since the front is monotonic, we may use phase space variables and define $p(q) = -\mathrm{d}q/\mathrm{d}z$, where the minus sign is introduced to have $p > 0$. A simple calculation shows that monotonic fronts obey

$$\gamma p \frac{\mathrm{d}}{\mathrm{d}q}\left[p\frac{\mathrm{d}}{\mathrm{d}q}\left(p\frac{\mathrm{d}p}{\mathrm{d}q}\right)\right] - p\frac{\mathrm{d}p}{\mathrm{d}q} + cp - f(q) = 0 \tag{3}$$

with

$$p(0) = p(1) = 0 \quad \text{and} \quad p > 0\,.$$

Next, we obtain the upper bound. Let $g(q)$ be an arbitrary positive decreasing function. Multiplying Eq. (3) by g/p and integrating with respect to q between 0 and 1 we obtain the identity

$$c\int_0^1 g(q)\,\mathrm{d}q = \int_0^1 \frac{gf}{p}\,\mathrm{d}q + \int_0^1 ph\,\mathrm{d}q + \gamma\int_0^1\left(\frac{1}{3}g'''p^3 + hpp'^2\right)\mathrm{d}q\,, \tag{4}$$

where primes denote derivatives with respect to q and where we have defined $h(q) = -g'(q) > 0$. In obtaining this expression several integrations by parts were performed. Surface terms vanish due to the boundary conditions on p. Furthermore, we assume that the function g does not diverge in a manner that prevents the vanishing of surface terms.

Consider now the functional

$$S_g(p) = \int_0^1 \frac{gf}{p}\,\mathrm{d}q + \int_0^1 ph\,\mathrm{d}q + \gamma\int_0^1\left(\frac{1}{3}g'''p^3 + hpp'^2\right)\mathrm{d}q\,. \tag{5}$$

It can be shown (details will be given elsewhere) that for $g \in C^3([0,1]), g' < 0, g''' > 0$, this functional has a unique minimizer which we call $\hat{p}$. Therefore,

$$S_g(p) \geqslant \min_p\, S_g(p) = S_g(\hat{p})\,.$$

This implies in Eq. (4) that

$$c\int_0^1 g(q)\,\mathrm{d}q \geqslant S_g(\hat{p})\,. \tag{6}$$

The minimizing p, $\hat{p}$, can be obtained by solving the Euler–Lagrange equation for $S_g(p)$,

$$\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{\partial L}{\partial p'}\right) - \frac{\partial L}{\partial p} = 0\,.$$

Recalling that the arbitrary function g is a function of q, we obtain

$$2\gamma\frac{\mathrm{d}}{\mathrm{d}q}[h\hat{p}\hat{p}'] + \frac{gf}{\hat{p}^2} - h - \gamma(h\hat{p}'^2 + g'''\hat{p}^2) = 0\,. \tag{7}$$

To obtain the minimizing p for each function $g(q)$, we should solve this equation. This is not an easy task since $g(q)$ is an arbitrary unspecified function. However, it follows from this equation, multiplying by $\hat{p}(q)$ and integrating in q that

$$\int_0^1 \left(\frac{gf}{\hat{p}} - h\hat{p} - \gamma g'''\hat{p}^3 - 3\gamma h \hat{p}'^2 \hat{p} \right) \mathrm{d}q = 0 \, .$$

Using this result we find that $S_g(\hat{p})$ can be written as

$$S_g(\hat{p}) = \frac{4}{3} \int_0^1 \frac{fg}{\hat{p}} + \frac{2}{3} \int_0^1 \hat{p} h \, \mathrm{d}q \, .$$

Inequality (6) is then

$$c \int_0^1 g(q) \, \mathrm{d}q \geqslant \frac{4}{3} \int_0^1 \frac{f(q)g(q)}{\hat{p}(q)} \, \mathrm{d}q + \frac{2}{3} \int_0^1 \hat{p}(q) h(q) \, \mathrm{d}q \, .$$

Finally, since $f > 0, g > 0$ and $h = -g' > 0$ we use the inequality $a^2 + b^2 \geqslant 2ab$ in the expression above to obtain our main result

$$c \geqslant \frac{4\sqrt{2}}{3} \frac{\int_0^1 \sqrt{fgh} \, \mathrm{d}q}{\int_0^1 g \, \mathrm{d}q} \, . \tag{8}$$

Notice that this expression is similar in form to the bound obtained previously for the speed of fronts of the reaction diffusion equation. In that case we proved that the bound is sharp and that it follows from a variational principle. In the present case this bound does not saturate; however, a variational principle will follow from (6) if we succeed in proving that there is a certain function $g(q)$ for which $\hat{p}$ is the solution of the differential (3). This point will be addressed in future work.

3. Conclusion

A lower bound on the speed of monotonic fronts of the EFK equation has been obtained. This bound allows to determine the range in which an asymptotic front may propagate with the linear speed c_L. We conjecture that there is a variational principle for the minimal speed of the fronts from which its exact value could be calculated.

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