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Evolution equation for bidirectional surface waves in a convecting fluid

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Surface waves in a heated viscous fluid exhibit a long-wave oscillatory instability. The nonlinear evolution of unidirectional waves is known to be described by a modified Korteweg-deVries-Kuramoto-Sivashinsky equation. In the present work, we eliminate the restriction of unidirectional waves and find that the evolution of the wave is governed by a modified Boussinesq system. A perturbed Boussinesq equation of the form $y_{tt}-y_{xx}-\epsilon^2[y_{xxtt}+(y^2)_{xx}]+\epsilon^3[y_{xxt}+y_{xxxxt}+(y^2)_{xxt}]=0$, which includes instability and dissipation, can be derived from this system.
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I. INTRODUCTION

Long-wave oscillatory instabilities arise in different contexts. They have been found in the study of unstable drift waves in plasmas,[1] fluid flow along an inclined plane,[2,3] convection in fluids with a free surface,[4–7] the Eckhaus instability of traveling waves,[8] and, more recently, in solar dynamo waves.[9] The nonlinear evolution of the instability is generically described by a Korteweg-deVries-Kuramoto-Sivashinsky (KdV-KS) equation

$$u_t+uu_x+\delta u_{xxx}+ru_{xx}+u_{xxxx}=0,$$

in some cases with different or additional nonlinearities.[6,8,9] The solutions to this equation have been studied extensively, focusing mainly on the formation and interaction of solitary pulses.[10–12] It exhibits parameter regimes where different behavior is found, numerical studies have shown the appearance of modulated traveling waves, period doubling, and chaos.[12,13] Exact solutions have been constructed by different methods.[14–18] The limits of large and small dispersion δ have been studied rigorously.[19,20]

The derivation of this equation in the physical situations mentioned above involves the assumption of unidirectional waves. This is natural in the situations where a preferred direction exists: the electron diamagnetic direction in the case of plasma waves, toward the equator in the case of solar dynamo waves, downhill for flow along the inclined plane. However, in the case of surface waves in a convecting fluid, this assumption is not natural. The assumption of unidirectional waves for surface waves was abandoned in Refs. 21 and 22 and a modified Boussinesq system of equations was derived. In Ref. 21, the scaling introduced led to a system that, when reduced to unidirectional waves, led to the KdV-Burgers equation. It did not account for the balance between instability and dissipation present in the KdV-KS equation, which is indispensable for the formation of solitary pulses. In Ref. 22, surface waves sustained by Marangoni convection were studied. The Boussinesq system derived does reduce to the KdV-KS equation for unidirectional waves, however no attempt was made to derive a single evolution equation for bidirectional waves.

The purpose of this article is to study the evolution of surface waves sustained by Rayleigh-Benard convection to find the equation that governs the nonlinear evolution of the surface. This equation will be the natural extension of the KdV-KS equation to bidirectional waves. A higher-order Boussinesq system is derived that, following the procedure for ideal water waves,[24] can be reduced to a modified Boussinesq equation including instability and dissipation. The evolution equation that we derive systematically in this work has been suggested as one of two possible heuristic extensions of the KdV-KS equation to bidirectional waves without reference to any physical system.[25] Since the equation is derived systematically, it becomes clear, from the asymptotics and the physical interpretation, that the equation found is the relevant extension of the Boussinesq equation to dissipative systems. In the present work, we consider surface waves sustained by buoyancy-driven convection. The results found also hold when the convection is driven by the Marangoni effect.

It is worth mentioning that the Boussinesq equation with added instability and dissipation,

$$N_{TT}-N_{YY}-\epsilon^2\lambda_1 N_{YYYY}-\epsilon^2\lambda_2(N^2)_{YY}+\epsilon^3(\lambda_3 N_{YYT}+\lambda_4 N_{YYYYT})+\lambda_5\epsilon^3(N^2)_{YYT}=0,$$

which we derive below, arises as the phase equation in the study of the stability of one-dimensional periodic patterns in systems with Galilean invariance.[26] The same equation, but with cubic nonlinearities, $(N^3)_{YY}$, $(N^3)_{YYT}$, describes the oscillatory instability of convective rolls.[26,27]

In our case, we may choose either the equation above or the improved Boussinesq form

$$N_{TT}-N_{YY}-\epsilon^2\lambda_1 N_{YYTT}-\epsilon^2\lambda_2(N^2)_{YY}+\epsilon^3(\lambda_3 N_{YYT}+\lambda_4 N_{YYYYT})+\lambda_5\epsilon^3(N^2)_{YYT}=0,$$

as equally valid evolution equations for the surface displacement.

II. FORMULATION OF THE PROBLEM

We consider a layer of fluid that, at rest, lies between $z=0$ and d. Upon it acts a gravitational field $\vec{g}=-g\hat{z}$. The fluid is described by the Boussinesq equations

$$\nabla\cdot\vec{v}=0, \tag{1}$$

$$\rho_0\frac{d\vec{v}}{dt}=-\vec{\nabla}p+\mu\nabla^2\vec{v}+\vec{g}\rho, \tag{2}$$

$$\frac{dT}{dt}=\kappa\nabla^2 T, \tag{3}$$

where the density ρ depends linearly on the temperature

$$\rho=\rho_0[1-\alpha(T-T_0)]. \tag{4}$$

Here $\frac{d}{dt}=\frac{\partial}{\partial t}+\vec{v}\cdot\vec{\nabla}$ is the convective derivative; p, T, and $\vec{v}$ denote the pressure, temperature, and fluid velocity, respectively. The quantities ρ_0 and T_0 are reference values. The fluid properties, namely its viscosity μ, thermal diffusivity κ, and coefficient of thermal expansion α, are constant. Furthermore, we restrict ourselves to two-dimensional motion so that $\vec{v}=(u,0,w)$.

The fluid is bounded above by a free surface on which the heat flux is fixed and upon it a constant pressure p_a is exerted. As the fluid moves, the free surface is deformed. We shall denote its position by $z=d+\eta(x,t)$. The boundary conditions on the upper surface are

$$\eta_t+u\eta_x=w, \tag{5}$$

$$p-p_a-\frac{2\mu}{N^2}[w_z+u_x\eta_x^2-\eta_x(u_z+w_x)]=0, \tag{6}$$

$$\mu(1-\eta_x^2)(u_z+w_x)+2\mu\eta_x(w_z-u_x)=0, \tag{7}$$

$$\hat{n}\cdot\vec{\nabla}T=-F/k, \tag{8}$$

on $z=d+\eta$. Here, subscripts denote derivatives, $N=(1+\eta_x^2)^{1/2}$, $\hat{n}=(-\eta_x,0,1)/N$ is the unit normal to the free surface, F is the prescribed normal heat flux, and k is the thermal conductivity.

On the lower surface, the fluid is bounded by an idealized plane free surface maintained at constant temperature. Denoting by T_b the fixed temperature of the lower surface, the boundary conditions on the lower surface $z=0$ are then

$$w=u_z=0,\quad T=T_b. \tag{9}$$

The static solution to these equations is given by $T_s=-F(z-d)/k+T_0$, $\rho_s=\rho_0[1+(\alpha F/k)(z-d)]$, and $p_s=p_a-g\rho_0[(z-d)+(\alpha F/2k)(z-d)^2]$. We have chosen the reference temperature T_0 as the value of the static temperature on the upper surface. The temperature on the lower surface is then $T_b=T_0+Fd/k$. Equations (1)–(9) constitute the problem to be solved. We shall adopt d as unit of length, d^2/κ as the unit of time, $\rho_o d^3$ as the unit of mass, and Fd/k as the unit of temperature. Then there are three dimensionless parameters involved in the problem: the Prandtl number $\sigma=\mu/\rho_o\kappa$, the Rayleigh number $R=\rho_o g\alpha Fd^4/k\kappa\mu$, and the Galileo number $G=gd^3\rho_0^2/\mu^2$.

The linear stability theory has been studied elsewhere.[4,5] The result of interest in the present context is the existence of an oscillatory instability at vanishing wave number $a_c=0$ with critical Rayleigh number $R_c=30$. The frequency along the marginal curve is given by $\omega=a\sigma\sqrt{G}+\cdots$, which vanishes at criticality. In dimensional quantities, this frequency is that of long surface waves in an ideal fluid. In this problem, the damping due to viscosity is compensated by heating.

III. SMALL-AMPLITUDE NONLINEAR EXPANSION

To study the evolution of the oscillatory instability slightly above onset, we let

$$R=R_c+\epsilon^2R_2$$

and, since the instability occurs at vanishing wave number and the frequency along the marginal curve is of the order of the wave number, we introduce scaled time and horizontal coordinates

$$\xi=\epsilon x,\quad \tau=\epsilon t.$$

Next we look for a perturbative solution expanding the dependent variables as

$$p=p_s(z)+\epsilon^2(p_0+\epsilon p_1+\epsilon^2p_2+\cdots),$$

$$T=T_s(z)+\epsilon^3(\theta_0+\epsilon\theta_1+\epsilon^2\theta_2+\cdots),$$

$$u=\epsilon^2(u_0+\epsilon u_1+\epsilon^2u_2+\cdots),$$

$$w=\epsilon^3(w_0+\epsilon w_1+\epsilon^2w_2+\cdots),$$

$$\eta=\epsilon^2(\eta_0+\epsilon\eta_1+\epsilon^2\eta_2+\cdots),$$

and proceed to solve at each order. Our main interest is to obtain a modified Boussinesq system including instability and dissipation, from which a single evolution equation for the surface can be derived. This requires that the calculation be carried out to order ϵ^3. The details are given in the Appendix. Here we quote the relevant results.

In leading order, the horizontal velocity is given by

$$u_0=\phi_0(\xi,\tau),$$

where $\phi_0(\xi,\tau)$ is an arbitrary function to be determined. At each higher order a new arbitrary function appears in the solution for the horizontal speed. We will find a coupled system of equations for the surface height and horizontal velocity that are a modified Boussinesq system. The first equation of this system is obtained from the kinematic boundary condition for the free surface, the second equation from the solvability condition of Eqs. (1)–(9).

The kinematical boundary condition is, at each order,

$$\eta_{0\tau}+\phi_{0\xi}=0, \tag{10}$$

$$\eta_{1\tau}+\phi_{1\xi}=0, \tag{11}$$

$$\eta_{2\tau}+\phi_{2\xi}+\tilde{\alpha}\phi_{0\xi\xi\xi}+(\phi_0\eta_0)_\xi=0, \tag{12}$$

$$\eta_{3\tau}+\phi_{3\xi}+\tilde{\alpha}\phi_{1\xi\xi\xi}-\beta\phi_{0\xi\xi\xi\tau}+(\phi_0\eta_1+\phi_1\eta_0)_\xi=0. \quad (13)$$

The functions ϕ_1, ϕ_2, and ϕ_3 are arbitrary functions of ξ and τ. The coefficients $\tilde{\alpha}$ and β are given in the Appendix. The second equation needed to close the system comes, as mentioned above, from a solvability condition. At each order we find

$$\sigma^2G\eta_{0\xi}+\phi_{0\tau}=0, \quad (14)$$

$$\sigma^2G\eta_{1\xi}+\phi_{1\tau}=0, \quad (15)$$

$$\sigma^2G\eta_{2\xi}+\phi_{2\tau}-\gamma\phi_{0\xi\xi\tau}+\phi_0\phi_{0\xi}+30\sigma\eta_0\eta_{0\xi}=0, \quad (16)$$

$$\sigma^2G\eta_{3\xi}+\phi_{3\tau}-\gamma\phi_{1\xi\xi\tau}+r\phi_{0\xi\xi}+\tilde{\mu}\phi_{0\xi\xi\xi\xi}+\tilde{\nu}\phi_{0\xi\xi\tau\tau} + 16\sigma(\eta_0\phi_{0\xi})_\xi+30\sigma(\eta_0\eta_1)_\xi+(\phi_0\phi_1)_\xi=0. \quad (17)$$

In these equations $r=2\sigma R_2/15$ is the measure of instability. The rest of the coefficients are given in the Appendix.

Finally we construct a single system, correct to order ϵ^3 for the surface elevation $\eta=\eta_0+\epsilon\eta_1+\epsilon^2\eta_2+\epsilon^3\eta_3$ and arbitrary function $\phi=\phi_0+\epsilon\phi_1+\epsilon^2\phi_2+\epsilon^3\phi_3$ by simply adding the equations above. In doing so, some ambiguity arises in the choice of terms of order ϵ^2 and ϵ^3, where it is possible to replace $\phi_\tau=-\sigma^2G\eta_\xi$ or to replace $\phi_\xi=-\eta_\tau$ committing an error of $O(\epsilon^4)$. In the theory of ideal water waves, this leads to the existence of different families of Boussinesq systems.[23] At this stage we use this freedom to minimize the type of terms. With this criterion, the kinematic condition, correct to order ϵ^3, becomes

$$\eta_\tau+\phi_\xi+\tilde{\alpha}\epsilon^2\phi_{\xi\xi\xi}-\beta\epsilon^3\phi_{\xi\xi\xi\tau}+\epsilon^2(\phi\eta)_\xi=0. \quad (18)$$

The solvability condition correct to order ϵ^3 is

$$\sigma^2G\eta_\xi+\phi_\tau-\epsilon^2\gamma\phi_{\xi\xi\tau}+\epsilon^3r\phi_{\xi\xi}-\epsilon^3(\tilde{\mu}+\tilde{\nu})\eta_{\xi\xi\xi\tau}+\epsilon^2\phi\phi_\xi + 30\sigma\epsilon^2\eta\eta_\xi+16\sigma\epsilon^3(\eta\phi_\xi)_\xi=0. \quad (19)$$

This is a modified Boussinesq system including instability and dissipation in the linear terms of order ϵ^3. It describes the propagation of bidirectional waves. In the following section, we reduce this system to a single equation for the surface elevation, obtaining a modified Boussinesq equation.

IV. EVOLUTION EQUATION FOR THE FREE SURFACE

If we search for unidirectional waves, the system (10)–(17) leads to the modified KS-KdV equation (20),[6]

$$\eta_t+\delta_1\eta\eta_x+\delta_2\eta_{xxx}+\epsilon[r\eta_{xx}+\delta_3\eta_{xxxx}+\delta_4(\eta\eta_x)_x]=0. \quad (20)$$

Our purpose here is to obtain the evolution equation for the surface without this restriction. In this process, some ambiguity arises as explained in the previous section. In reducing to a single equation, we adopt the criterion of choosing the form that gives the simplest physically meaningful linear theory.

A. Linear theory

Keeping linear terms in Eqs. (18) and (19), we find that the surface displacement satisfies the equation

$$\eta_{\tau\tau}-\sigma^2G\eta_{\xi\xi}-\epsilon^2A\eta_{\xi\xi\tau\tau}+\epsilon^3(r\eta_{\xi\xi\tau}+B\eta_{\xi\xi\xi\xi\tau})=0,$$

where

$$A=\tilde{\alpha}+\gamma=\frac{1}{3}+\frac{34\sigma}{21},$$

$$B=\tilde{\mu}+\tilde{\nu}+\sigma^2G\beta =\frac{2798\sigma}{2079}+(G\sigma^2-1)\left(\frac{1265}{12096}+\frac{3973}{60480\sigma}\right).$$

Searching for plane-wave solutions $e^{s\tau}e^{ik\xi}$, we obtain the characteristic polynomial

$$s^2(1+\epsilon^2Ak^2)-\epsilon^3(rk^2-Bk^4)s+k^2=0.$$

The growth rate is given by

$$\lambda=\Re(s)=\frac{1}{2}\frac{\epsilon^3(rk^2-Bk^4)}{1+\epsilon^2Ak^2}$$

and the frequency is given by

$$\omega^2=\Im(s)^2=\frac{k^2-\frac{1}{4}\epsilon^6(rk^2-Bk^4)^2}{1+\epsilon^2Ak^2}.$$

On the marginal curve $\lambda=0$, $r=Bk^2$. Small wave numbers are unstable; large wave numbers are stable as it occurs in the KdV-KS equation.

B. Nonlinear equation

Taking time and space derivatives of Eqs. (18) and (19), respectively, and subtracting, we obtain

$$\eta_{\tau\tau}-\sigma^2G\eta_{\xi\xi}-\epsilon^2A\eta_{\xi\xi\tau\tau}+\epsilon^3(r\eta_{\xi\xi\tau}+B\eta_{\xi\xi\xi\xi\tau})+\epsilon^2 \times\left[(\phi\eta)_\tau-\tfrac{1}{2}(\phi^2)_\xi\right]_\xi-15\sigma\epsilon^2(\eta^2)_{\xi\xi}+8\sigma\epsilon^3(\eta^2)_{\xi\xi\tau}=0. \quad (21)$$

We recognize the order ϵ^2 containing ϕ as already appearing in the derivation of the standard Boussinesq equation, therefore we only outline the procedure[24] to obtain the modified Boussinesq equation. We write

$$[(\phi\eta)_\tau-\phi\phi_\xi]_\xi=-\frac{\sigma^2G}{2}(\eta^2)_{\xi\xi}+(\phi^2)_{\xi\xi}+O(\epsilon^2)$$

and

$$(\phi^2)_{\xi\xi}=2(\eta_\tau^2-\phi\eta_{\tau\xi})+O(\epsilon^2).$$

Assuming that ϕ vanishes at $-\infty$, we may write

$$\phi=-\int_{-\infty}^{\xi}\eta_\tau d\xi'+O(\epsilon^2).$$

Replacing these expressions in (21), we obtain the evolution equation for bidirectional surface waves,

$$\eta_{\tau\tau}-\sigma^2 G\eta_{\xi\xi}-\epsilon^2 A\eta_{\xi\xi\tau\tau}+\epsilon^3(r\eta_{\xi\xi\tau}+B\eta_{\xi\xi\xi\xi\tau})-2\epsilon^2 \times\left(\eta_\tau^2+\eta_{\tau\xi}\int_{-\infty}^{\xi}\eta_\tau d\xi'\right)-\Lambda\epsilon^2(\eta^2)_{\xi\xi}+8\sigma\epsilon^3(\eta^2)_{\xi\xi\tau}=0,$$

where $\Lambda=15\sigma+\sigma^2G/2$. Terms of order ϵ^4 have been neglected.

The standard form of the Boussinesq equation is achieved by changing to the Lagrangian description. Defining new variables

$$Y=\frac{1}{\sigma\sqrt{G}}\left(\xi+\epsilon^2\int_{-\infty}^{\xi}\eta d\xi'\right),\quad T=\tau$$

and

$$N=\eta-\epsilon^2\eta^2,$$

we obtain our main result, a single evolution equation for bidirectional waves including instability and dissipation,

$$N_{TT}-N_{YY}-\epsilon^2\lambda_1 N_{YYTT}-\epsilon^2\lambda_2(N^2)_{YY} +\epsilon^3(\lambda_3 N_{YYT}+\lambda_4 N_{YYYYT})+\lambda_5\epsilon^3(N^2)_{YYT}=0, \qquad (22)$$

where

$$\lambda_1=\frac{A}{\sigma^2G},\quad \lambda_2=1+\frac{\Lambda}{\sigma^2G},\quad \lambda_3=\frac{r}{\sigma^2G},$$

$$\lambda_4=\frac{B}{\sigma^4G^2},\quad \text{and}\quad \lambda_5=\frac{8}{\sigma G}.$$

Up to order ϵ^2, we recognize the improved Boussinesq equation. Instability and dissipation are included in the linear terms of order ϵ^3. The nonlinearity of order ϵ^3 is an additional nonlinearity that is characteristic of this problem [see Eq. (20)].

Finally, we point out that to the same order of approximation we may replace $N_{TT}=N_{YY}+O(\epsilon^2)$ in the linear term of order ϵ^2 and obtain the equally valid form

$$N_{TT}-N_{YY}-\epsilon^2\lambda_1 N_{YYYY}-\epsilon^2\lambda_2(N^2)_{YY} +\epsilon^3(\lambda_3 N_{YYT}+\lambda_4 N_{YYYYT})+\lambda_5\epsilon^3(N^2)_{YYT}=0, \qquad (23)$$

which has been found in the study of instabilities of periodic patterns.

V. SUMMARY

We have studied the nonlinear development of a long-wave oscillatory instability of surface waves. When the assumption of unidirectional waves is made, the surface evolution is determined by a modified KdV-KS equation. In the present study, we allow bidirectional propagation of waves and are led to a modified Boussinesq system, where instability and dissipation are present at high order. A single equation is obtained for the evolution of the surface following the method used in ideal surface waves. In the case of ideal surface waves, one is led to a Boussinesq equation. Here we obtain a modified Boussinesq equation including instability and dissipation.

In this work, we considered surface waves driven by Rayleigh-Benard convection. Surface waves can also be driven by Marangoni convection. For Marangoni driven waves, the Boussinesq type system, that is, the analog of Eqs. (18) and (19), and hence the equation for the surface derived from it, contains additional nonlinearities. It is straightforward to show that by the same change of variables used here, it reduces to Eq. (22).

Previous studies have dealt with the damped Boussinesq equation ($r<0$, $B=0$) where stable structures, solitons, and periodic solutions must be sustained by external forcing.[28,29] In the present case, $r>0$ and it represents the destabilizing term. The equation we find has been proposed as one of two possible heuristic generalizations of the KdV-KS equation to bidirectional waves. Here we show that it is indeed the adequate bidirectional generalization of the KdV-KS equation by deriving it systematically, just as the Boussinesq equation is derived for surface waves in an ideal fluid.

Numerical integrations in extended domains[25,30] show that this equation has solutions with soliton-like characteristics qualitatively similar to those observed in experiments.[31] Unidirectional traveling-wave solutions $N(x-ct)$ satisfy the same third-order differential equation as traveling-wave solutions of the generalized KdV-KS equation (15) for which exact periodic and solitary wave solutions have been found.[14–16,18] Numerical and analytical results for the evolution of solutions in small domains will be the subject of future work.

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APPENDIX: PERTURBATION EXPANSION

In this appendix, the equations to be solved at each order and their solution are given. Terms that vanish have been omitted.

In leading order, the equations to be solved are

$$u_{0zz}=0,$$

$$w_{0z}=-u_{0\xi},$$

$$p_{0z}=0,$$

$$\theta_{0zz}=-w_0,$$

subject to

$$w_0(0)=u_{0z}(0)=\theta_0(0)=0,$$

$$u_{0z}(1)=\theta_{0z}(1)=0,$$

$$p_0(1)=\sigma^2G\eta_0,$$

$$\eta_{0\tau}=w_0(1).$$

The solution is given by

$$u_0=\phi_0(\xi,\tau),\quad w_0=-\phi_{0\xi}z,$$

$$\theta_0 = \phi_{0\xi} T_0(z), \quad p_0 = \sigma^2 G \eta_0,$$

where we have defined $T_0(z)=(z^3-3z)/6$. The kinematical boundary condition is

$$\eta_{0\tau} + \phi_{0\xi} = 0. \qquad \text{(A1)}$$

At order ϵ the equations to be solved are

$$u_{1zz} = \frac{1}{\sigma} p_{0\xi} + \frac{1}{\sigma} u_{0\tau},$$

$$w_{1z} = - u_{1\xi},$$

$$p_{1z} = 30\sigma\theta_0,$$

$$\theta_{1zz} = \theta_{0\tau} - w_1,$$

subject to

$$w_1(0) = u_{1z}(0) = \theta_1(0) = 0,$$

$$u_{1z}(1) = \theta_{1z}(1) = 0,$$

$$p_1(1) = \sigma^2 G \eta_1 + 2\sigma w_{0z}(1),$$

$$\eta_{1\tau} = w_1(1).$$

The solvability condition $\int_0^1 u_{1zz} dz = u_{1z}(1) - u_{1z}(0) = 0$ implies

$$\sigma^2 G \eta_{0\xi} + \phi_{0\tau} = 0. \qquad \text{(A2)}$$

The solution at this order is

$$u_1 = \phi_1(\xi,\tau),$$

$$w_1 = - \phi_{1\xi} z,$$

$$\theta_1 = \phi_{1\xi} T_0(z) + \phi_{0\xi\tau} T_1(z),$$

$$p_1 = \sigma^2 G \eta_1 + 30\sigma\phi_{0\xi} P_1(z) - 2\sigma\phi_{0\xi},$$

where we have defined $T_1(z)=(25z-10z^3+z^5)/120$ and $P_1(z)=T_{1z}(z)=(5-6z^2+z^4)/24$. The kinematical boundary condition at this order becomes

$$\eta_{1\tau} + \phi_{1\xi} = 0. \qquad \text{(A3)}$$

At order ϵ^2 the equations to be solved are

$$u_{2zz} = \frac{1}{\sigma} p_{1\xi} + \frac{1}{\sigma} u_{1\tau} - u_{0\xi\xi},$$

$$w_{2z} = - u_{2\xi},$$

$$p_{2z} = 30\sigma\theta_1 + \sigma w_{1zz} - w_{0\tau},$$

$$\theta_{2zz} = \theta_{1\tau} - w_2 - \theta_{0\xi\xi},$$

subject to

$$w_2(0) = u_{2z}(0) = \theta_2(0) = 0,$$

$$u_{2z}(1) = - w_{0\xi}(1), \quad \theta_{2z}(1) = - \eta_0 \theta_{0zz}(1),$$

$$p_2(1) = \sigma^2 G \eta_2 + 2\sigma w_{1z}(1) + 15\sigma\eta_0^2,$$

$$\eta_{2\tau} = w_2(1) - (u_0\eta_0)_\xi.$$

The solvability condition $\int_0^1 u_{2zz} dz = u_{2z}(1)$ implies

$$\sigma^2 G \eta_{1\xi} + \phi_{1\tau} = 0, \qquad \text{(A4)}$$

and the solution to this order is

$$u_2 = \phi_2(\xi,\tau) + \phi_{0\xi\xi} U_2(z),$$

$$w_2 = - \phi_{2\xi} z + \phi_{0\xi\xi\xi} W_2(z),$$

$$\theta_2 = \phi_{2\xi} T_0(z) + \phi_{1\xi\tau} T_1(z) + \phi_{0\xi\tau\tau} T_2(z) + \phi_{0\xi\xi\xi} T_3(z) - z\eta_0\phi_{0\xi},$$

$$p_2 = \sigma^2 G \eta_2 + 15\sigma\eta_0^2 + 30\sigma\phi_{1\xi} Q_1(z) + \phi_{0\xi\tau} P_2(z) - 2\sigma\phi_{1\xi},$$

where we have defined

$$U_2(z) = z^2(39 - 15z^2 + z^4)/24,$$

$$W(z) = (- 91z^3 + 21z^5 - z^7)/168,$$

$$P_2(z) = \sigma(- 61 + 75z^2 - 15z^4 + z^6)/24 + (z^2 - 1)/2,$$

$$T_2(z) = (- 427z + 175z^3 - 21z^5 + z^7)/5040,$$

$$T_3(z) = (5z^9 - 180z^7 + 1134z^5 + 5040z^3 - 19575z)/60480.$$

The kinematical boundary condition becomes

$$\eta_{2\tau} + \phi_{2\xi} + \tilde{\alpha}\phi_{0\xi\xi\xi} + (\phi_0\eta_0)_\xi = 0, \qquad \text{(A5)}$$

where $\tilde{\alpha}=71/368$.

At order ϵ^3 we only need to calculate the solvability condition for u_3 and the kinematical boundary condition. The equation for the horizontal velocity is

$$u_{3zz} = \frac{1}{\sigma} p_{2\xi} + \frac{1}{\sigma} u_{2\tau} - u_{1\xi\xi} + \frac{1}{\sigma} u_0 u_{0\xi}$$

subject to

$$u_{3z}(0) = 0, \quad u_{3z}(1) = - w_{1\xi}(1).$$

The solvability condition $\int_0^1 u_{3zz} dz = u_{3z}(1)$ yields

$$\sigma^2 G \eta_{2\xi} + \phi_{2\tau} - \gamma\phi_{0\xi\xi\tau} + \phi_0\phi_{0\xi} + 30\sigma\eta_0\eta_{0\xi} = 0, \qquad \text{(A6)}$$

where $\gamma=34\sigma/21-5/56$. The solution for the horizontal velocity is

$$u_3 = \phi_3(\xi,\tau) + \phi_{1\xi\xi} U_2(z) + \phi_{0\xi\xi\tau} U_3(z).$$

In the expression above, $U_3(z)=\tilde{U}_3(z)+U_3^*(z)/\sigma$, where

$$\tilde{U}_3(z) = (- 620z^2 + 350z^4 - 28z^6 + z^8)/1344,$$

$$U_3^*(z) = (- 396z^2 + 238z^4 - 28z^6 + z^8)/1344.$$

The kinematical boundary condition at this order,

$$\eta_{3\tau} = w_3(1) - (u_0\eta_1)_\xi - (u_1\eta_0)_\xi,$$

yields

$$\eta_{3\tau} + \phi_{3\xi} + \tilde{\alpha}\phi_{1\xi\xi\xi} - \beta\phi_{0\xi\xi\xi\tau} + (\phi_0\eta_1 + \phi_1\eta_0)_\xi = 0, \quad \text{(A7)}$$

where $\beta = 1265/12096 + 3973/(60480\sigma)$.

The pressure at this order satisfies the equation

$$p_{3z} = 30\sigma\theta_2 + \sigma w_{2zz} - w_{1\tau} + \sigma w_{0\xi\xi} + \sigma R_2\theta_0$$

with

$$p_3(1) = \sigma^2 G\eta_3 + 30\sigma\eta_0\eta_1 + 2\sigma w_{2z}(1) - \eta_0 p_{1z}(1).$$

Its explicit solution is not indispensable to continue the calculation. Finally, the last equation needed is obtained from the solvability condition of the horizontal velocity at order ϵ^4. We have

$$u_{4zz} = \frac{1}{\sigma}p_{3\xi} + \frac{1}{\sigma}u_{3\tau} - u_{2\xi\xi} + \frac{1}{\sigma}(u_0u_1)_\xi,$$

with

$$u_{4z}(0) = 0, \quad u_{4z}(1) = -\eta_0 u_{2zz}(1) - w_{2\xi}(1) - \eta_0 w_{0\xi z}(1) - 2\eta_{0\xi}[w_{0z}(1) - u_{0\xi}(1)].$$

The solvability condition $\int_0^1 u_{4zz}dz = u_{4z}(1)$ is the last equation needed to complete the system. We obtain

$$\sigma^2 G\eta_{3\xi} + \phi_{3\tau} - \gamma\phi_{1\xi\xi\tau} + r\phi_{0\xi\xi} + \tilde{\mu}\phi_{0\xi\xi\xi\xi} + \tilde{\nu}\phi_{0\xi\xi\tau\tau} + 16\sigma(\eta_0\phi_{0\xi})_\xi + 30\sigma(\eta_0\eta_1)_\xi + (\phi_0\phi_1)_\xi = 0, \quad \text{(A8)}$$

where $r = 2\sigma R_2/15$ is proportional to the excess of the Rayleigh number over its critical value, $\tilde{\mu} = 478\sigma/693$, and $\tilde{\nu} = 124\sigma/189 - 3973/(60480\sigma) - 1265/12096$.

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