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Rigorous results for the minimal speed of Kolmogorov–Petrovskii–Piscounov monotonic fronts with a cutoff[a]

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We study the effect of a cutoff on the speed of pulled fronts of the one-dimensional reaction diffusion equation. To accomplish this, we first use variational techniques to prove the existence of a heteroclinic orbit in phase space for traveling wave solutions of the corresponding reaction diffusion equation under conditions that include discontinuous reaction profiles. This existence result allows us to prove rigorous upper and lower bounds on the minimal speed of monotonic fronts in terms of the cut-off parameter ε. From these bounds we estimate the range of validity of the Brunet–Derrida formula for a general class of reaction terms. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4770248]

I. INTRODUCTION

The reaction diffusion equation

$$u_t = u_{xx} + f(u) \tag{1}$$

is one of the simplest models which shows how a small perturbation to an unstable state develops into a moving front joining a stable to an unstable state. For the reaction term $f(u)$ one can adopt different expressions depending on the physical problem under consideration. One of the most studied cases, is the Fisher reaction term[13] $f(u) = u(1 - u)$ for which the asymptotic speed of the propagating front is $c = 2$, a value determined from linear considerations. A more general case was studied by Kolmogorov, Petrovskii, and Piscounov (KPP)[14] who showed that for all reaction terms which satisfy the so called KPP condition

$$f(u) > 0, \qquad f(0) = f(1) = 0, \qquad f(u) < f'(0)u \tag{2}$$

the asymptotic speed of the front joining the stable $u = 1$ point to the unstable $u = 0$ point is given by

$$c_{KPP} = 2\sqrt{f'(0)}.$$

The evolution of localized initial conditions to the front of minimal speed was established in Ref. 1 for a general class of smooth (in fact $C^1[0, 1]$) reaction terms. Recent work has dealt with effects not included in the classical reaction diffusion equation (1), namely the effects of noise and of the finiteness in the number n of diffusive particles. It was suggested by Brunet and Derrida[9] that such effects can be simulated by introducing a cutoff in the reaction term. In the case of noise the cut-off parameter measures the amplitude of the noise while in the case of finite number of n

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diffusing particles the cut-off parameter $\varepsilon = 1/n$. They presented numerical evidence to support their conjecture. By means of an asymptotic matching Brunet and Derrida showed that for a reaction term $f(u) = u(1 - u^2)$ a small cutoff changes the speed of the front to

$$c \approx 2 - \frac{\pi^2}{(\log \varepsilon)^2}. \tag{3}$$

In recent work it has been shown that the Brunet-Derrida formula for the speed is correct to $\mathcal{O}((\log \varepsilon)^{-3})$ for a wider class of pulled reaction terms and cut-off functions.[11] A completely different behavior is found when a cutoff is applied to a bistable reaction term or to a pushed front, in these two cases the cutoff changes the speed by an amount which has a power law dependence of the cut-off parameter.[5,12] The validity of representing the finiteness in the number of particles in the diffusion process by a reaction diffusion equation with a cutoff, and the effect of noise in the reaction diffusion equation with a cutoff was proved rigorously in Refs. 7 and 16, respectively.

The purpose of this work is to prove rigorous upper and lower bounds for the minimal speed of monotonic fronts for reaction terms of the form $f(u)\Theta(u - \varepsilon)$ where f satisfies conditions analogous to the KPP condition Eq. (2) and Θ is the step function. More precisely, we require the profile to satisfy

$$f(u) = 0, \text{ for } u \leq \varepsilon, \text{ and } 0 \leq f(u) \leq f'(\varepsilon^+)u \text{ for } \varepsilon < u \leq 1, \text{ and } f(1) = 0, \tag{4}$$

as well as some regularity conditions (see Sec. II below). The results obtained are valid for all ε, and in the limit of $\varepsilon \to 0$ the upper and lower limits coincide and give the Brunet–Derrida value.

In the Aronson and Weinberger article alluded to before,[1] it is proven (so far only for $C^1[0, 1]$ reaction profiles) that:

(i) There exists a minimal value of c (say $c_{\min}$ for which there exists a monotone traveling front solution of the form $q(x - ct)$ for the reaction diffusion equation (1), joining the $q = 1$ and the $q = 0$ states. This function is a solution of the ordinary differential equation,

$$q'' + cq' + f(q) = 0, \tag{5}$$

with $q' < 0$ and $q(z) \to 1$ as $z \to -\infty$ and $q(z) \to 0$ and $z \to +\infty$, and

(ii) that any sufficiently localized initial condition to (1) evolves into the solution q of Eq. (5) with $c = c_{\min}$.

Profiles exhibiting a cutoff, like the one considered by Brunet and Derrida[9] and others, certainly are not in $C^1[0, 1]$, and therefore the Aronson and Weinberger scheme does not necessarily apply to them. For particular reaction profiles with cutoff, (e.g., for $u(1 - u^2)\Theta(u - \varepsilon)$, $u^m(1 - u)\Theta(u - \varepsilon)$, for $m \geq 2$, and others) the existence of heteroclinic orbits has been proven using geometric methods (see, e.g., Refs. 11 and 15).

In this manuscript we partially extend the Aronson and Weinberger scheme, in the sense that under rather weak conditions on the reaction profile (in particular allowing discontinuous profiles), we prove the existence of a unique monotone traveling front solution, with minimal speed, of the form $q(x - ct)$ for the reaction diffusion equation (1), joining the $q = 1$ and the $q = 0$ states, and that this function satisfies (5). Under somewhat stronger assumptions we show that they are unique. Our proof relies on variational techniques. Notice that making the change of variable $z \to s = \exp(-cz)$ in (5) and writing $q(z) = u(s)$, u should satisfy,

$$c^2 \frac{d^2u}{ds^2} + \frac{f(u)}{s^2} = 0 \tag{6}$$

and given the conditions on q above, we are interested in solutions of (6), such that $u(s) > 0$, $u'(s) > 0$ on the half–line $[0, \infty)$. Intuitively, (6) is the Euler equation of the variational principle,

$$c^2 = 2 \sup \frac{\int_0^\infty F(u)/s^2\, ds}{\int_0^\infty u'^2(s)\, ds} \tag{7}$$

with $F(u) = \int_0^u f(x)\, dx$, and the supremum should be considered on an appropriate function space.

Our strategy here (see Sec. II below) is to prove that under weak regularity conditions on the profile $f(u)$, the supremum of (7) is attained and that the maximizer satisfies precisely (6). For C^1 profiles (in which case the existence of the heteroclinic orbit had already been established by Aronson and Weinberger), the connection between the variational principle and the Euler equation (5) was established in Refs. 2 and 4. In Sec. III, we explicitly determine the maximizing function for the profile

$$f_L(u) = 0, \text{ for } u \leq \varepsilon, \text{ and } 0 \leq f_L(u) = u \text{ for } \varepsilon < u < 1. \tag{8}$$

In view of the KPP type conditions (4) required on $f(u)$ we have that

$$f(u) \leq f_L(u),$$

for all $0 \leq u \leq 1$. The profile $f_L(u)$ falls into the class studied in Sec. II, and thus the existence of a minimizer follows at once. If we denote by c_L^2 the value of the supremum of (7) when we replace F by $F_L(u) = \int_0^u f_L(x)\, dx$, one can explicitly compute (see Sec. III below),

$$c_L \equiv 2 \sin \phi_*, \tag{9}$$

where ϕ_* is the first positive solution of the equation,

$$\phi_* \tan \phi_* = \frac{1}{2} |\log \varepsilon|. \tag{10}$$

In particular, for $\varepsilon \to 0$, we have,

$$c_L = 2 - \frac{\pi^2}{|\log \varepsilon|^2} + o\left(\frac{1}{|\log \varepsilon|^2}\right).$$

Since $f(u) \leq f_L(u)$, for all $0 \leq u \leq 1$ it follows from the variational principle (7) that

$$0 \leq c^2 \leq c_L^2. \tag{11}$$

On the other hand, using proper trial functions for u in (7) one can get sharp lower bounds on c^2. Our main result, concerning upper and lower bounds for the minimal speed for profiles satisfying (4) is the following theorem:

Theorem 1.1: *Consider the reaction diffusion equation (1) where the reaction profile satisfies (4). Let* $N(u) = f_L(u) - f(u)$, *all* $u \in [0, 1]$, *where* $f_L(u)$ *is given by (8). Moreover, assume* $N(u) \leq B(u - \varepsilon)^{1+\eta}$, *for* $\varepsilon \leq u \leq 1$, *where* $\eta > 0$. *Then, the minimal speed of propagation of monotonic fronts of the reaction diffusion equation (1),* c, *satisfies,*

$$0 \leq c_L^2 - c^2 \leq o\left(\frac{1}{|\log \varepsilon|^2}\right). \tag{12}$$

Here, c_L *is given explicitly by (9) and (10). In particular, for* $\varepsilon \to 0$, *we have,*

$$c_L = 2 - \frac{\pi^2}{|\log \varepsilon|^2} + o\left(\frac{1}{|\log \varepsilon|^2}\right)$$

Remark: Although in principle the interest is focused on small values of the parameter ε, our expression for $c_L = 2\sin \phi_*$ is valid for any $0 \leq \varepsilon < 1$. In fact, one can also consider the interesting case $\varepsilon \to 1$. In that case, the profile f_L is peaked around $u = 1$, which is the typical situation that arises in the propagation of flames (first studied in Ref. 17). For the case of profiles $f(u)$ peaked around $u = 1$, the speed of fronts is approximately given by

$$c_{ZFK} = \sqrt{2 \int_0^1 f(u) du},$$

(see, Ref. 17; it turns out that this expression c_{ZFK} for the speed of the traveling fronts is actually a lower bound to the actual speed c, see Refs. 8 and 3). Using the ZFK expression for the profile $f_L(u)$, one has,

$$c_{ZFK} = \sqrt{1 - \varepsilon^2} \approx \sqrt{2(1 - \varepsilon)},$$

as $\varepsilon \to 1$. On the other hand, as $\varepsilon \to 1$, $|\log \varepsilon| = |\log(1 + (\varepsilon - 1)| \approx 1 - \varepsilon$, approaches zero. Using (10), we see that also $\tan\phi_* \approx 0$ in this case, and we have $\sin\phi_* \approx \tan\phi_* \approx \phi_*$. Hence, from (10),

$$\phi_*^2 \approx \frac{1}{2}(1 - \varepsilon)$$

and thus,

$$c = 2\sin(\phi_*) \approx 2\phi_* = \sqrt{2(1-\varepsilon)}$$

which coincides with the ZFK value.

The rest of the paper is organized as follows: in Sec. II, we put in a rigorous mathematical framework the maximizing principle (7), for a rather general class of profiles with minimal regularity properties (in particular allowing for jumps, which are necessary when one considers the Brunet–Derrida type problems). We prove the existence of a unique maximizer, and we prove that this maximizer satisfies the Euler equation (6). As we mentioned above, the mathematical results proven in Sec. II, let us show the existence of the appropriate heteroclinic orbit characterizing the minimal speed of monotone traveling fronts. In Sec. III, we give an explicit expression for the minimizer of the *linear problem*, i.e., of the variational principle associated to the *linear profile* $f_L(u)$, and we compute in closed form the value of the maximum, c_L^2, in this case. Finally, in Sec V, we prove our main result (i.e., Theorem 1.1) and, in particular we provide error bounds for c. A preliminary report on these results was announced in Ref. 6.

II. EXISTENCE OF TRAVELING WAVES

In this section we prove the existence and uniqueness of the traveling wave under suitable assumptions on the profile f. As we shall see, the existence of a maximizer which is distinct from the existence of a traveling wave solution can be achieved under rather general assumptions. To prove existence and uniqueness of traveling waves is more difficult and we are able to do this only under more restrictive assumptions that nevertheless include discontinuous profiles.

We look for the existence of a maximizer for the functional

$$\mathcal{F}(u) = 2\frac{\int_0^\infty \frac{F(u(s))}{s^2}ds}{\int_0^\infty u'(s)^2 ds} \tag{13}$$

in the class of functions $u : \mathbb{R}_+ \to [0, 1]$ with $u(0) = 0$, u increasing, $\lim_{s\to\infty} u(s) = 1$ and such that the weak derivative $u' \in L^2([0, \infty))$. We denote this set of functions by $\mathcal{C}$. Note that the functional $\mathcal{F}(u)$ is scale invariant in the sense that if we replace $u(s)$ by $v(s) = u(\alpha s)$ then $\mathcal{F}(u) = \mathcal{F}(v)$. It is convenient to relax the domain and consider $\mathcal{F}$ on the set $\mathcal{C}_<$ that consists of all monotone increasing functions $u(s)$ with $u(0) = 0$, $\lim_{s\to\infty} u(s) \le 1$ and $u' \in L^2([0, \infty))$.

One of the main assumptions is that the function

$$F(u) := \int_0^u f(v)dv$$

satisfies

$$\int_0^1 \frac{F(\sqrt{s})}{s^2} ds < \infty.$$

Note that f of the form given in (4) satisfies these assumptions. This condition is violated in cases where the KPP criterion is satisfied, i.e., $f(u) \le f'(0)u$ for all $u \in [0, 1]$. For these cases there does not exist a maximizer. Indeed, consider the sequence

$$u_n(s) = \begin{cases} \left(\frac{s}{n}\right)^\alpha & \text{if } 0 \le s \le n \\ 1 & \text{if } n \le s \end{cases}$$

where $\alpha > 1/2$. In this case it is easily seen that

$$\lim_{n\to\infty} \mathcal{F}(u_n) = \frac{2}{\alpha} f'(0).$$

Hence, the supremum over $\alpha > 1/2$ is $4f'(0)$ giving the precise KPP speed. There is, however no maximizer. It is interesting to note that the maximum is entirely determined by the behavior of the $f(u)$ for small values of u, in which case the functional reduces to

$$f'(0)\frac{\int_0^\infty \frac{u(s)^2}{s^2} ds}{\int_0^\infty u'(s)^2 ds},$$

which is bounded above by $4f'(0)$ by Hardy's inequality (see Ref. 10).

Theorem 2.1: *Assume that $f(u)$ is non-negative and lower semicontinuous and that the function $F(u) = \int_0^u f(v)dv$ satisfies*

$$\int_0^1 \frac{F(\sqrt{s})}{s^2} ds < \infty.$$

Then there exists a function $u \in \mathcal{C}_<$ such that

$$\mathcal{F}(u) = \sup_{v\in\mathcal{C}_<} \mathcal{F}(v) =: M. \tag{14}$$

Moreover, for any smooth non-negative function h with support in the open set

$$Z = \{s \in (0, \infty) : u(s) < 1\},$$

$$-2M \int_0^\infty u'(s)h'(s)ds + \int_0^\infty \frac{f(u(s))h(s)}{s^2} ds \le 0. \tag{15}$$

In particular the function u is a concave function on Z.

Proof: Let u_n be a maximizing sequence, i.e.,

$$\mathcal{F}(u_n) \to M, \tag{16}$$

which at this moment we do not assume to be finite. By scaling we may assume that $\int_0^\infty u_n'(s)^2 ds = 1$. It is a standard estimate that

$$|u_n(x) - u_n(y)| \le |x - y|^{1/2}\sqrt{\int_0^\infty u_n'(s)^2 ds} = |x - y|^{1/2}, \tag{17}$$

in particular

$$u_n(s) \le \sqrt{s}.$$

Therefore,

$$0 \le F(u_n(s)) \le \frac{F(\sqrt{s})}{s^2},$$

where the right side, by assumption, is integrable on [0, 1]. Hence

$$\int_0^\infty \frac{F(u_n(s))}{s^2} ds \le \int_0^1 \frac{F(\sqrt{s})}{s^2} ds + \int_1^\infty \frac{F(1)}{s^2} ds = \int_0^1 \frac{F(\sqrt{s})}{s^2} ds + F(1)$$

and $M < \infty$. By (17), the functions u_n are uniformly continuous. Since the functions u_n are uniformly bounded, by the Theorem of Arzela and Ascoli we can pass to a subsequence, again denoted by u_n, which converges uniformly on any finite subinterval of $[0, \infty)$ to some function u. This function is in $\mathcal{C}_<$ since the point-wise limit of monotone functions is monotone. The sequence $F(u_n(s))$ is uniformly bounded by an integrable function and using the dominated convergence theorem

$$\lim_{n\to\infty} \int_0^\infty \frac{F(u_n(s))}{s^2} ds = \int_0^\infty \frac{F(u(s))}{s^2} ds, \tag{18}$$

and

$$\liminf_{n\to\infty}\int_0^\infty u_n'^2 ds \geq \int_0^\infty u'^2 ds \tag{19}$$

by the weak lower semicontinuity of the L^2-norm. Thus

$$M = \lim_{n\to\infty}\mathcal{F}(u_n) \leq \mathcal{F}(u) \tag{20}$$

and u is a maximizer in $\mathcal{C}_<$. In particular $\int_0^\infty u'^2 ds = 1$.

By the assumption on h we have for t sufficiently small $0 < u + th < 1$ on Z. The following general remark is useful. Let v be a function with $0 \leq v(s) \leq 1$, $v(0) = 0$, $\int_0^\infty v'(s)^2 ds < \infty$, but not necessarily monotone. Consider the function

$$u_v(s) = \min\{\int_0^s \max\{v'(t), 0\}dt\ , 1\}. \tag{21}$$

Clearly, this function is in $\mathcal{C}_<$. Further, $u_v \geq v$ pointwise, so that $F(v) \leq F(u_v)$. By construction $\|u_v'\|_2 \leq \|v'\|_2$ and hence

$$\mathcal{F}(v) \leq \mathcal{F}(u_v). \tag{22}$$

Thus,

$$\mathcal{F}(u + th) \leq \mathcal{F}(u_{u+th}) \leq \mathcal{F}(u)$$

and the function $t \to \mathcal{F}(u + th)$ has a maximum at $t = 0$. Since f is lower semi-continuous we have that

$$\frac{1}{t}\int_{u(s)}^{u(s)+th(s)} f(v)dv \geq f(v(s,t))h(s),$$

where

$$f(v(s,t)) = \min_{u(s)\leq v\leq u(s)+th(s)} f(v).$$

As $t \to 0$, $v(s, t)$ converges to $u(s)$ and hence again by the lower semi-continuity of f

$$\liminf_{t\to 0}\frac{1}{t}\int_{u(s)}^{u(s)+th(s)} f(v)dv \geq f(u(s))h(s).$$

By Fatou's lemma

$$\liminf_{t\to 0}\int_0^\infty \frac{1}{t}\int_{u(s)}^{u(s)+th(s)} f(v)dv\frac{1}{s^2}ds \geq \int_0^\infty f(u(s))h(s)\frac{1}{s^2}ds,$$

and hence,

$$0 \geq \liminf_{t\to 0}\frac{\mathcal{F}(u+th)-\mathcal{F}(u)}{t} \geq -2M\int_0^\infty u'(s)h'(s)ds + \int_0^\infty \frac{f(u(s))h(s)}{s^2}ds, \tag{23}$$

where we have used, conveniently, that $\int_0^\infty u'(s)^2 ds = 1$. Thus, the second derivative of u in the weak sense is non-positive on Z and from this it follows that u is concave on Z. □

Remark 2.2: The inequality (15) has another interesting consequence. It is a priori not clear that the function $\frac{f(u(s))}{s^2}$ is locally integrable. As an example, consider the function

$$f(v) = \begin{cases} 1 & \text{if } v = 3/4 \\ \left(\frac{1}{|v-3/4|^{1/2}} - 3/2\right)_+ & \text{if } v \neq 3/4 \end{cases}. \tag{24}$$

This function satisfies all the conditions required. Assume now that near $s = 1$ the function $u(s)$ is of the form

$$u(s) = \frac{3}{4} + (s-1)^3.$$

Then

$$f(u(s)) = (\frac{1}{|s-1|^{3/2}} - 3/2)_+$$

which is not integrable at $s = 1$. Inequality (15) states that the maximizer $u(s)$ 'adjusts' itself as to render $f(u(s))$ integrable and hence the function $u(s)$ displayed above cannot be a maximizer.

Theorem 2.3: *With the same assumptions on f as in Theorem 2.1, let u be a maximizer. If $u(s) < 1$ for all $s \in [0, \infty)$, then $\liminf_{v<a, v\to a} f(v) = f(a) = 0$ where $a = \lim_{s\to\infty} u(s) \le 1$. If in addition we assume that the support of f (which is defined since f is lower semi-continuous), is an interval of the form $[\varepsilon, 1]$ for some $0 \le \varepsilon < 1$. Then $\lim_{s\to\infty} u(s) = 1$ and hence $u \in \mathcal{C}$. In particular*

$$\mathcal{F}(u) = \sup_{v\in\mathcal{C}} \mathcal{F}(v) = \sup_{v\in\mathcal{C}_<} \mathcal{F}(v) =: M. \tag{25}$$

Proof: Since $u(s) < 1$ for all $s \in [0, \infty)$ the inequality (15) applies with $Z = (0, \infty)$. Pick $0 \le \phi_\varepsilon \in C_c^\infty(-\varepsilon, \varepsilon)$ with $\int \phi_\varepsilon dx = 1$ and set $w_\varepsilon = \phi_\varepsilon * u$. Then for all $k \in C_c^\infty((\varepsilon, \infty))$ we have

$$2M \int_0^\infty w_\varepsilon'(s) k'(s) ds + \int_0^\infty (\phi_\varepsilon * \frac{f(u)}{s^2})(s) k(s) ds \le 0$$

or

$$2M w_\varepsilon'' + (\phi_\varepsilon * \frac{f(u)}{s^2})(s) \le 0.$$

Integrating this inequality from s to ∞ using the fact that u' and hence w' tends to zero, we find

$$2M w_\varepsilon'(s) \ge \int_s^\infty (\phi_\varepsilon * \frac{f(u(s))}{s^2})(t) dt = \int_{\mathbb{R}} \phi_\varepsilon(t) \int_s^\infty \frac{f(u(\sigma - t))}{(\sigma - t)^2} d\sigma dt.$$

Since $u(s)$ is increasing towards its limit a

$$\liminf_{s\to\infty} f(u(s)) = \liminf_{v<a, v\to a} f(v),$$

and thus

$$\liminf_{s\to\infty} s w_\varepsilon'(s) \ge \frac{\liminf_{v<a, v\to a} f(v)}{2M}.$$

If the right side was not zero this would imply that $w_\varepsilon(s)$ diverges logarithmically fast as $s \to \infty$ which is contradiction since $\lim_{s\to\infty} w_\varepsilon(s) = \lim_{s\to\infty} u(s) = a$. Hence $\liminf_{v<a, v\to a} f(v) = 0$. Since f is lower semi-continuous, $\liminf_{v<a, v\to a} f(v) \ge f(a)$ and since f is non-negative, $\liminf_{v<a, v\to a} f(v) = f(a) = 0$. The other statements are an immediate consequence of this and Theorem 2.1, since $\lim_{s\to\infty} u(s) > \varepsilon$ for otherwise $M = 0$. □

Thus, we have established the existence of an optimizer. In practice one has compute these optimizers by solving a non-linear second order differential equation. Thus, our goal is to establish this equation and to show that the solutions are essentially unique. We shall do so by assuming that the profile f vanishes on the interval $[0, \varepsilon]$ and is strictly positive, bounded and lower semi-continuous on the interval $(\varepsilon, 1)$. It is easy to see that all the assumptions of Theorems 2.1 and 2.3 are satisfied under these new assumptions.

Lemma 2.4: Assume that f vanishes on the interval $[0, \varepsilon]$ for some $0 < \varepsilon < 1$ and is strictly positive, bounded and lower semi-continuous on the interval $(\varepsilon, 1)$. Then the optimizer $u \in \mathcal{C}$ satisfies the equation

$$\int_s^t \frac{f(u(y))}{y^2} dy = 2M[u'(s) - u'(t)], \tag{26}$$

for all $s, t \in Z = \{x \in (0, \infty) : u(x) < 1\}$. In particular the function u' is continuous on Z. Moreover, on every compact subset C of Z there exists a constant $c > 0$ such that $u'(s) \ge c$ for all $s \in C$.

Proof: Pick the "bump" function $0 \le \phi_\epsilon \in C_c^\infty(-\epsilon, \epsilon)$ with $\int \phi(x)dx = 1$ and consider the function

$$h(x) = \int_{-\infty}^{x} [\phi_\epsilon(s-y) - \phi_\epsilon(t-y)]dy,$$

which is non-negative and for ε sufficiently small in $C_c^\infty(Z)$. Using (15) and a simple limiting argument as $\varepsilon \to 0$ we find that

$$\int_s^t \frac{f(u(y))}{y^2} dy \le 2M[u'(s) - u'(t)] \tag{27}$$

holds for almost all s, $t \in Z$. From this it follows that the function $f(u(y))/y^2$, which is a lower semi-continuous function, is integrable and hence the left side of (27) for fixed s is an absolutely continuous, monotone function in t or vice versa. Moreover, the function $u'(s)$ is decreasing and hence can have at most a countable number of points where it is discontinuous. At all the other points (27) is satisfied. Write $Z = (0, s_0)$ where s_0 could be infinity. It follows from (27) that

$$2Mu'(s) \ge \int_s^t \frac{f(u(y))}{y^2} dy$$

and by letting $t \to s_0$ we have that

$$2Mu'(s) \ge \int_s^{s_0} \frac{f(u(y))}{y^2} dy.$$

The right side is strictly positive for all $s < s_0$ and since $u'(s)$ is decreasing we find that $u'(s) > c > 0$ on any compact subset of Z. To derive the differential equation for u we have to compute

$$\lim_{t\to 0} \int_0^\infty \frac{1}{t} \int_{u(s)}^{u(s)+th(s)} f(v)dv \frac{1}{s^2} ds,$$

with $h \in C_c^\infty(Z)$ not necessarily positive. The problem is that for u constant the limit

$$\lim_{t\to 0} \frac{1}{t} \int_u^{u+th(s)} f(v)dv = f(u)h(s) \tag{28}$$

might not exist. It exists only for *almost every u*. By Lemma 2.4 $u'(s) \ge c > 0$ on any compact subset of Z. Thus, the inverse function of u exists and is Lipschitz. If $A \subset (0, 1)$ is the set of zero measure where (28) fails, then

$$B = \{s \in Z : u(s) \in A\}$$

is also a set of zero measure (Sard's theorem). Hence

$$\lim_{t\to 0} \frac{1}{t} \int_{u(s)}^{u(s)+th(s)} f(v)dv = f(u(s))h(s)$$

for almost every $s \in Z$. It remains to justify the interchange of this limit with the s-integration. Since f is bounded, the integrand

$$\frac{1}{t} \int_{u(s)}^{u(s)+th(s)} f(v)dv \frac{1}{s^2} \le \frac{\text{const.}}{s^2},$$

and we have by the dominated convergence theorem that

$$\lim_{t\to 0} \int_0^\infty \frac{1}{t} \int_{u(s)}^{u(s)+th(s)} f(v)dv \frac{1}{s^2} ds = \int_0^\infty f(u(s))h(s) \frac{1}{s^2} ds.$$

A straightforward computation shows that

$$-2M \int u'(s)h'(s)ds + \int_0^\infty f(u(s))h(s) \frac{1}{s^2} ds = 0$$

for all functions $h \in C_c^\infty(Z)$. Using the "bump" function argument as before, we obtain

$$\int_s^t \frac{f(u(y))}{y^2} dy = 2M[u'(s) - u'(t)],$$

for almost all $s, t \in Z$. Assume that $u(s)$ is discontinuous at some $z_0 \in Z$, i.e, $\lim_{s\to z_0, s>z_0} u(s) - \lim_{s\to z_0, s<z_0} u(s) = C$ where C is a positive constant. Since the number of points where $u(s)$ is not continuous is countable and since (26) holds for almost all $s, t \in Z$ we find a sequence $t_n > s_0$ with $t_n \to s_0$ and $s_n < s_0$ and $s_n \to s_0$ along which (26) holds. Since the left side of (26) is continuous we find that C must vanish. Hence (26) holds for all s, t in Z. □

Lemma 2.5: Assume that f vanishes in the interval $[0, \varepsilon]$ *for some* $0 < \varepsilon < 1$ *and is strictly positive, bounded, and lower semi-continuous on* $(\varepsilon, 1)$. *Further assume that* $\lim_{v<1, v\to 1} f(v)$ *exists and is positive. Then there exists* $z_0 < \infty$*such that* $u(s) < 1$ *on* $[0, s_0)$, $u(s) = 1$ *for* $s \geq s_0$ *and* $u'(s_0) = 0$.

Proof: By Theorem 2.3 we know that there must exist a finite number s_0 beyond which $u(s) = 1$. It remains to show that $u'(s_0) = 0$. By (26) we know that $c := \lim_{s<s_0, s\to s_0} u'(s)$ exists. Assume that this limit is strictly positive. Then since $u'(s)$ is decreasing, we have that $u'(s) \geq c$ for all $s \in [0, s_0]$. Let $h \in C_c^\infty(0, \infty)$ be non-negative. Then for $t > 0$ we know by the arguments in the proof of Theorem 2.3 that $\mathcal{F}(u) \geq \mathcal{F}(u - th)$ and hence by a simple computation

$$0 \leq \lim_{t\to 0} \int_0^\infty \frac{1}{t} \int_{u(s)-th(s)}^{u(s)} f(v) dv \frac{1}{s^2} ds - 2M \int_0^\infty u'(s)h'(s) ds \tag{29}$$

provided the limit exists. Since the limit $\lim_{v<1, v\to 1} f(v)$ exists we have that

$$\lim_{t\to 0} \int_{s_0}^\infty \frac{1}{t} \int_{1-th(s)}^1 f(v) dv \frac{1}{s^2} ds = \int_{s_0}^\infty f(u(s))h(s) \frac{1}{s^2} ds.$$

Likewise, by the same arguments as in the proof of Lemma 2.4 the limit

$$\lim_{t\to 0} \int_0^{s_0} \frac{1}{t} \int_{u(s)-th(s)}^{u(s)} f(v) dv \frac{1}{s^2} ds$$

exists since $u'(s) \geq c > 0$. Hence

$$\int_0^\infty f(u(s))h(s) \frac{1}{s^2} ds - 2M \int_0^\infty u'(s)h'(s) ds \geq 0.$$

Next we pick h. Let $\phi_\epsilon(y)$ be the usual "bump" function, i.e., smooth, non-negative, supported in $[-\epsilon, \epsilon]$ with integral 1 and define for $s < s_0 < t$

$$h_\epsilon(x) = \int_{-\infty}^x [\phi_\epsilon(s - y) - \phi_\epsilon(t - y)] dy$$

which is in $C_c^\infty(0, \infty)$. As $\epsilon \to 0$ we find that

$$\int_0^\infty f(u(s))h_\epsilon(s) \frac{1}{s^2} ds \to \int_s^t \frac{f(u(s))}{s^2} ds$$

for almost all s, t and that

$$2M \int_0^\infty u'(y)h_\epsilon'(y) dy = 2M \int_0^\infty u'(y)[\phi_\epsilon(s - y) - \phi_\epsilon(t - y)] dy \to 2Mu'(s).$$

We use here the fact that $u'(y)$ is continuous for $y < s_0$ and that $u'(t) = 0$. Hence

$$\int_s^t \frac{f(u(s))}{s^2} ds \geq 2Mu'(s)$$

for almost all $t > s_0$. Letting t tend to s_0 we find using (26) that

$$0 \geq 2Mu'(s) - \int_s^{s_0} \frac{f(u(s))}{s^2} ds \geq 2Mu'(s_0)$$

which contradicts the assumption that $u'(s_0) > 0$. □

To obtain uniqueness of the solution of Eq. (26) we need some drastic assumptions on f which, however, include our discontinuous profiles.

Theorem 2.6: *Let f be a function that vanishes on the interval $[0, \varepsilon]$ and is strictly positive and uniformly Lipschitz on the interval $(\varepsilon, 1)$. Then there exists a unique solution $u \in \mathcal{C}$ of Eq. (26) which satisfies the normalization $u(\varepsilon) = \varepsilon$.*

Proof: Since the functional (13) as well as $\mathcal{C}$ are invariant under scaling we may assume that the solution u satisfies $u(\varepsilon) = \varepsilon$. Since f vanishes on the interval $[0, \varepsilon]$ we infer from (26) that u' is constant on the interval $[0, \varepsilon]$ and hence, since $u(\varepsilon) = \varepsilon$, we have that $u'(s) = 1$ on $[0, \varepsilon]$. Since f is uniformly Lipschitz on $(\varepsilon, 1)$ Eq. (26) has a unique solution with the initial conditions $u(\varepsilon) = \varepsilon$, $u'(\varepsilon) = 1$ on the set Z. Note that the set Z is uniquely specified by the initial conditions. □

Lemma 2.7: Let u be a maximizer of the functional $\mathcal{F}(u)$ with the profile given by the function

$$f(v) = \begin{cases} 0 \text{ if } 0 \leq v \leq \varepsilon \\ v \text{ if } \varepsilon < v < 1 \end{cases}. \tag{30}$$

Then

$$s_0 = \inf\{s : u(s) = 1\} \tag{31}$$

is finite and $u'(s_0) = 0$. If we assume that u is normalized such that $u(\varepsilon) = \varepsilon$ then on the interval $(0, \varepsilon]$, $u(s) = s$ and on the interval $(\varepsilon, s_0]$the function $u(s)$ is of the form

$$u(s) = \sqrt{s} A \cos(\frac{1}{2}\sqrt{2/M - 1}) \log s + \delta) \tag{32}$$

for suitable constants A and δ. Finally, on (s_0, ∞), $u(s) \equiv 1$.

Proof: The existence follows from Theorem 2.3. That s_0 is finite follows from Lemma 2.5. Finally the form of u given in (32) follows from a straightforward calculation. □

III. THE MAXIMIZER

In this section we determine explicitly the optimizer, whose existence was established in Sec. II.

Theorem 3.1: *The unique maximizer is given by*

$$u(s) = \begin{cases} s \text{ if } 0 \leq s \leq \varepsilon \\ A\sqrt{s}\cos(\phi(s)) \text{ if } \varepsilon < s < s_0 \end{cases} \tag{33}$$

with

$$A = \frac{\sqrt{\varepsilon}}{\cos(\phi_*)} \quad s_0 = \frac{1}{\varepsilon} \tag{34}$$

and

$$\phi(s) = \frac{1}{2}\cot(\phi_*)\log(\frac{s}{\varepsilon}) - \phi_*. \tag{35}$$

Here ϕ_* is the first positive solution of the equation

$$\phi_* \tan(\phi_*) = \frac{1}{2}|\log(\varepsilon)|. \tag{36}$$

Moreover, we have

$$M =: \sup_{v\in\mathcal{C}} \mathcal{F}(v) = \mathcal{F}(u) = 2\sin^2\phi_*. \tag{37}$$

Proof. In a first step we show that

$$s_0 = \frac{1}{\varepsilon}. \tag{38}$$

We know by Lemmas 2.1 and 2.7 that there exists a maximizer with the following properties:

$$u(\varepsilon) = \varepsilon \quad u(s_0) = 1 \quad u'(s_0) = 0. \tag{39}$$

Since $u'(s)$ is continuous and $u(s) = s$ for $s \leq \varepsilon$ we also have

$$u'(\varepsilon) = 1. \tag{40}$$

Moreover, on the interval $[\varepsilon, s_0]$ the function $u(s)$ is positive and increasing and has the form

$$u(s) = \sqrt{s}A\cos\left(\frac{1}{2}\sqrt{\frac{2}{M}-1}\log s + \delta\right), \tag{41}$$

where $M = \mathcal{F}(u)$, the maximal value of the functional. Note that by (36) and (37) $M < 2$.

Since $u(\varepsilon) = \varepsilon$ and $u'(\epsilon) = 1$ we have, using (41)

$$\sqrt{\varepsilon} = A\cos\left(\frac{1}{2}\sqrt{\frac{2}{M}-1}\log\varepsilon + \delta\right) \tag{42}$$

$$-\sqrt{\varepsilon} = A\sqrt{\frac{2}{M}-1}\sin\left(\frac{1}{2}\sqrt{\frac{2}{M}-1}\log\varepsilon + \delta\right). \tag{43}$$

Similarly, from the fact that $u(s_0) = 1$ and $u'(s_0) = 0$ we get from (41)

$$1 = \sqrt{s_0}A\cos(\frac{1}{2}\sqrt{\frac{2}{M}-1}\log s_0 + \delta), \tag{44}$$

$$1 = \sqrt{s_0}A\sqrt{\frac{2}{M}-1}\sin(\frac{1}{2}\sqrt{\frac{2}{M}-1}\log s_0 + \delta). \tag{45}$$

Next we prove (36), i.e., we calculate $\mathcal{F}(u) = M$. A straightforward calculation yields for the numerator

$$\int_0^\infty \frac{[u(s)^2 - \varepsilon^2]}{s^2}ds \tag{46}$$

$$= \frac{A^2}{2}\left(\log\frac{s_0}{\varepsilon} + \frac{1}{\sqrt{\frac{2}{M}-1}}(\sin(\sqrt{\frac{2}{M}-1}\log s_0 + 2\delta) - \sin(\sqrt{\frac{2}{M}-1}\log\varepsilon + 2\delta)\right). \tag{47}$$

Likewise, for the denominator

$$2\int_0^\infty u'(s)^2 ds = 2\varepsilon + \frac{A^2}{2M}\log\frac{s_0}{\varepsilon} \tag{48}$$

$$+\frac{A^2}{2}(1-\frac{1}{M})\frac{1}{\sqrt{\frac{2}{M}-1}}\left((\sin(\sqrt{\frac{2}{M}-1}\log s_0+2\delta)-\sin(\sqrt{\frac{2}{M}-1}\log\varepsilon+2\delta)\right) \tag{49}$$

$$+\frac{A^2}{2}\left(\cos(\sqrt{\frac{2}{M}-1}\log s_0+2\delta)-\cos(\sqrt{\frac{2}{M}-1}\log\varepsilon+2\delta)\right). \tag{50}$$

The equation

$$\int_0^\infty \frac{[u(s)^2-\varepsilon^2]}{s^2}ds = M\left(2\int_0^\infty u'(s)^2 ds\right)$$

then reduces to

$$0 = 2\varepsilon - \frac{A^2}{2}\sqrt{\frac{2}{M}-1}\left(\sin(\sqrt{\frac{2}{M}-1}\log s_0+2\delta)-\sin(\sqrt{\frac{2}{M}-1}\log\varepsilon+2\delta)\right) \tag{51}$$

$$+\frac{A^2}{2}\left(\cos(\sqrt{\frac{2}{M}-1}\log s_0+2\delta)-\cos(\sqrt{\frac{2}{M}-1}\log\varepsilon+2\delta)\right). \tag{52}$$

Using (42)–(45) together with the double angle formulas for cosine and sine one easily sees that the above equation reduces to

$$\left(\varepsilon-\frac{1}{s_0}\right)\left(\frac{2}{2-M}\right)=0, \tag{53}$$

and hence (38) is proved.

The next step is to calculate M. Note that (44) and (45) now read

$$1=\frac{1}{\sqrt{\varepsilon}}A\cos(-\frac{1}{2}\sqrt{\frac{2}{M}-1}\log\varepsilon+\delta), \tag{54}$$

$$1=\frac{1}{\sqrt{\varepsilon}}A\sqrt{\frac{2}{M}-1}\sin(-\frac{1}{2}\sqrt{\frac{2}{M}-1}\log\varepsilon+\delta). \tag{55}$$

from which we deduce that

$$\tan(\frac{1}{2}\sqrt{\frac{2}{M}-1}\log\varepsilon-\delta)=-\frac{1}{\sqrt{\frac{2}{M}-1}}. \tag{56}$$

Likewise from (42) and (43) we obtain

$$\tan(\frac{1}{2}\sqrt{\frac{2}{M}-1}\log\varepsilon+\delta)=-\frac{1}{\sqrt{\frac{2}{M}-1}}. \tag{57}$$

Using the addition formula for the tangent function yields

$$\tan(\sqrt{\frac{2}{M}-1}\log\varepsilon)=-\tan(\sqrt{\frac{2}{M}-1}|\log\varepsilon|)=-\frac{\sqrt{\frac{2}{M}-1}}{\frac{1}{M}-1}. \tag{58}$$

If we set

$$\phi_* = \frac{1}{2}\sqrt{\frac{2}{M} - 1}|\log\varepsilon| \tag{59}$$

and note that

$$\tan(\sqrt{\frac{2}{M} - 1}|\log\varepsilon|) = \frac{2\tan\phi_*}{1 - (\tan\phi_*)^2}, \tag{60}$$

we learn that

$$\tan\phi_* = \frac{1}{\sqrt{\frac{2}{M} - 1}} = \frac{|\log\varepsilon|}{2\phi_*}, \tag{61}$$

which yields (36). Since $\phi_* > 0$ and M is the maximum of our functional, we have to choose ϕ_* to be the first positive solution of (36). In particular we have that $\phi_* < \pi/2$.

It remains to determine δ and A. Subtracting (56) from (57) we find that

$$\tan(2\delta) = 0 \tag{62}$$

and hence $\delta = N\pi/2$ where $N \in \mathbb{Z}$. Note that as s ranges from ε to $1/\varepsilon$, the function

$$\frac{1}{2}\sqrt{\frac{2}{M} - 1}\log s$$

varies from $-\phi_*$ to ϕ_*. The function $u(s)$ is positive and increasing and hence, if we choose the constant A positive, we find that $\delta = 2\pi N$ where $N \in \mathbb{Z}$. Hence we may choose $\delta = 0$. The function $\frac{1}{2}\sqrt{\frac{2}{M} - 1}\log s$ can be conveniently be written as

$$\frac{1}{2}\sqrt{\frac{2}{M} - 1}\log s = \frac{1}{2}\cot\phi_* \log\frac{s}{\varepsilon} - \phi_* \tag{63}$$

and the condition that $u(\varepsilon) = \varepsilon$ yields the value for the constant A stated in Theorem 3.1. Finally, Eq. (37) for the value of M follows immediately from the first equality in (61). □

IV. ERROR ESTIMATES: PROOF OF THEOREM 1.1

In Sec. III we have determined the exact value of the minimal speed, c_L say, of monotonic traveling fronts of Eq. (1) for a linear profile with a cutoff. In fact, if the profile is given piecewise by $f(u) = 0$, for $u < \varepsilon$, and $f(u) = u$ for $\varepsilon \le u \le 1$, we have shown that c_L is given exactly by

$$c_L = 2\sin\phi_*, \tag{64}$$

where ϕ_* is the first positive solution of the equation

$$\phi_* \tan\phi_* = \frac{1}{2}|\log\varepsilon|. \tag{65}$$

Solving (65) for ϕ_* in power series on $1/|\log\varepsilon|$, and replacing it in (64) one finds that

$$c_L = 2 - \frac{\pi^2}{|\log\varepsilon|^2} + o\left(\frac{1}{|\log\varepsilon|^2}\right),$$

where the leading two terms account precisely for the Brunet and Derrida result (i.e., Eq. (3) in the Introduction).

Here, we would like to determine error bounds when the profile $f(u)$ is a KPP profile with a cutoff, in other words, when the profile is given piecewise by $f(u) = 0$, for $0 \le u < \varepsilon$, and $f(u) \le u$ for $\varepsilon \le u \le 1$. If we write $f(u) = u - N(u)$, for $\varepsilon \le u \le 1$, the KPP criterion amounts to requiring that $N(u) \ge 0$. For such a reaction profile, we have that

$$F(u) \equiv \int_0^u f(q)\,dq,$$

is such that $F(u) = 0$ for $0 \le u \le \varepsilon$, whereas

$$F(u) = \frac{1}{2}(u^2 - \varepsilon^2) - \int_\varepsilon^u N(q)\, dq,$$

for $\varepsilon \le u \le 1$. For a KPP profile $N(u) \ge 0$, thus,

$$F(u) \le G(u), \tag{66}$$

where

$$G(u) = \frac{1}{2}\left(u^2 - \varepsilon^2\right)_+ .$$

Hence, using (66) in (13), and, taking into account (37), we see that in general for a KPP profile with a cutoff, the speed of propagation of fronts for an initially localized disturbance of (1), say c, satisfies,

$$c \le c_L.$$

On the other hand, we can also use the variational principle embodied in (13) to obtain a lower bound on c. For that purpose we use as a trial function in (13) the minimizer $\hat{u}$ of the functional G. After some simple computations, we obtain,

$$c_L^2 - c^2 \le \frac{\int_\varepsilon^{1/\varepsilon} N(\hat{u}(s))\hat{u}'(s)(1/s)\, ds}{\int_0^{1/\varepsilon} (\hat{u}'(s))^2\, ds}. \tag{67}$$

Here, we will find estimates on the difference $c_L^2 - c^2$ for profiles that satisfy the bound,

$$0 \le N(x) \le B(x - \varepsilon)^{1+\eta} \tag{68}$$

for $\varepsilon \le x \le 1$, where $\eta > 1$. The denominator can be calculated in closed form as follows:

$$\begin{aligned}
\text{Den} &= \textstyle\int_0^{1/\varepsilon} (\hat{u}'(s))^2\, ds = \varepsilon + \int_\varepsilon^{1/\varepsilon} (\hat{u}'(s))^2\, ds \\
&= \varepsilon + \tfrac{\varepsilon}{4\cos^2\phi_* \sin^2\phi_*} \tfrac{\sin\phi_*}{\cos\phi_*} \textstyle\int_{-\phi_*}^{\phi_*} 2\sin(\phi_* - t)^2\, dt \\
&= \varepsilon + \tfrac{\varepsilon}{4\cos^2\phi_* \sin^2\phi_*} \tfrac{\sin\phi_*}{\cos\phi_*} \textstyle\int_{-\phi_*}^{\phi_*} (1 - \cos(2\phi_* - 2t))\, dt \\
&= \varepsilon + \tfrac{\varepsilon}{4\cos^2\phi_* \sin^2\phi_*} \tfrac{\sin\phi_*}{\cos\phi_*} (2\phi_* - (1/2)\sin 4\phi_*) \\
&= \varepsilon \tfrac{1}{4\cos^3\phi_* \sin\phi_*} (2\phi_* + \sin 2\phi_*)\, .
\end{aligned} \tag{69}$$

On the other hand, using (68) in the numerator of (67), the properties of the trial function $\hat{u}(s)$ (in particular the fact that this function is increasing), we can estimate the numerator as,

$$\text{Num} \le \frac{B\sqrt{\varepsilon}}{2\cos\phi_* \sin\phi_*} \int_\varepsilon^{1/\varepsilon} \left(\sqrt{\varepsilon s}\frac{\cos\phi}{\cos\phi_*} - \varepsilon\right)^{1+\eta} \sin(\phi_* - \phi)\, \frac{ds}{s^{3/2}}. \tag{70}$$

Using the fact that $s = \exp(2\phi\, \tan\phi_*)$ (which follows from (35) and (36) above) and that we can write $\sqrt{\varepsilon} = \exp((\log\varepsilon)/2) = \exp(-|\log\varepsilon|/2) = \exp(-\phi_* \tan\phi_*)$, we have that

$$\sqrt{s\varepsilon} = \exp[(\phi - \phi_*)\tan\phi_*],$$

and also that

$$\frac{ds}{s^{3/2}} = 2\sqrt{\varepsilon}\tan\phi_* \exp((\phi_* - \phi)\tan\phi_*).$$

Changing the variable of integration from s to ϕ in (70), making use of these last two expressions, we find

$$\text{Num} \le \frac{B\varepsilon}{\cos^2\phi_*} \int_{-\phi_*}^{\phi_*} \left(\exp[-(\phi_* - \phi)\tan\phi_*]\frac{\cos\phi}{\cos\phi_*} - \varepsilon\right)^{1+\eta} \sin(\phi_* - \phi) \exp(\phi_* - \phi)\tan\phi_*)\, d\phi. \tag{71}$$

Finally making the change of variables $\phi \to \sigma = \phi_* - \phi$ we get

$$\mathrm{Num} \leq \frac{B\varepsilon}{\cos^2\phi_*}\int_0^{2\phi_*}\left(\exp\left[-\sigma\tan\phi_*\right]\frac{\cos(\phi_*-\sigma)}{\cos\phi_*}-\varepsilon\right)^{1+\eta}\sin\sigma\exp\left(\sigma\tan\phi_*\right)d\sigma. \tag{72}$$

Hence, from (69) and (72), we have,

$$\frac{\mathrm{Num}}{\mathrm{Den}} \leq 4\,B\frac{\cos\phi_*\sin\phi_*}{2\phi_*+\sin(2\phi_*)}I, \tag{73}$$

with,

$$I = \int_0^{2\phi_*}\left(\exp\left[-\sigma\tan\phi_*\right]\frac{\cos(\phi_*-\sigma)}{\cos\phi_*}-\varepsilon\right)^{1+\eta}\sin\sigma\exp\left(\sigma\tan\phi_*\right)d\sigma. \tag{74}$$

When $\varepsilon \to 0$, we have from (36) that $\phi_* \approx \pi/2$, $\sin\phi_* \approx 1$, $\sin(2\phi_*) \approx 0$ and $\cos\phi_* = O(1/|\log\varepsilon|)$. Thus, in order to control the difference $c_L^2 - c^2$, all we have to prove is that

$$I \leq o\left(1/|\log\varepsilon|\right).$$

We can estimate I from above by dropping the ε inside the factor in the integral above. Moreover, we write $\cos(\phi_* - \sigma)/\cos\phi_* = \cos\sigma + \tan\phi_*\sin\sigma$. Thus, we have

$$I \leq J \equiv \int_0^{2\phi_*}\left(\cos\sigma+\tan\phi_*\sin\sigma\right)^{1+\eta}\sin\sigma\exp\left(-\sigma\eta\tan\phi_*\right).\,d\sigma. \tag{75}$$

We now split the integral over σ into two parts. We denote by J_1 the integral between 0 and α and by J_2 the integral between α and $2\phi_*$. The value of α will be conveniently chosen later on. We will first estimate J_1. We use: (i) $\exp(-\sigma\eta\tan\phi_*) \leq 1$ (since $\tan\phi_* > 0$); (ii) $\cos\sigma \leq 1$, and (iii) $\sin\sigma \leq \sigma \leq \alpha$ to get

$$J_1 \leq \int_0^{\alpha}\left(1+\alpha\tan\phi_*\right)^{1+\eta}\sigma\,d\sigma = \frac{\alpha^2}{2}\left(1+\alpha\tan\phi_*\right)^{1+\eta}, \tag{76}$$

and, using the convexity of $x \to x^{1+\eta}$ (since $\eta > 0$), we have

$$J_1 \leq 2^{\eta-1}\left[\alpha^2+\alpha^{3+\eta}(\tan\phi_*)^{1+\eta}\right]. \tag{77}$$

On the other hand, in order to estimate J_2, we use the fact that $0 \leq \cos\sigma, \sin\sigma \leq 1$ in the interval $[0, \phi_*]$ (recall that $\phi_* \leq \pi/2$). We also use that $\exp(-x)$ is decreasing, and we get at once

$$J_2 \leq \left(1+\tan\phi_*\right)^{1+\eta}\exp(-\alpha\eta\tan\phi_*)\,2\phi_*, \tag{78}$$

Pick any $0 < r < 1$, and then choose α to be

$$\alpha = (\tan\phi_*)^{-(2+\eta+r)/(3+\eta)}. \tag{79}$$

The idea behind this choice is that it will make $J_1 = o(1/|\log\varepsilon|)$, and at the same time it will make J_2 of smaller order. Now, one can easily check that, for any $0 < r < 1$,

$$2\left(\frac{2+\eta+r}{3+\eta}\right) > 1+r. \tag{80}$$

Now, since for ε small, $\tan\phi_* > 1$, it follows from (77), our choice of α (i.e., Eq. (79)), and (80) that,

$$J_1 \leq 2^{\eta}(\tan\phi_*)^{-(1+r)}. \tag{81}$$

Finally, using (36) in (81), and the fact that $r > 0$, we get the desired estimate,

$$J_1 = o\left(\frac{1}{|\log\varepsilon|}\right). \tag{82}$$

Also, with our choice of α, (79),

$$\alpha\tan\phi_* = (\tan\phi_*)^{(1-r)/(3+\eta)},$$

where $r > 1$ and $\eta > 0$. Using (78), we see that J_2 is exponentially small as a function of $1/|\log \varepsilon|$ (the first factor grows polynomially as a function of $\tan\phi_*$, while the second factor is exponentially small). Summarizing, we have proven that

$$0 \le c_L^2 - c^2 \le o\left(\frac{1}{|\log\varepsilon|^2}\right)$$

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APPENDIX: ERROR ESTIMATES

For the sake of completeness, in this appendix we prove bounds on $c_L^2 - c^2$ in terms of the parameter η, for KPP profiles. These bounds allow us to show that as $\eta \to \infty$, $c_L^2 - c^2 \to 0$. Consider,

$$J = \int_0^{2\phi_*} (\cos\sigma + \tan\phi_* \sin\sigma)^{1+\eta} \sin\sigma \exp(-\sigma\eta\tan\phi_*)\, d\sigma. \tag{A1}$$

Denote by

$$H \equiv \sigma\tan\phi_* - \log(\cos\sigma + \tan\phi_*\sin\sigma) \tag{A2}$$

and notice that

$$H_\sigma \equiv \frac{dH}{d\sigma} = \sin\sigma \frac{(1+\tan^2\phi_*)}{(\cos\sigma + \tan\phi_*\sin\sigma)} > 0.$$

Using (A1) and (A2), we can write,

$$J = \int_0^{2\phi_*} (\cos\sigma + \tan\phi_*\sin\sigma)\exp(-\eta H)\, \sin\sigma\, d\sigma, \tag{A3}$$

which can be rewritten as,

$$J = \frac{1}{1+\tan\phi_*^2}\int_0^{2\phi_*} (\cos\sigma + \tan\phi_*\sin\sigma)^2 \exp(-\eta H)\, H_\sigma\, d\sigma. \tag{A4}$$

We have remarked before that on the interval $(0, 2\phi_*)$, both $0 < \cos\sigma, \sin\sigma < 1$, thus $(\cos\sigma + \tan\phi_*\sin\sigma)^2 \le (1 + \tan\phi_*)^2$. Moreover, using that $(1 + x)^2/(1 + x^2) \le 2$ for $x \ge 0$, we can finally write,

$$J \le 2\int_0^{2\phi_*} e^{-H\eta} H_\sigma\, d\sigma.$$

Recalling that $H_\sigma > 0$, making the change of variables $\phi \to H$, and computing $H(0) = 0$ and $H(2\phi_*) = 2\phi_*\tan\phi_*$ we get

$$J \le \int_0^{2\phi_*\tan\phi_*} e^{-H\eta}\, dH \le \frac{2}{\eta}(1 - \exp(-2\eta\phi_*\tan\phi_*)) \le \frac{2}{\eta}. \tag{A5}$$

Hence, $J \to 0$ as $\eta \to \infty$.

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