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# Fifth order evolution equation for long wave dissipative solitons

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ABSTRACT

Third and fifth order nonlinear wave equations which arise in the theory of water waves possess solitary and periodic traveling waves. Solitary waves also arise in systems with dissipation and instability where a balance between these effects allows the existence of dissipative solitons. Here we search for a model equation to describe long wave dissipative solitons including fifth order dispersion. The equation found includes quadratic and cubic nonlinearities. For periodic solutions in a small box we characterize the rate of growth, and show that they do not blow up in finite time. Analytic solutions are constructed for special parameter values.



## 1. Introduction

Dissipative solitons or solitary waves in systems with dissipation and instability, arise in fluid dynamics, optics, waves in solids and other physical problems [1]. In cases where a long wavelength oscillatory instability is present, the evolution of small amplitude perturbations can be described by the Kawahara equation or by the Korteweg–deVries–Kuramoto–Sivashinsky (KdV–KS) equation. These are fourth order evolution equations including as a single nonlinearity a convective term. However, the KdV equation with its characteristic nonlinearity is one of many long wave evolution equations that have been derived from the theory of water waves [2–4]. Many of these equations count among its solutions traveling solitary waves. The purpose of this work is to find a higher order evolution equation for dissipative solitons, that is, to construct the analog of higher order water wave equations, for problems that include instability and dissipation. The addition of a higher order dispersion term to the KdV–KS equation has been done previously including a fifth order derivative to the equation [5,6]. In the present work we include high order dispersion taking into account the fact that in a physical problem the presence of a higher order linear term will be accompanied by additional nonlinearities to be determined by a consistent asymptotic expansion. We choose as a model problem that of flow along an inclined plane, and find that the amplitude $\eta(x,t)$ of long surface waves is determined by the perturbed KdV equation

$$\begin{aligned}&\eta_\tau + 6\eta\eta_\xi + \left(\Lambda_1 + \epsilon^2 R_2 \Lambda_1'\right)\eta_{\xi\xi\xi} \\ &\quad + \epsilon\left(\frac{6}{5}R_2\eta_{\xi\xi} + \Lambda_2\eta_{\xi\xi\xi\xi} + \Lambda_3(\eta\eta_\xi)_\xi\right) \\ &\quad + \epsilon^2\left(3\eta^2\eta_\xi + \Lambda_4\eta_{\xi\xi\xi\xi\xi} + \Lambda_5\eta_\xi\eta_{\xi\xi} + \Lambda_6\eta\eta_{\xi\xi\xi}\right) = 0, \qquad (1)\end{aligned}$$

where the coefficients $\Lambda_i$ depend on parameter values, and the parameter $R_2$ measures the excess of the Reynolds number over the critical value. The ordinary differential equation for traveling wave solutions $\eta(x-ct)$ of (1) was considered in [7] where it is shown to be the nonlinear differential equation of fourth order that has exact periodic solutions expressed in terms of Weierstrass functions. The fifth order evolution equation (1) is applicable to problems which exhibit oscillatory long wave instabilities such as surface waves on convecting fluids and others. This equation represents a consistent extension of the KdV–KS equation to include fifth order dispersion.

The problem of flow of a film along an inclined plane has been studied extensively after the first experiments of Kapitsa and Kapitsa [8] due to its relevance in different physical phenomena and interesting mathematical properties. Linear stability analysis is described by the Orr–Sommerfeld equation which predicts that a steady flow becomes unstable to long wave perturbations beyond a critical angle of inclination The nonlinear evolution of these unstable long waves was first studied by Benney [9] by means of asymptotic expansions making use of the fact the ratio of depth to wavelength is a natural small parameter. This result was extended to higher order by Lin [10] to obtain what is usually referred to as Benney–Lin equation. The two equations differ not only in their order of expansion but also in some scaling assumptions and give

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different predictions on the nature of the bifurcation from the basic steady state [11]. We refer to review articles [12,13] for additional references to the problem of flow along an inclined plane. We will show the connection between the Benney–Lin equation and the fifth order Eq. (1).

In Section 2 we formulate the problem. In Section 3 we perform a long wave asymptotic expansion without any assumption on the size of the parameters to obtain the fifth order evolution equation. Here we also show the connection between the equation found with the Benney–Lin equation. In Section 4 we study periodic solutions in a small box and show that in a box where KdV–KS solutions die away, the present model allows growth. The maximal rate of growth is obtained by means of a variational principle from which it will follow that finite time blowup does not occur. Finally analytic solitary waves and front solutions are exhibited for special parameter values. This equation also exhibits periodic solutions in terms of Weierstrass functions which have been given elsewhere.

## 2. Mathematical formulation

Consider a thin layer of incompressible fluid of density $\rho$, viscosity $\mu$ flowing down an inclined plane with an angle $\beta$. The motion of the fluid is described by the Navier–Stokes equations

$$\nabla\cdot\vec{u}=0, \tag{2}$$

$$\rho\big(\vec{u}_t+(\vec{u}\cdot\nabla)\vec{u}\big)=-\nabla p+\rho\vec{g}+\mu\nabla^2\vec{u}. \tag{3}$$

We choose coordinates such that $\vec{g}=g(\sin\beta,0,-\cos\beta)$ and we consider two dimensional motion so that the velocity is given by $\vec{u}=(u,0,w)$.

The fluid is bounded below by a rigid surface on which

$$u=0,\qquad w=0, \tag{4}$$

and the upper surface $z=h(x,t)$ is free. The laminar flow solution is given by

$$p_s=p_a-\rho g\cos\beta(z-d),\qquad U_s=\frac{g\sin\beta}{\nu}\left(dz-\frac{1}{2}z^2\right), \tag{5}$$

where $d$ is the undisturbed depth and $p_a$ the atmospheric pressure. On the free surface $h(x,t)=d+\eta(x,t)$ the boundary conditions are

$$\eta_t+u\eta_x=w, \tag{6}$$

$$p-p_a-\frac{2\mu}{N^2}\big[w_z+u_x\eta_x^2-\eta_x(u_z+w_x)\big] = -\sigma\frac{n_{xx}}{(1+\eta_x)^{3/2}}, \tag{7}$$

$$\big(1-\eta_x^2\big)(u_z+w_x)+2\eta_x(w_z-u_x)=0, \tag{8}$$

where $\sigma$ is the surface tension. Eqs. (2)–(8) constitute the problem to be solved. Scaling length with depth $d$, velocities with the mean stationary speed $u_0=gd^2\sin\beta/(3\nu)$ and pressure with the mean stationary pressure $p_0=\rho gd\cos\beta/2$ and time with $t_0=d/u_0$, the equations can be written in dimensionless form in terms of three parameters, the angle $\beta$, the Reynolds number $R$ and the Weber number $W$ defined as

$$R=\frac{gd^3\sin\beta}{3\nu^2},\qquad W=\frac{\sigma}{\rho gd^2\cos\beta}.$$

## 3. Long wave asymptotic expansion

We consider the case of long small amplitude waves for Weber number of order unity. Following the usual procedure we introduce the small parameter $\epsilon$ which measures the ratio of depth to wavelength and introduce new variables

$$\xi=\epsilon(x-ct),\qquad \tau=\epsilon^3 t,$$
$$u=U_s+\epsilon^2\hat{u},\qquad w=\epsilon^3\hat{w},\qquad p=p_s+\epsilon^2\hat{p},\qquad \eta=\epsilon^2\hat{\eta}.$$

With this scaling the equations and boundary conditions are written as

$$\hat{u}_\xi+\hat{w}_z=0, \tag{9}$$

$$\epsilon\big((U_s-c)\hat{u}_\xi+U_{sz}\hat{w}\big)+\epsilon^3(\hat{u}_\tau+\hat{u}\hat{u}_\xi+\hat{w}\hat{u}_z) = \frac{1}{R}\left[-\frac{3}{2}\epsilon\cot\beta\hat{p}_\xi+\epsilon^2\hat{u}_{\xi\xi}+\hat{u}_{zz}\right], \tag{10}$$

$$\epsilon^2(U_s-c)\hat{w}_\xi+\epsilon^4(\hat{w}_\tau+\hat{u}\hat{w}_\xi+\hat{w}\hat{w}_z) = \frac{1}{R}\left[-\frac{3}{2}\cot\beta\hat{p}_z+\epsilon^3\hat{w}_{\xi\xi}+\epsilon\hat{w}_{zz}\right]. \tag{11}$$

The boundary conditions become

$$\hat{u}=\hat{w}=0\quad\text{on } z=0, \tag{12}$$

and, on $z=1+\epsilon^2\hat{\eta}$,

$$\left(\frac{3}{2}-c\right)\hat{\eta}_\xi+\epsilon^2(\hat{\eta}_\tau+\hat{u}\hat{\eta}_\xi)-\frac{3}{2}\epsilon^4\hat{\eta}^2\eta_\xi=\hat{w}, \tag{13}$$

$$\hat{p}=2\hat{\eta} + \frac{4\epsilon\tan\beta}{3(1+\epsilon^6\hat{\eta}_\xi^2)}\big[\hat{w}_z+\epsilon^6\hat{u}_\xi\hat{\eta}_\xi^2-\epsilon^2\hat{\eta}_\xi\big(-3\hat{\eta}+\hat{u}_z+\epsilon^2\hat{w}_\xi\big)\big] - \frac{2\epsilon^2W\hat{\eta}_{\xi\xi}}{(1+\epsilon^6\hat{\eta}_\xi^2)^{3/2}}, \tag{14}$$

$$\big(1-\epsilon^6\hat{\eta}_\xi^2\big)\big(-3\eta+\hat{u}_z++\epsilon^2\hat{w}_\xi\big)+2\epsilon^4(\hat{w}_z-\hat{u}_\xi)=0, \tag{15}$$

where we have used that, in dimensionless variables, $U_s(1+\epsilon^2\hat{\eta})=\frac{3}{2}-\frac{3}{2}\epsilon^4\hat{\eta}^2$, $p_z(1+\epsilon^2\hat{\eta})=p_a-2\epsilon^2\hat{\eta}$ and $U_{sz}(1+\epsilon^2\hat{\eta})=-3\epsilon^2\hat{\eta}$.

Next we proceed to solve perturbatively the system by expanding all quantities in $\epsilon$ as $\hat{f}=f_0+\epsilon f_1+\epsilon^2 f_2+\cdots$ where $\hat{f}$ represents any of the independent variables $\hat{u}$, $\hat{w}$, $\hat{p}$, $\hat{\eta}$. The Reynolds number is expanded as $R=R_0+\epsilon^2R_2+\cdots$. Our aim is to construct a fifth order evolution equation for the surface deformation. At leading order the equations to be solved are

$$u_{0\xi}+w_{0z}=0,\qquad u_{0zz}=0,\qquad p_{0z}=0$$

subject to $w_0(0)=u_0(0)=0$ and

$$\left(\frac{3}{2}-c\right)\eta_{0\xi}=w_0(1),\qquad p_0(1)=2\eta_0,\qquad -3\eta_0+u_{0z}=0.$$

The solubility condition for this system determines the speed to be $c=3$. We omit giving the details of derivation at higher orders which is lengthy but straightforward. We will only state the solubility conditions at each order. At order $\epsilon$ the solubility condition determines the critical Reynolds number

$$R_0=\frac{5}{6}\cot\beta.$$

At the following order the solubility condition determines that the evolution equation for the leading order surface deformation is the Korteweg–deVries equation

$$\eta_{0\tau}+6\eta_0\eta_{0\xi}+\Lambda_1\eta_{0\xi\xi\xi}=0,$$

where $\Lambda_1=3$. At order $\epsilon^3$ the solubility condition yields

$$\eta_{1\tau}+6(\eta_0\eta_1)_\xi+\Lambda_1\eta_{1\xi\xi\xi}+\frac{6}{5}R_2\eta_{0\xi\xi}+\Lambda_2\eta_{0\xi\xi\xi\xi} + \Lambda_3(\eta_0\eta_{0\xi})_\xi=0,$$

where

$$\Lambda_2 = W\cot\beta + f_1(\beta), \qquad f_1(\beta) = \frac{7743}{2240}\cot\beta - \frac{1}{36036}\cot^3\beta,$$

and $\Lambda_3 = 3\cot\beta$.

The procedure can be halted at this stage and a generalized Kuramoto Sivashinsky equation constructed for $\eta = \eta_0 + \epsilon\eta_1$. Our aim is to determine a higher order correction which will provide a more accurate description of the linear theory and determine the nonlinearities that accompany a linear fifth order derivative term consistently. Proceeding to the next order the solubility condition yields

$$\eta_{2\tau} + 6(\eta_0\eta_2)_\xi + 6\eta_1\eta_{1\xi} + \Lambda_1\eta_{2\xi\xi\xi} + R_2\Lambda_1'\eta_{0\xi\xi\xi} + \frac{6}{5}R_2\eta_{1\xi\xi} + \Lambda_2\eta_{1\xi\xi\xi\xi} + \Lambda_3(\eta_0\eta_1)_{\xi\xi} + 3\eta_0^2\eta_{0\xi} + \Lambda_4\eta_{0\xi\xi\xi\xi\xi} + \Lambda_5\eta_{0\xi}\eta_{0\xi\xi} + \Lambda_6\eta_0\eta_{0\xi\xi\xi} = 0,$$

where

$$\Lambda_4 = \frac{25}{21}W + f_2(\beta),$$
$$f_2(\beta) = \frac{11}{2} + \frac{2198765}{532224}\cot^2\beta - \frac{1385}{38594556}\cot^4\beta,$$
$$\Lambda_1' = \frac{10}{7}\cot\beta, \qquad \Lambda_5 = 26 + \frac{449}{42}\cot^2\beta,$$
$$\Lambda_6 = 12 + \frac{25}{7}\cot^2\beta.$$

Finally a single evolution equation for the surface displacement $\eta = \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2$, obtained combining the solubility conditions, is given by

$$\eta_\tau + 6\eta\eta_\xi + (\Lambda_1 + \epsilon^2 R_2\Lambda_1')\eta_{\xi\xi\xi} + \epsilon\left(\frac{6}{5}R_2\eta_{\xi\xi} + \Lambda_2\eta_{\xi\xi\xi\xi} + \Lambda_3(\eta\eta_\xi)_\xi\right) + \epsilon^2(3\eta^2\eta_\xi + \Lambda_4\eta_{\xi\xi\xi\xi\xi} + \Lambda_5\eta_\xi\eta_{\xi\xi} + \Lambda_6\eta\eta_{\xi\xi\xi}) = 0. \quad (16)$$

This fifth order equation constitutes the evolution equation we searched for. The inclusion of fifth order dispersion has been obtained by a systematic perturbation method which shows that together with the high order dispersive term additional quadratic and cubic nonlinearities have to be considered. Several fifth order water wave models have been studied, most of them correspond to particular cases of (16) without instability and dissipation, that is without the terms that appear at order $\epsilon$. Particular cases of the equation $u_t + 6uu_x + \beta_1 u_{xxx} + \epsilon^2(\beta_2 u^2 u_x + +\beta_3 u_{xxxxx} + \beta_4 u_x u_{xx} + \beta_5 u u_{xxx}) = 0$ are well studied fifth order model equations. We refer to the recent work [3,4].

A natural question which arises is the relation of this equation to the classical large amplitude equations derived by Benney and Lin. Benney's equation includes derivatives up to the fourth order, and therefore the present equation cannot be derived from it. We may consider however Lin's equation

$$h_t + A(h)h_x + \alpha\frac{\partial}{\partial x}[B(h)h_x + C(h)h_{xxx}] + \alpha^2\frac{\partial}{\partial x}[D(h)h_x^2 + E(h)h_{xx} + F(h)h_{xxxx} + G(h)h_x h_{xxx} + H(h)h_{xx}^2 + I(h)h_x^2 h_{xx}] + O(\alpha^3) = 0 \quad (17)$$

where the coefficients are those given in [10] and $\alpha$ is the small wavenumber. In the derivation of this equation it is assumed that the Weber number $W$ is large so that $\alpha^2 W$ is of order one. The coefficients $C(h)$, $F(h)$, $G(h)$, $H(h)$, $I(h)$ are proportional to $\alpha^2 W$. In the present calculation the Weber number is of order one, and these terms become of order $\alpha^3$ and $\alpha^4$, orders which were neglected in Lin's equation. As we show below, we may obtain (16) from (17) but with modified linear terms which arise from the higher order terms not included in (17).

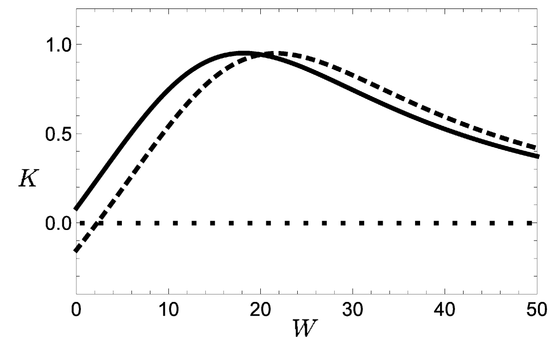


**Fig. 1.** $K$ as function of the Weber number for $\epsilon = 0.1$, $\beta = \pi/16$ with and without the angle factors $f_i$.

Let $h = 1 + \epsilon^2 s(x,t)$, $\alpha = \epsilon$, $R = 5\cot\beta/4 + \epsilon^2\tilde{R}_2$, and search for solutions $s(x,t) = \tilde{\eta}(z, \epsilon^2\tilde{\tau})$ with $z = x - A(1)t$. We then expand keeping terms up to order $\epsilon^4$ and taking into account that the Weber number is of order one. In this process we see that linear terms of order $\alpha^3$ and $\alpha^4$ should be included. Since they are neglected in (17), we expect to find a difference in the linear coefficients. It is found that

$$\tilde{\eta}_{\tilde{\tau}} + A'(1)\tilde{\eta}\tilde{\eta}_z + E(1)\tilde{\eta}_{zzz} + \epsilon\left[\frac{8}{15}R_2\tilde{\eta}_{zz} + \frac{2}{3}W\tilde{\eta}_{zzzz} + B'(1)(\tilde{\eta}\tilde{\eta}_z)_z\right] + \epsilon^2\left[\frac{1}{2}\tilde{\eta}_z\tilde{\eta}^2 A''(1) + (\tilde{\eta}\tilde{\eta}_{zz})_z E'(1) + \frac{40}{63}R_c W\tilde{\eta}_{zzzzz} + (\tilde{\eta}_z^2)_z D(1)\right] = 0. \quad (18)$$

In obtaining this expression we used the fact that $B(1) = 8(R - R_c)/15 = 8\epsilon^2 R_2/15$, $C(1) = 2\epsilon^2 W/3$ and $F(1) = 40\epsilon^2 R_c W/63$. One verifies that, after introducing the scalings $\tau = 2\tilde{\tau}/3$, $R_2 = 2\tilde{R}_2/3$, $W = \tilde{W}\tan\beta$ in (16) to account for the different choice of dimensionless variables, the two Eqs. (16) and (18) coincide except in the linear terms. Eq. (18) does not contain the functions $f_1(\beta)$, $f_2(\beta)$ which appear in the coefficients accompanying the linear terms involving fourth and fifth order derivatives. The reason for the discrepancy is as explained above. The additional angle dependent coefficients $f_1(\beta)$, $f_2(\beta)$ become important when analyzing the bifurcation from the basic state for small Weber number. In effect, the nature of the bifurcation from the steady state $\eta = 0$ changes from subcritical to supercritical when these factors are included. Letting $\eta = \delta(\tilde{\eta}_0 + \delta\eta_1 + \cdots)$ and expanding in the small parameter $\delta$ and solving to the third order, we find that the modulus of the amplitude for the bifurcating branch is given by

$$\frac{d|A|}{dt} = |A| - K|A|^3$$

where the coefficient $K$ is positive for all values of the Weber number when $f_1(\beta)$, $f_2(\beta)$ are included. In Fig. 1 the solid line depicts the bifurcation coefficient $K$ including $f_1$ and $f_2$, the dashed line without including them. As can be seen from the graph, at low values of the Weber number the nature of the bifurcation changes from subcritical to supercritical when the angle dependent functions $f_i$ are included.

## 4. Finite amplitude instability in a small box

The evolution equation (16) contains nonlinearities which are not present in the KS–KdV equation, in particular the quadratic term $(\eta^2)_{xx}$ is known to be destabilizing. We will show that in a sufficiently small box, where KdV–KS solutions die away, the higher order equation derived has finite amplitude instability, and the rate of growth of the instability will be determined. In particular we show that solutions cannot blow up in finite time.

Let us rewrite the equation as

$$\eta_t + \lambda_2\eta_{xx} + \lambda_3\eta_{xxx} + \lambda_4\eta_{xxxx} + \lambda_5\eta_{xxxxx} + \beta_1(\eta^2)_x + \beta_2(\eta^2)_{xx} + \beta_3(\eta^3)_x + \beta_4(\eta_x^2)_x + \beta_5(\eta\eta_{xx})_x = 0 \tag{19}$$

and consider periodic solutions in a box of size $L$. One can show that

$$\frac{dI}{dt} = \lambda_2\int_0^L \eta_x^2\,dx - \lambda_4\int_0^L \eta_{xx}^2\,dx + \beta_2\int_0^L \eta\eta_x^2\,dx + \omega\int_0^L \eta_x^3\,dx, \tag{20}$$

where $\omega = \beta_4 - \beta_5/2$ and

$$I(t) = \frac{1}{2}\int_0^L \eta^2(x,t)\,dx. \tag{21}$$

For the KdV–KS equation only the first two terms on the right side are present and it follows that all solutions die away for [14] $L \leqslant 2\pi\sqrt{\lambda_4/\lambda_2}$. In what follows we will show that

$$\lim_{t\to+\infty} I(t)e^{-2\gamma\beta_2 t} = 0, \quad \text{for } 0 < L < 2\pi\sqrt{\frac{\lambda_4}{\lambda_2}} \tag{22}$$

for any arbitrary $\gamma > 0$. In effect, since

$$\eta\eta_x^2 \leqslant \frac{\gamma}{2}\eta^2 + \frac{1}{2\gamma}\eta_x^4$$

for any $\gamma > 0$, it follows from Eq. (20) that

$$e^{2\gamma\beta_2 t}\frac{d}{dt}\left(Ie^{-2\gamma\beta_2 t}\right) \leqslant \lambda_2\int_0^L \eta_x^2\,dx - \lambda_4\int_0^L \eta_{xx}^2\,dx + \frac{\beta_2}{\gamma}\int_0^L \eta_x^4\,dx + \omega\int_0^L \eta_x^3\,dx. \tag{23}$$

We will prove that the right side is negative for $L < 2\pi\sqrt{\lambda_4/\lambda_2}$ using an auxiliary variational principle.

Define the functional $J[v]$ as

$$J[v] = \int_0^L \left(\lambda_4 v_x^2 - \lambda_2 v^2 - \frac{\beta_2}{\gamma}v^4 - \omega v^3\right)dx \tag{24}$$

and search for an extremum among solutions of zero average,

$$\int_0^L v\,dx = 0.$$

$J[v]$ has an extremum for $v = \hat{v}$ which satisfies the Euler–Lagrange equation

$$2\lambda_4\hat{v}_{xx} + 2\lambda_2\hat{v} + 4\frac{\beta_2}{\gamma}v^3 + 3\omega\hat{v}^2 - \mu = 0, \tag{25}$$

where $\mu$ is a Lagrange multiplier to be determined by enforcing the constraint. The extremum is given then by

$$J[\hat{v}] = \int_0^L \left(\frac{\beta_2}{\gamma}\hat{v}^4 + \frac{\omega}{2}\hat{v}^3\right)dx.$$

Notice that the Euler–Lagrange is the equation of motion of a particle of mass $2\lambda_4$ moving in the potential $V(v) = \lambda_2 v^2 + (\beta_2/\gamma)v^4 + \omega v^3 - \mu v$. The problem can be solved numerically without difficulty, we find $J[\hat{v}] \geqslant 0$. The period $L$ is a decreasing function of energy and $J[\hat{v}]$ increases with energy. The minimum of $J$ is attained when the energy tends to zero at which value $L \to 2\pi\sqrt{\lambda_2/\lambda_4}$, and $J_{\min} \to 0$. In summary,

$$J[\hat{v}] \geqslant 0 \quad \text{for } 0 \leqslant L \leqslant 2\pi\sqrt{\frac{\lambda_2}{\lambda_4}}.$$

The original inequality (23) can be written as

$$e^{2\gamma\beta_2 t}\frac{d}{dt}\left(Ie^{-2\gamma\beta_2 t}\right) \leqslant -J[\eta_x] \leqslant 0 \quad \text{for } 0 \leqslant L \leqslant 2\pi\sqrt{\frac{\lambda_2}{\lambda_4}} \tag{26}$$

which implies (22), therefore, in a small box solutions do not blow up in finite time.

## 5. Analytic solutions

In this section we will study the evolution equation and construct analytic traveling wave solutions for special values of the parameters. To our knowledge there is only one work where (16) has been considered. In [7] a search was made for nonlinear ordinary differential equations with exact solutions that can be expressed in terms of elliptic Weierstrass equations. There it is found that the most general fourth order ODE with second degree singularities that has this property is the equation that describes traveling wave solutions of (16).

We search for the traveling wave solutions $\eta(\xi,\tau) = \eta(z)$ with $z = \xi - C_0\tau$ so that (19) becomes an ordinary differential equation which can be integrated once to yield

$$-C_0\eta + \lambda_2\eta' + \lambda_3\eta'' + \lambda_4\eta''' + \lambda_5\eta'''' + \beta_1\eta^2 + 2\beta_2\eta\eta' + \beta_3\eta^3 + \beta_4(\eta')^2 + \beta_5\eta\eta'' = -C_1 \tag{27}$$

where $C_1$ is an integration constant and primes denote derivatives with respect to $z$. Following [15] we search for solutions expressible in terms of solutions

$$Y(z) = \frac{a}{2} + \frac{1}{2}\sqrt{a^2+4b}\tanh\left(\frac{1}{2}\sqrt{a^2+4b}\,z + C_2\right)$$

of Ricatti's equation $Y_z = -Y^2 + aY + b$. Introducing the Ansatz

$$\eta(z) = A_0 + A_1 Y(z) + A_2 Y^2(z) \tag{28}$$

in (27) one finds an algebraic system of equations

$$\sum_{j=1}^{6} W_j Y^j = 0.$$

The solution of each $W_j = 0$ determines relations among the parameters.

It is useful to notice that (28) can be rewritten as

$$\eta = p_0 + p_1\tanh(Kz) + p_2\,\mathrm{sech}^2(Kz),$$

where

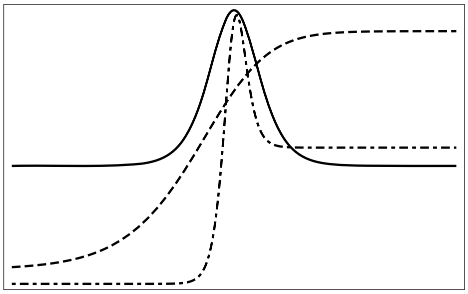

**Fig. 2.** Graph of the three analytic solutions found for different parameter values.

$$p_0 = A_0 + \frac{1}{2}aA_1 + \left(\frac{1}{2}a^2 + b\right)A_2,$$
$$p_1 = K(A_1 + aA_2),$$
$$p_2 = -K^2 A_2,$$

where $K = (1/2)\sqrt{a^2 + 4b}$.

We will choose parameters that lead to simple analytic solutions. Choose $\beta_1 = 3$, $\lambda_4 = \lambda_5 = \beta_5 = 1$, $\beta_3 = 3/40$, $\beta_4 = 0$. For this choice the system of equations $W_j = 0$ has simple solutions.

For example, for $\lambda_2 = -5$, $\lambda_3 = 85/3$, $\beta_2 = 3/20$ an exact solution is the solitary wave

$$\eta(z) = 50\,\mathrm{sech}^2\left(\sqrt{\frac{5}{2}}z\right),$$

depicted with a solid line in Fig. 2.

Choosing $\lambda_2 = 11543/128$, $\lambda_3 = 139/64$, $\beta_2 = 1/2$, an exact solution is the kink

$$\eta(z) = \frac{245}{32}\,\mathrm{sech}^2\left(\frac{7z}{16}\right) + \frac{245}{16}\tanh\left(\frac{7z}{16}\right),$$

solution depicted with a dashed line in Fig. 2.

Finally, for $\lambda_2 = 155/2$, $\lambda_3 = 169/16$, $\beta_2 = 1/2$ we find the solitary wave depicted with the dot-dashed line,

$$\eta(z) = \frac{175}{2}\tanh\left(\frac{5z}{2}\right) + 250\,\mathrm{sech}^2\left(\frac{5z}{2}\right) - 64.$$

Analytic solutions exist for different parameter values including physical values for flow along the inclined plane. It is also possible to verify that a sech$^2$ solution does not exist for arbitrary values of the parameters. We do not pursue any further the construction of analytic solutions, several methods exist to construct exact solutions, the examples show that solitary waves and fronts are among the solutions of this equation, as well as the periodic solutions found in previous work [7].

## 6. Summary

The purpose of this work was to construct a fifth order description of small amplitude dissipative solitons. A consistent asymptotic approach in the model problem chosen leads to a fifth order evolution equation which, in addition to the convective nonlinearity present in the KdV–KS equation, includes other nonlinearities present in fifth order water wave models. We show that the form of the equation can be obtained as a small amplitude limit of Lin's equation, but that higher order linear corrections must be included for small Weber numbers. The evolution equation found here is applicable not only to the present problem, it will be valid for other problems with long wave oscillatory instability in dissipative systems such as surface waves in convecting fluids [16], waves in plasmas and others. For the purpose of illustration single humped homoclinic ant heteroclinic solutions are constructed for special parameter values. In previous work [7] analytic periodic solutions were found, in that work however the equation was not derived from a physical problem but constructed as a model ordinary differential equation with exact analytic solutions in terms of Weierstrass functions. Periodic solutions in a box of length smaller than the critical length at which linear instability of the basic state $\eta = 0$ occurs may grow, but always slower than exponentially. Stability of large period solutions will be the subject of future work.

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