Solitary Waves in a Shallow Viscous Fluid Sustained by an Adverse Temperature Gradient

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We show that shallow water waves in a viscous fluid are not damped if the fluid is adequately heated from below. The critical temperature gradient needed to sustain the wave as well as its frequency are determined analytically. The nonlinear evolution of the wave is governed by the Korteweg–de Vries equation.

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It is well known that in an inviscid shallow fluid solitary waves, described by the Korteweg–de Vries equation, may propagate.[1] In any physical situation, however, viscosity will damp these waves.[2] On the other hand, and without any apparent relation to this problem, we know that if a fluid in a gravitational field is heated from below, at a critical value of the temperature gradient or equivalently of the Rayleigh number, the static state becomes unstable and convection sets in. At the critical Rayleigh number the energy gain from buoyancy exactly balances the loss due to viscous dissipation.[3] The problem we address here is whether an adverse temperature gradient may provide enough energy to compensate the viscous losses that damp solitary waves in shallow fluids, in a manner similar to what is observed in convection. We shall show that under certain thermal and mechanical boundary conditions this is in fact possible and, therefore, we may expect solitary waves to propagate in nonideal circumstances. We must mention that this is not an original idea; previous attempts, though, were unsuccessful.[4]

A previous numerical study of the linear stability theory of a fluid bounded above by a free surface and subject to a temperature gradient showed that oscillatory instabilities, which are not present when surface deflection is not allowed, exist.[5] In the case when the lower surface is stress-free and at constant temperature, and the free surface is maintained at fixed heat flux, a long-wavelength oscillatory instability occurs at a critical Rayleigh number $R_c = 30$, which is well below its value for the onset of steady convection. This long-wave instability corresponds to shallow gravity waves. The main features that permit this identification are the value of the frequency which corresponds to that of standard shallow gravity waves, and the flow pattern, which, in leading order, consists of horizontal streamlines rather than of closed loops as is usual in convective motion. In this Letter we study the nonlinear evolution of this instability and show that undamped solitary waves described by the Korteweg–de Vries equation may propagate in viscous fluids.

Let us consider a fluid bounded above by a free surface on which the normal heat flux is prescribed and below by a plane stress-free perfect thermally conducting medium. At rest the fluid lies between $z=0$ and d. Upon it acts gravity $\mathbf{g} = -g\hat{\mathbf{z}}$. In the Boussinesq approximation the equations that describe the motion of the fluid are

$$\nabla \cdot \mathbf{v} = 0\,, \tag{1}$$

$$\rho_0 \frac{d\mathbf{v}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{g}\rho\,, \tag{2}$$

$$\frac{dT}{dt} = \kappa \nabla^2 T\,, \tag{3}$$

$$\rho = \rho_0[1-\alpha(T-T_0)]\,, \tag{4}$$

where $d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$ is the convective derivative, $\mathbf{v} = (u,0,w)$ is the fluid velocity, p is the pressure, and T is the temperature. T_0 and ρ_0 are a reference temperature and density, respectively. The viscosity, μ, thermal diffusivity, κ, and coefficient of thermal expansion, α, are constant.

On the upper free surface $z = d + \eta(x,t)$ the boundary conditions are[6,7]

$$\eta_t + u\eta_x = w\,, \tag{5}$$

$$p - p_a - \frac{2\mu}{N^2}[w_z + u_x\eta_x^2 - \eta_x(u_z + w_x)] = 0\,, \tag{6}$$

$$\mu(1-\eta_x^2)(u_z + w_x) + 2\mu\eta_x(w_z - u_x) = 0\,, \tag{7}$$

and

$$\hat{\mathbf{n}}\cdot\nabla T = -F/k\,. \tag{8}$$

Subscripts x and z denote derivatives with respect to the horizontal and vertical coordinates, respectively. Here $N = (1+\eta_x^2)^{1/2}$, $\hat{\mathbf{n}} = (-\eta_x, 0, 1)/N$ is the unit normal to the free surface, F is the normal heat flux, k is the thermal conductivity, and p_a is a constant pressure exerted on the upper free surface.

We shall assume that the lower surface is stress-free and at constant temperature T_b. The boundary conditions on the lower surface $z=0$ are then

$$w = u_z = 0, \quad T = T_b\,. \tag{9}$$

The static solution to these equations is given by

$T_s = -F(z-d)/k + T_0$, $\rho_s = \rho_0[1+(\alpha F/k)(z-d)]$, and

$$p_s = p_a - g\rho_0[(z-d) + (\alpha F/2k)(z-d)^2] .$$

It is convenient to adopt d as unit of length, d^2/κ as unit of time, $\rho_0 d^3$ as unit of mass, and Fd/k as unit of temperature. Then only three dimensionless parameters are involved in the problem; the Prandtl number $\sigma = \mu/\rho_0\kappa$, the Rayleigh number $R = \rho_0 g\alpha F d^4/k\kappa\mu$, and the Galileo number $G = gd^3\rho_0^2/\mu^2$.

In order to obtain the nonlinear evolution of the perturbations to the static solution we introduce slow variables defined by

$$\xi = \epsilon(x-ct) , \quad \tau = \epsilon^3 t ,$$

and introduce the expansions

$$u = \epsilon^2(u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots) ,$$

$$w = \epsilon^3(w_0 + \epsilon w_1 + \epsilon^2 w_2 + \cdots) ,$$

$$p - p_s = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \cdots ,$$

$$T - T_s = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \cdots ,$$

$$\eta = \epsilon^2(\eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + \cdots) ,$$

and proceed to solve to each order in ϵ.

In leading order the asymptotic solution to Eqs. (1)-(9) yields

$$\theta_0 = p_0 = 0 .$$

In the next order we find

$$\theta_1 = p_1 = 0 , \quad u_0 = f(\xi,\tau) ,$$

$$w_0 = -f_\xi(\xi,\tau)z , \quad \eta_0 = f(\xi,\tau)/c ,$$

where $f(\xi,\tau)$ is an arbitrary function. In order ϵ^2 the solution is given by

$$\theta_2 = 0 , \quad p_2 = \sigma^2 G\eta_0 , \quad u_1 = g(\xi,\tau) ,$$

$$w_1 = -g_\xi(\xi,\tau)z , \quad \eta_1 = g(\xi,\tau)/c ,$$

and the solubility condition determines the speed

$$c^2 = \sigma^2 G . \tag{10}$$

Again $g(\xi,\tau)$ is an arbitrary function to be determined at higher order. That this solution corresponds to shallow gravity waves is best seen when the speed is expressed in terms of the physical parameters of the problem. Recalling that we have used d^2/κ as unit of time and d as unit of length we obtain that the dimensional speed is $C = (gd)^{1/2}$, which corresponds to the usual shallow water wave speed.[1] In the following order we find

$$\theta_3 = f_\xi(z^3 - 3z)/6 ,$$

$$p_3 = \sigma^2 G\eta_1 - 2\sigma f_\xi + \sigma R f_\xi(z^4 - 6z^2 + 5)/24 ,$$

$$u_2 = f_{\xi\xi}(z^6 - 15z^4 + 39z^2)/24 + h(\xi,\tau) ,$$

$$w_2 = -f_{\xi\xi\xi}(z^7 - 21z^5 + 91z^3)/168 - h_\xi z .$$

The solubility condition in this order determines the critical Rayleigh number $R = 30$. Here too, $h(\xi,\tau)$ is an arbitrary function. The boundary condition for the position of the free surface, Eq. (5), in this order determines a relation between the arbitrary functions $f(\xi,\tau)$, $h(\xi,\tau)$, and η_2 which is

$$ch_\xi - c^2(\eta_2)_\xi = -f_\tau - 2ff_\xi - \frac{71c}{168}f_{\xi\xi\xi} . \tag{11}$$

Finally, in order ϵ^4 we obtain

$$\theta_4 = -cf_{\xi\xi}(z^5 - 10z^3 + 25z)/120 + g_\xi(z^3 - 3z)/6 ,$$

$$p_4 = \sigma^2 G\eta_2 + \sigma g_\xi(5z^4 - 30z^2 + 17)/4 + 15\sigma f^2/c^2 - cf_{\xi\xi}(z^2-1)/2 - \sigma cf_{\xi\xi}(z^6 - 15z^4 + 75z^2 - 61)/24 .$$

The solubility condition in this order is

$$ch_\xi - c^2(\eta_2)_\xi = f_\tau + \frac{30+\sigma G}{\sigma G}ff_\xi + \frac{(272\sigma - 15)c}{168}f_{\xi\xi\xi} . \tag{12}$$

We must request that Eqs. (11) and (12) be compatible. This requirement determines the evolution equation for the function $f(\xi,\tau)$ which is

$$f_\tau + \frac{3(10+\sigma G)}{2\sigma G}ff_\xi + \sigma\sqrt{G}\left(\frac{17\sigma}{21} + \frac{1}{6}\right)f_{\xi\xi\xi} = 0 . \tag{13}$$

The evolution equation for $f(\xi,\tau)$ is the Korteweg–de Vries equation, whose properties are well established. Higher-order corrections, which do not interest us here, determine evolution equations for the arbitrary functions g and h. A particular solution for f is a single-soliton solution

$$f = A\,\mathrm{sech}^2[(\xi - U\tau)/\lambda] ,$$

where the amplitude-dependent propagation speed U and pulse width λ are given by $U = A(10+\sigma G)/2\sigma G$ and

$$\lambda = 2\sigma G^{3/4}(7+34\sigma)^{1/2}/[21A(10+\sigma G)]^{1/2} .$$

The existence of undamped solitary waves is possible due to the fact that the energy released by buoyancy balances exactly the amount of kinetic energy dissipated by viscosity. The detailed balance is best seen if we consider the rate of change of kinetic energy given by

$$\frac{\partial}{\partial t}\frac{v^2}{2} + \mathbf{v}\cdot(\mathbf{v}\cdot\nabla\mathbf{v}) = -\mathbf{v}\cdot\nabla p + \sigma\mathbf{v}\cdot\nabla^2\mathbf{v} + \sigma R\theta w . \tag{14}$$

To see how the present solution achieves this balance, we introduce the slow variables ξ and τ, and the scaling $u=\epsilon^2\hat{u}=\epsilon^2(u_0+\epsilon u_1+\epsilon^2 u_2+\cdots)$, $w=\epsilon^3\hat{w}=\epsilon^3(w_0+\epsilon w_1+\epsilon^2 w_2+\cdots)$, $p=\epsilon^2\hat{p}=\epsilon^2(p_2+\epsilon p_3+\cdots)$, and $\theta=\epsilon^3\hat{\theta}=\epsilon^3(\theta_3+\epsilon\theta_4+\cdots)$. Then Eq. (14) becomes, after dropping the hat from all variables,

$$\epsilon^5 ww_\tau+\epsilon^3 uu_\tau-\epsilon cuu_\xi+\epsilon^3 uu_\xi u+\epsilon^5 wuw_\xi+\epsilon^3 uwu_z+\epsilon^5 ww_z$$
$$=-\epsilon up_\xi-\epsilon p_z w+\sigma\epsilon^2 uu_{\xi\xi}+\sigma\epsilon^2 ww_{zz}+\sigma uu_{zz}+\sigma\epsilon^4 ww_{\xi\xi}+\sigma R\epsilon^2\theta w\,.$$

The leading orders of an expansion in ϵ of this equation are

$$u_0 u_{0zz}=0,\quad -cu_0 u_{0\xi}=-u_0 p_{2\xi}-p_{2z}w_0\,,$$
$$0=-u_1 p_{2\xi}-u_0 p_{3\xi}-p_{2z}w_1-p_{3z}w_0+\sigma u_0 u_{0\xi\xi}+\sigma w_0 w_{0zz}+\sigma u_0 u_{2zz}+\sigma u_2 u_{0zz}+\sigma R\theta_3 w_0\,.$$

The solution satisfies these conditions provided that the speed c and Rayleigh number are the critical values found as solubility conditions. This shows that the amount of energy released by buoyancy is exactly the amount dissipated by viscosity.

We have shown that a solitary wave in a viscous medium may persist if the fluid is subject to an adverse temperature gradient. Crucial to this effect are the choice of the thermal boundary conditions on the upper and lower surfaces and of the mechanical boundary condition on the lower surface. Surface tension does not modify this result, provided that surface tension inhomogeneities are sufficiently small.[8]

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