Evolution equation of surface waves in a convecting fluid

H. Aspe and M. C. Depassier
Facultad de Física, Universidad Católica de Chile, Casilla 6177, Santiago 22, Chile
(Received 3 October 1989)

We study the evolution of long shallow waves in a convecting fluid when the critical Rayleigh number slightly exceeds its critical value. The surface displacement is found to obey a perturbed Korteweg–de Vries equation that includes diffusion and instability effects.

I. INTRODUCTION

Convecting fluids whose first instability from the static state is oscillatory have received much attention recently. Such behavior is found, for example, in binary fluid convection[1] and in electrohydrodynamic convection in nematic liquid crystals.[2] A common feature of these two systems is that the transition occurs with finite critical wave number and frequency. General arguments,[3] as well as asymptotic expansion in particular cases,[4] show that the nonlinear behavior near the transition is governed by coupled Landau-Newell-type equations. Traveling waves are governed by the Ginzburg-Landau equation. Much work has been devoted to the study of the behavior of the solutions to these equations and their comparison with experimental results.[5–8] In this article we study the evolution of a system which exhibits an oscillatory instability from the static state with vanishing wave number and frequency.[9] This instability corresponds to the appearance of long surface waves.[10] We show that the nonlinear evolution of the system near the transition is governed by the perturbed Korteweg–de Vries equation

$$u_t+\lambda_1 uu_x+\lambda_2 u_{xxx}+\epsilon\left[\frac{\sigma R_2}{15}u_{xx}+\lambda_3 u_{xxxx}+\lambda_4(uu_x)_x\right]=0\ ,$$

where σ is the Prandtl number and ϵ is a small parameter such that the excess of the Rayleigh number above its critical value is given by ϵ^2R_2. The coefficients λ_i, $i=1$–4, are functions of the parameters of the problem. Subscripts denote derivatives with respect to the time t and horizontal coordinate x. This equation without the effect of instability, that is, with $R_2=0$, has been found recently to be the generic equation that describes the evolution of marginally diffusively stable wave trains.[11] A similar equation but with $\lambda_4=0$ and with instability and diffusion of the same order as dispersion arises in the study of fluid flow along an inclined plane.[12] Numerical studies of this equation have shown that in the dispersion-dominated regime, for periodic boundary conditions, regular arrangements of solitonlike pulses appear.[13] The overall evolution of the system for a sufficiently long periodicity interval is apparently governed by the interaction of these pulses.[14] In the present case, dispersion dominates over diffusion and instability; therefore similar behavior is expected. However, the presence of the nonlinearity $(uu_x)_x$ has an additional destabilizing effect, as will be shown below. If this nonlinearity predominates over the diffusion term u_{xxxx}, the leading-order Korteweg–de Vries (KdV) soliton will not be stable and we do not expect the appearance of periodic arrangements of solitonlike pulses. Numerical studies are needed to determine the nature of the solution in this case.

II. MATHEMATICAL FORMULATION

Let us consider a layer of fluid which, at rest, lies between $z=0$ and $z=d$. Upon it acts a gravitational field $\mathbf{g}=-g\hat{\mathbf{z}}$. The fluid is described by the Boussinesq equations

$$\nabla\cdot\mathbf{v}=0\ , \tag{1}$$

$$\rho_0\frac{d\mathbf{v}}{dt}=-\nabla p+\mu\nabla^2\mathbf{v}+\mathbf{g}\rho\ , \tag{2}$$

$$\frac{dT}{dt}=\kappa\nabla^2T\ , \tag{3}$$

$$\rho=\rho_0[1-\alpha(T-T_0)]\ , \tag{4}$$

where $d/dt=\partial/\partial t+\mathbf{v}\cdot\nabla$ is the convective derivative; p, T, ρ, and $\mathbf{v}$ denote the pressure, temperature, density, and fluid velocity, respectively. The quantities ρ_0 and T_0 are reference values. The fluid properties, that is, its viscosity μ, thermal diffusivity, κ, and coefficient of thermal expansion α are constant. Furthermore we restrict ourselves to two-dimensional motion so that $\mathbf{v}=(u,0,w)$.

The fluid is bounded above by a free surface on which the heat flux is fixed and upon it a constant pressure p_a is exerted. Below, it is bounded by a plane stress-free surface which is maintained at constant temperature. As the fluid moves the free surface is deformed, we shall denote its position by $z=d+\eta(x,t)$. The boundary conditions on the upper surface are[15]

$$\eta_t+u\eta_x=w\ , \tag{5}$$

$$p-p_a-\frac{2\mu}{N^2}[w_z+u_x\eta_x^2-\eta_x(u_z+w_x)]=0\ , \tag{6}$$

$$\mu(1-\eta_x^2)(u_z+w_x)+2\mu\eta_x(w_z-u_x)=0\ , \tag{7}$$

$$\hat{\mathbf{n}}\cdot\nabla T=-F/k \ , \tag{8}$$

on $z=d+\eta$. Here, subscripts denote derivatives $N=(1+\eta_x^2)^{1/2}$, $\hat{\mathbf{n}}=(-\eta_x,0,1)/N$ is the unit normal to the free surface, F is the prescribed normal heat flux, and k is the thermal conductivity.

Denoting by T_b the fixed temperature of the lower surface, the boundary conditions on the lower surface $z=0$ are

$$w=u_z=0, \quad T=T_b \ . \tag{9}$$

The static solution to these equations is given by $T_s=-F(z-d)/k+T_0$, $\rho_s=\rho_0[1+(\alpha F/k)(z-d)]$, and $p_s=p_a-g\rho_0[(z-d)+(\alpha F/2k)(z-d)^2]$. We have chosen the reference temperature T_0 as the value of the static temperature on the upper surface. The temperature on the lower surface is then $T_b=T_0+Fd/k$. Equations (1)–(9) constitute the problem to be solved. We shall adopt d as unit of length, d^2/κ as unit of time, $\rho_0 d^3$ as unit of mass, and Fd/k as unit of temperature. Then there are three dimensionless parameters involved in the problem, the Prandtl number $\sigma=\mu/\rho_0\kappa$, the Rayleigh number $R=\rho_0 g\alpha F d^4/k\kappa\mu$, and the Galileo number $G=gd^3\rho_0^2/\mu^2$.

III. ASYMPTOTIC SOLUTION

In order to obtain the nonlinear evolution of the perturbations to the static solution we introduce the following scaling: $\eta(x,t)=\epsilon^2\hat{\eta}(x,t)$, $u(x,z,t)=\epsilon^2\hat{u}(x,z,t)$, $w(x,z,t)=\epsilon^3\hat{w}(x,z,t)$, $p(x,z,t)=p_s(z)+\epsilon^2\hat{p}(x,z,t)$, and $T(x,z,t)=T_s(z)+\epsilon^3\hat{\theta}(x,z,t)$. In addition, we introduce slow space and time variables defined by

$$\xi=\epsilon(x-ct), \quad \tau=\epsilon^3 t \ .$$

The nondimensional scaled equations read, after dropping the hat from all variables,

$$u_\xi+w_z=0 \ , \tag{10}$$

$$-\epsilon c u_\xi+\epsilon^3 u_\tau+\epsilon^3 u u_\xi+\epsilon^3 w u_z = -\epsilon p_\xi+\epsilon^2\sigma u_{\xi\xi}+\sigma u_{zz} \ , \tag{11}$$

$$-\epsilon^2 c w_\xi+\epsilon^4 w_\tau+\epsilon^4 u w_\xi+\epsilon^4 w w_z = -p_z+\epsilon^3\sigma w_{\xi\xi}+\epsilon\sigma w_{zz}+\epsilon\sigma R\theta \ , \tag{12}$$

$$-\epsilon c\theta_\xi+\epsilon^3\theta_\tau+\epsilon^3 u\theta_\xi+\epsilon^3 w\theta_z-w=\theta_{zz}+\epsilon^2\theta_{\xi\xi} \ , \tag{13}$$

where subscripts denote derivatives. The boundary conditions become

$$\theta=w=u_z=0 \ , \tag{14}$$

on $z=0$, and

$$w=-c\eta_\xi+\epsilon^2\eta_\tau+\epsilon^2 u\eta_\xi \ , \tag{15}$$

$$\theta_z=\epsilon^4\eta_\xi\theta_\xi+\epsilon^{-3}[1-(1+\epsilon^6\eta_\xi^2)^{1/2}] \ , \tag{16}$$

$$p=G\sigma^2\eta+\epsilon^2\frac{\sigma R}{2}\eta^2 + \frac{2\sigma\epsilon}{N^2}(w_z-\epsilon^2\eta_\xi u_z-\epsilon^4\eta_\xi w_\xi+\epsilon^6 u_\xi\eta_\xi^2) \ , \tag{17}$$

$$u_z=-\epsilon^2 w_\xi+\epsilon^6(u_z+\epsilon^2 w_\xi)\eta_\xi^2+2\epsilon^4\eta_\xi(u_\xi-w_z) \ , \tag{18}$$

on $z=1+\epsilon^2\eta$. We then seek an asymptotic solution to Eqs. (11)–(19) of the form $u=u_0+\epsilon u_1+\epsilon^2 u_2+\cdots$, $w=w_0+\epsilon w_1+\epsilon^2 w_2+\cdots$, $\theta=\theta_0+\epsilon\theta_1+\epsilon^2\theta_2+\cdots$, $p=p_0+\epsilon p_1+\epsilon^2 p_2+\cdots$, $\eta=\eta_0+\epsilon\eta_1+\epsilon^2\eta_2+\cdots$. The Rayleigh number R is slightly above its critical value, so we let $R=R_c+\epsilon^2 R_2$. The wave speed c and R_c are the eigenvalues of the problem which will be found from the solubility conditions. The equations to be solved at each order and their solution are given in the Appendix. Here we shall only quote the results.

The horizontal velocity $u(\xi,\tau)$ is, in the first two leading orders, independent of z; its solution is given by $u(\xi,\tau)=f(\xi,\tau)+\epsilon g(\xi,\tau)$, where f and g are arbitrary functions whose evolution equation we seek. The solubility conditions at order ϵ and ϵ^2 determine the critical speed $c^2=\sigma^2 G$ and the critical Rayleigh number $R_c=30$, respectively. In the following orders, the solubility conditions yield evolution equations for the functions f and g; they are

$$f_\tau+\lambda_1 f f_\xi+\lambda_2 f_{\xi\xi\xi}=0 \ , \tag{19}$$

$$g_\tau+\lambda_1(fg)_\xi+\lambda_2 g_{\xi\xi\xi}+\frac{\sigma R_2}{15}f_{\xi\xi} +\lambda_3 f_{\xi\xi\xi\xi}+\lambda_4(ff_\xi)_\xi=0 \ , \tag{20}$$

where the coefficients are given by

$$\lambda_1=\frac{3}{2\sigma G}(10+\sigma G), \quad \lambda_2=\frac{\sigma\sqrt{G}}{2}\left[\frac{1}{3}+\frac{34\sigma}{21}\right] \ ,$$

$$\lambda_3=\sigma\left[\frac{682\sigma^2 G+717}{2079}\right], \quad \lambda_4=\frac{8}{\sqrt{G}} \ .$$

An improved approximate equation for the horizontal velocity $u=f+\epsilon g$ may now be obtained by recombining Eqs. (19) and (20) above:

$$(f+\epsilon g)_\tau+\lambda_1[ff_\xi+\epsilon(fg)_\xi]+\lambda_2(f+\epsilon g)_{\xi\xi\xi} +\epsilon\left[\frac{\sigma R_2}{15}f_{\xi\xi}+\lambda_3 f_{\xi\xi\xi\xi}+\lambda_4(ff_\xi)_\xi\right]=0 \ .$$

Then, the equation for u, correct to order ϵ, is

$$u_\tau+\lambda_1 u u_\xi+\lambda_2 u_{\xi\xi\xi} +\epsilon\left[\frac{\sigma R_2}{15}u_{\xi\xi}+\lambda_3 u_{\xi\xi\xi\xi}+\lambda_4(uu_\xi)_\xi\right]=0 \ . \tag{21}$$

As mentioned in the Introduction, this equation with $\lambda_4=0$ has been extensively studied. In the dispersion-dominated regime, which is the case here, for periodic boundary conditions, regular arrays of solitonlike pulses appear for all initial conditions. The additional nonlinearity $(uu_\xi)_\xi$ has a destabilizing effect; this can be readily seen by means of an asymptotic expansion around the solution of the leading-order KdV equation.[16] Following the standard procedure, we let $u=u_0+\epsilon u_1+\cdots$, with u_0 the solution of the unperturbed KdV equation,

$$u_0 = N\,\mathrm{sech}^2\left[\left[\frac{\lambda_1 N}{12\lambda_2}\right]^{1/2}\left[\xi - \frac{\lambda_1 N}{3}\tau\right]\right],$$

where the amplitude N is allowed to depend on a slow time $T=\epsilon\tau$. The slow time dependence of the amplitude is governed by

$$\frac{dN}{dT} = \frac{\lambda_1 N^2}{315\lambda_2^2}\left[\frac{7\sigma\lambda_2 R_2}{5} - \lambda_1(5\lambda_3 - 12\lambda_4)N\right].$$

The case $\lambda_4=0$ yields a limiting value for the amplitude which agrees well with numerical results;[13] in the present case the coefficient $5\lambda_3-12\lambda_4$ is positive for values of the Prandtl and Galileo numbers within the limits of validity of the Boussinesq approximation; therefore we expect the appearance of solitonlike pulses with an amplitude $N_c = 7\sigma R_2 \lambda_2/[5\lambda_1(5\lambda_3-12\lambda_4)]$.

IV. SUMMARY

We have studied the nonlinear evolution of shallow surface waves in a convecting fluid. The first instability from the static state for the system studied is oscillatory; however, the general arguments that predict the weakly nonlinear evolution of such systems are not valid in this case, as the transition occurs at zero wave number and with vanishing frequency. In order to determine the nonlinear behavior near the transition we have performed an asymptotic expansion. The evolution of the surface displacement is governed by a perturbed Korteweg–de Vries equation. The excess of the Rayleigh number above its critical value as well as nonlinear terms have a destabilizing effect which is balanced by diffusion. Based on previous results on similar equations, the appearance of periodic solitonlike structures with the amplitude fixed by the Rayleigh number is expected. Numerical solutions of the evolution equation (21) would be desirable to determine the nature of the solutions for the case of diffusion weaker than the nonlinear destabilizing effects; we have not addressed this problem here, for within the parameter range of interest in this problem this situation does not occur.

APPENDIX: SOLUTION OF THE ASYMPTOTIC EXPANSION

In this appendix the equations to be solved at each order and their solution are given. Terms that vanish have been omitted.

In leading order the system to be solved is

$$w_{0z} = -u_{0\xi}, \quad u_{0zz}=0, \quad p_{0z}=0, \quad \theta_{0zz} = -w_0,$$

subject to

$$w_0(0)=u_{0z}(0)=\theta_0(0)=0, \quad u_{0z}(1)=\theta_{0z}(1)=0,$$

$$c\eta_{0\xi} = -w_0(1), \quad p_0(1)=\sigma^2 G\eta_0.$$

The solution is given by

$$u_0 = f(\xi,\tau), \quad \eta_0 = f/c, \quad w_0 = -f_\xi z,$$

$$\theta_0 = f_\xi T_0(z), \quad p_0 = \sigma^2 G \eta_0,$$

where we have defined $T_0(z)=(z^3-3z)/6$.

At order ϵ the system to be solved is

$$w_{1z} = -u_{1\xi}, \quad u_{1zz} = \frac{1}{\sigma}(p_{0\xi} - cu_{0\xi}),$$

$$p_{1z} = \sigma R_c \theta_0, \quad \theta_{1zz} = -c\theta_{0\xi} - w_1,$$

subject to

$$w_1(0)=u_{1z}(0)=\theta_1(0)=0, \quad u_{1z}(1)=\theta_{1z}(1)=0,$$

$$c\eta_{1\xi} = -w_1(1), \quad p_1(1) = \sigma^2 G\eta_1 + 2\sigma w_{0z}(1).$$

The solubility condition $u_{1z}(1)=\int_0^1 u_{1zz}dz=0$ determines the critical speed $c^2=\sigma^2 G$. The solution is given by

$$u_1 = g(\xi,\tau), \quad \eta_1 = g/c, \quad w_1 = -g_\xi z,$$

$$\theta_1 = -cf_{\xi\xi}T_1(z) + g_\xi T_0(z),$$

$$p_1 = \sigma^2 G\eta_1 - 2\sigma f_\xi + \sigma R_c f_\xi P_1(z),$$

where $T_1(z)=(z^5-10z^3+25z)/120$ and $P_1(z)=dT_1/dz = (z^4-6z^2+5)/24$. In the following order we have

$$w_{2z} = -u_{2\xi}, \quad u_{2zz} = \frac{1}{\sigma}(p_{1\xi} - cu_{1\xi}) - u_{0\xi\xi},$$

$$p_{2z} = cw_{0\xi} + \sigma R_c\theta_1, \quad \theta_{2zz} = -\theta_{0\xi\xi} - c\theta_{1\xi} - w_2,$$

subject to

$$w_2(0)=u_{2z}(0)=\theta_2(0)=0, \quad u_{2z}(1) = -w_{0\xi}(1),$$

$$\theta_{2z}(1) = -\eta_0\theta_{0zz}(1),$$

$$c\eta_{2\xi} = \eta_{0\tau} + u_0\eta_{0\xi} - w_2(1) - \eta_0 w_{0z}(1),$$

$$p_2(1) = \sigma^2 G\eta_2 + \frac{\sigma R_c}{2}\eta_0^2 + 2\sigma w_{1z}(1).$$

At this order, the solubility condition $u_{2z}(1)=\int_0^1 u_{2zz}dz$ yields the critical Rayleigh number $R_c=30$. Making use of this value, the solution may be written as

$$u_2 = f_{\xi\xi}U(z) + h(\xi,\tau), \quad w_2(z) = -h_\xi z - f_{\xi\xi\xi}W(z),$$

$$p_2 = p_2(1) + \sigma R_c g_\xi P_1(z) - \frac{c}{2}f_{\xi\xi}(z^2-1) - \sigma c R_c f_{\xi\xi}P_2(z),$$

$$\theta_2 = -\frac{1}{c}ff_\xi z + h_\xi T_0(z) - cg_{\xi\xi}T_1(z) + c^2 f_{\xi\xi\xi}T_2(z) + f_{\xi\xi\xi}T_3(z),$$

where

$$W(z) = \frac{1}{168}(z^7 - 21z^5 + 91z^3),$$

$$U(z) = \frac{dW}{dz} = \frac{1}{24}(z^6 - 15z^4 + 39z^2),$$

$$T_2(z) = \frac{1}{7!}(z^7 - 21z^5 + 175z^3 - 427z),$$

$$T_3(z) = \frac{6}{9!}(5z^9 - 180z^7 + 1134z^5 + 5040z^3 - 19\,575z),$$

$$P_2(z) = \frac{dT_2}{dz} = \frac{1}{720}(z^6 - 15z^4 + 75z^2 - 61).$$

At order ϵ^3 only the equations for u_3, p_3, and η_3 are needed. They are

$$u_{3zz}=\frac{1}{\sigma}(p_{2\xi}-cu_{2\xi}+u_{0\tau}+u_0u_{0\xi})-u_{1\xi\xi}\ ,$$

$$p_{3z}=cw_{1\xi}+\sigma w_{0\xi\xi}+\sigma w_{2zz}+\sigma R_c\theta_2+\sigma R_2\theta_0\ ,$$

subject to

$$u_{3z}(0)=0,\quad u_{3z}(1)=-w_{1\xi}(1)\ ,$$

$$c\eta_3=\eta_{1\tau}+u_0\eta_{1\xi}+u_1\eta_{0\xi}-w_3(1)-\eta_0w_{1z}(1)-\eta_1w_{0z}(1)\ ,$$

$$p_3(1)=-\eta_0p_{1z}(1)+\sigma^2G\eta_3+\sigma R_c\eta_0\eta_1+2\sigma w_{2z}(1)\ .$$

This time the solubility condition $u_{3z}(1)=\int_0^1u_{3zz}dz$ determines the evolution equation (19) for f. The solubility condition at order ϵ^4 will determine the evolution equation (20) for g. To obtain it only the form of p_3 is needed. We write it as

$$p_3=p_3(1)+\int_1^z(cw_{1\xi}+\sigma w_{0\xi\xi}+\sigma w_{2zz}+\sigma R_c\theta_2+\sigma R_2\theta_0)dz\ .$$

Finally at order ϵ^4 only the solubility condition is needed. The equation and boundary conditions for u_4 are

$$u_{4zz}=\frac{1}{\sigma}(p_{3\xi}-cu_{3\xi}+u_{1\tau}+u_0u_{1\xi}+u_1u_{0\xi})-u_{2\xi\xi}\ ,$$

subject to

$$u_{4z}(0)=0\ ,$$

$$u_{4z}(1)=-\eta_0u_{2zz}(1)-w_{2\xi}(1)-\eta_0w_{0\xi z}(1)+2\eta_{0\xi}u_{0\xi}(1)-2\eta_{0\xi}w_{0z}(1)\ .$$

The solubility condition $u_{4z}(1)=\int_0^1u_{4zz}dz$ yields Eq. (20).

ACKNOWLEDGMENTS

We thank R. Benguria for helpful discussions. This work was partially supported by Fondo Nacional de Ciencia y Tecnología (FONDECYT).

[1]B. J. A. Zielinska and H. R. Brand, Phys. Rev. A **35**, 4349 (1987).
[2]I. Rehberg, S. Rasenat, and V. Steinberg, Phys. Rev. Lett. **62**, 756 (1989).
[3]P. Coullet, S. Fauve, and E. Tirapegui, J. Phys. (Paris) Lett. **46**, 787 (1985).
[4]C. S. Bretherton and E. A. Spiegel, Phys. Lett. **96A**, 152 (1983).
[5]M. C. Cross, Phys. Rev. A **38**, 3593 (1988).
[6]R. Heinrichs, G. Ahlers, and D. S. Cannell, Phys. Rev. A **35**, 2761 (1987).
[7]E. Moses and V. Steinberg, Phys. Rev. A **34**, 693 (1986).
[8]H. T. Moon, P. Huerre, and L. G. Redekopp, Physica **7D**, 135 (1983).
[9]R. D. Benguria and M. C. Depassier, Phys. Fluids **30**, 1678 (1987).
[10]C. M. Alfaro and M. C. Depassier, Phys. Rev. Lett. **62**, 2597 (1989).
[11]A. J. Bernoff, Physica **D30**, 363 (1988).
[12]J. Topper and T. Kawahara, J. Phys. Soc. Jpn. **44**, 663 (1978).
[13]T. Kawahara, Phys. Rev. Lett. **51**, 381 (1983).
[14]T. Kawahara and S. Toh, Phys. Fluids **31**, 2103 (1988).
[15]S. H. Davis and G. M. Homsy, J. Fluid Mech. **98**, 527 (1980).
[16]E. Ott and R. N. Sudan, Phys. Fluids **12**, 2388 (1969).