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Oscillatory instabilities in the Rayleigh–Bénard problem with a free surface

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The linear stability theory of the Rayleigh–Bénard problem when the fluid is bounded above by a free deformable surface is studied numerically. When the free surface is thermally insulated and the lower surface is isothermic the marginal curve for the onset of oscillatory convection lies below the marginal curve for the exchange of stabilities.

I. INTRODUCTION

Many studies of convection assume that the boundaries of the fluid are free and plane. It can then be shown that the principle of exchange of stabilities is valid, that is, convection appears as a stationary instability.[1] In physical situations, however, the free surface is deformed due to the fluid motion, the monotonicity principle is no longer valid, and convection may appear as an oscillatory instability.

Previous work dealing with the effect of surface deformation has concentrated on thermocapillary convection[2,3] and in cases when buoyancy effects have been included, the exchange of stabilities has been assumed without proof.[4–6] The linear stability of buoyancy driven convection in a single fluid with a free deformable surface with respect to time-independent perturbations has been studied numerically.[7] In the absence of surface tension, the effect of surface deformability is measured by the Galileo number G. When the heat flux is fixed on the boundaries of the fluid, surface deformability has no effect on the linear stability theory with respect to time-independent perturbations.[8] When the temperature is fixed on the boundaries, for large values of the Galileo number, surface deflection is unimportant. For small values of the Galileo number, it is destabilizing, an effect which is stronger for the longest wavelengths.[6] Recent work has dealt with the two-layer Bénard problem. In a series of articles the linear stability of two nearly identical superimposed fluids has been considered. This system exhibits an oscillatory instability when the Rayleigh number is very close to the critical number for the one-fluid problem,[9,10] but no such instability is found when the Rayleigh number is below the critical value.[11] However, since complete marginal curves are not calculated, such instability must not be ruled out. Also related to the problem we study here is the new instability found by Yih[12] which occurs when two similar superimposed fluids flow down an inclined plane. When the upper fluid is less thermally conductive a long wavelength oscillatory instability is present.

In this article, we study the linear stability of a two-dimensional layer of fluid of infinite horizontal extent bounded above by a free deformable surface upon which a constant pressure is exerted. The lower surface of the fluid is plane. We have calculated numerically the marginal curves for oscillatory instability for different thermal boundary conditions. We find that when the temperature is fixed on the lower boundary and the heat flux is fixed on the free upper surface, the oscillatory instability occurs at values of the Rayleigh number lower than the critical value obtained when the upper surface is plane. We must remark that the instability found here is a new one and is not a limiting case of that found previously.[9–11] In Refs. 9–11 the instability appears for nearly identical superimposed fluids and for values of the Rayleigh number near criticality,[9] whereas we find such an instability for a fluid bounded above by a passive gas which does not interact with the fluid. The instability we have found occurs at values of the Rayleigh number significantly lower than the critical value for the exchange of stabilities and is due exclusively to the surface deformation. It is worthwhile mentioning, however, that the thermal boundary condition for which we find oscillatory instabilities at low Rayleigh numbers corresponds to an upper thermally insulating medium, which is a limiting case of the thermal boundary condition producing oscillatory instabilities in the system studied in Ref. 12.

II. MATHEMATICAL FORMULATION

Let us consider a two-dimensional layer of fluid of infinite horizontal extent which, at rest, lies between $z = 0$ and $-d$. A constant gravitational field $g = -g\hat{z}$ acts upon it. The fluid is described by the Boussinesq equations

$$\nabla\cdot\mathbf{v} = 0, \tag{1}$$

$$\rho_0\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{g}\rho, \tag{2}$$

$$\frac{\partial T}{\partial t} + (\mathbf{v}\cdot\nabla)T = \kappa\nabla^2 T, \tag{3}$$

$$\rho = \rho_0[1 - \alpha(T - T_0)], \tag{4}$$

where $\mathbf{v} = (u,w)$ is the fluid velocity, p is the pressure, and T is the temperature. The density ρ is considered a constant ρ_0 except in the external force term.

The viscosity μ, thermal diffusivity κ, and the coefficient of thermal expansion α are constant. Here T_0 is a reference temperature which we choose as the static temperature on the upper surface. The static solution to Eqs. (1)–(4) is

$$T_s = Az + T_0, \tag{5}$$

$$\rho_s = \rho_0(1 - \alpha A z), \tag{6}$$

$$p_s = p_a - g\rho_0(z - \alpha A z^2/2), \tag{7}$$

where p_a is a constant pressure exerted on the upper surface and A is a constant to be determined from the thermal boundary conditions. The boundary conditions on the lower surface $z = -d$ are

$$\mathbf{v} = 0 \tag{8a}$$

if it is a rigid surface, or

$$u_x = w_z = 0 \tag{8b}$$

if it is free but plane. Subscripts x and z denote derivatives with respect to the horizontal and vertical coordinates, respectively. The thermal boundary conditions are

$$T = T_b, \quad \text{a constant} \tag{9a}$$

if the fluid is bounded below by a thermally conducting medium, or

$$T_z = -F/k \tag{9b}$$

if it is bounded by a thermally insulating medium. Here F is the normal heat flux and k is the thermal conductivity. On the upper free surface $z = \eta(x,t)$ the boundary conditions are[1]

$$\eta_t + u\eta_x = w, \tag{10a}$$

$$p - p_a = 2\mu[\eta_x^2 u_x - \eta_x(u_z + w_x) + w_z](1 + \eta_x^2)^{-1}, \tag{10b}$$

$$(1 - \eta_x^2)(u_z + w_x) + 2\eta_x(w_z - u_x) = 0. \tag{10c}$$

We have not included surface tension since our main interest is to clarify the validity of slip boundary conditions. As will be seen later, the instabilities found are important for very long wavelengths, so surface tension has a small effect on them. Here, too, we have two types of thermal boundary conditions,

$$T = T_u, \tag{11a}$$

or

$$\hat{n}\cdot\nabla T = -F/kN, \tag{11b}$$

where $N = \sqrt{1 + \eta_x^2}$ and $\hat{n} = (-\eta_x, 1)/N$ is the normal unit vector to the free-surface. Thus according to the thermal boundary conditions we have the following cases.

Case (i):

$$T(-d) = T_b, \quad T(\eta) = T_u, \quad A = (T_u - T_b)/d.$$

Case (ii):

$$T_z(-d) = -F/k, \quad \mathbf{n}\cdot\nabla T(\eta) = -F/kN,$$
$$A = -F/k.$$

Case (iii):

$$T(-d) = T_b \mathbf{n}\cdot\nabla T(\eta) = -F/kN, \quad A = -F/k.$$

Case (iv):

$$T_z(-d) = -F/k, \quad T(\eta) = T_u, \quad A = -F/k.$$

We have solved cases (i)–(iv) numerically, each of them for rigid and slip boundary conditions on the lower surface.

Let us now linearize the equations around the static solution. Adopting d as the unit of length, d^2/κ as the unit of time, $\rho_0 d^3$ as the unit of mass, and $(-Ad)$ as the unit of temperature, the linear equations for the perturbations p', ρ', and T' to the static pressure, density, and temperature, respectively, are

$$\nabla\cdot\mathbf{v} = 0, \tag{12}$$

$$\mathbf{v}_t = -\nabla p' + \sigma\nabla^2\mathbf{v} - G\sigma^2\rho'\hat{z}, \tag{13}$$

$$T'_t - w = \nabla^2 T', \tag{14}$$

$$\rho' = (R/\sigma G)T', \tag{15}$$

subject to

$$u(-1) = w(-1) = 0 \tag{16a}$$

or

$$u_x(-1) = w_z(-1) = 0, \tag{16b}$$

and

$$T'(-1) = 0 \tag{17a}$$

or

$$T'_z(-1) = 0 \tag{17b}$$

on the lower surface; and

$$\eta_t = w(0), \tag{18}$$

$$u_z(0) + w_x(0) = 0, \tag{19}$$

$$-p'(0) + G\sigma^2\eta + 2\sigma w_z(0) = 0, \tag{20}$$

and

$$T'_z(0) = 0 \tag{21a}$$

or

$$T'(0) = \eta. \tag{21b}$$

The dimensionless numbers that appear are the Rayleigh number $R = (-A)d^4 g\alpha/\kappa\nu$, the Prandtl number $\sigma = \nu/\kappa$, and the Galileo number $G = gd^3/\nu^2$. Equations (12)–(21) completely describe the problem.

III. LINEAR STABILITY

Assuming that all perturbations evolve in the horizontal variable and time as $e^{iax}e^{\lambda t}$, the linear equations for the perturbations reduce to

$$(D^2 - a^2 - \lambda)\theta(z) = ia\psi(z), \tag{22}$$

$$(D^2 - a^2)(D^2 - a^2 - \lambda/\sigma)\psi(z) = iaR\theta(z), \tag{23}$$

where $\theta(z)$ is the amplitude of the temperature perturbation and $\psi(z)$ is the amplitude of a streamfunction ψ in terms of which the velocity is $v = (\psi_z, 0, -\psi_x)$. Here D denotes a derivative with respect to z. The boundary conditions for the amplitudes of the normal modes are $\psi(-1) = D\psi(-1)$ if the bottom is rigid, or $\psi(-1) = D^2\psi(-1) = 0$ if it is free. On the upper surface, they reduce to

$$\lambda\eta + ia\psi(0) = 0, \tag{24}$$

$$(D^2 + a^2)\psi(0) = 0, \tag{25}$$

$$\lambda D^3\psi(0) - \lambda(3a^2 + \lambda/\sigma)D\psi(0) - a^2\sigma G\psi(0) = 0. \tag{26}$$

The four cases for the thermal boundary conditions are as follows.

Case (i):

$$\theta(-1) = 0, \quad \theta(0) = -ia\psi(0)/\lambda.$$

Case (ii):

$$D\theta(-1)=0, \quad D\theta(0)=0.$$

Case (iii):

$$\theta(-1)=0, \quad D\theta(0)=0.$$

Case (iv):

$$D\theta(-1)=0, \quad \theta(0)=-ia\psi(0)/\lambda.$$

The equations are linear, and therefore the solution for ψ is of the form

$$\psi=\sum_{i=1}^{3}\{A_i \sinh[\alpha_i(z+1)]+B_i \cosh[\alpha_i(z+1)]\},$$

where the α_i's are the three different roots (neglecting changes of sign) of $(\alpha^2-a^2)(\alpha^2-a^2-\lambda/\sigma) \times(\alpha^2-a^2-\lambda)+a^2R=0$. The solution is obtained by requiring ψ to satisfy the boundary conditions. The eigenvalues R and λ are found as the roots of a 6×6 determinant which may be reduced to simple three-dimensional vector products. Details are given in the Appendix. We have solved this determinant for the four cases of thermal boundary conditions mentioned above, each of them for rigid and free bottom surfaces. The numerical method was checked against the known curves for the exchange of stabilities.

IV. NUMERICAL RESULTS

We have first examined the time-independent linear stability theory. The effect of surface deformation appears as a modification to the thermal boundary condition on the upper surface. Such an effect is present only when the temperature is fixed on the upper boundary. In agreement with previous results,[8] we find that surface deformation has a destabilizing effect, that is, the critical Rayleigh number when the Galileo number G is finite is lower than its value when G becomes infinite, in which case there is no surface deformation. In general, for fixed wavenumber, the marginal Rayleigh number has a lower value when G is finite. To conclude, for time-independent stability surface deformation has no effect in cases (ii) and (iii) and is destabilizing in cases (i) and (iv). Now, we describe the results for oscillatory instability. We find that when the temperature is fixed on the upper boundary [cases (i) and (iv)] there is no oscillatory instability for values of R, σ, and G within reasonable ranges. This is true for both rigid and stress-free boundary conditions on the lower surface. When the heat flux is fixed on both surfaces [case (ii)] there is an oscillatory instability, but it occurs at values of the Rayleigh number much higher than those for marginal stability and therefore will never be observed. When the temperature is fixed on the lower surface and the heat flux is fixed on the upper free surface [case (iii)] we find that there is an oscillatory instability which, depending on the mechanical boundary condition on the lower surface, occurs at values of R lower than the critical value for the exchange of stabilities. When the lower surface is rigid (Fig. 1) the marginal curves for overstability have

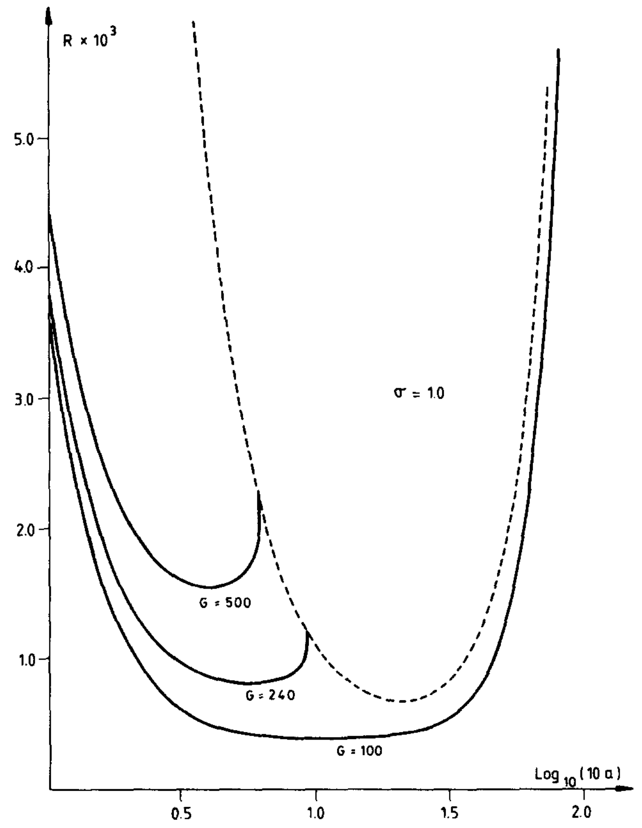

FIG. 1. Marginal curves for overstability for different values of G. The dashed line corresponds to the exchange of stabilities ($R_c=669$; $a_c=2.09$ is the critical point on the dashed line). The lower surface is rigid and isothermic and the upper surface is free and insulated.

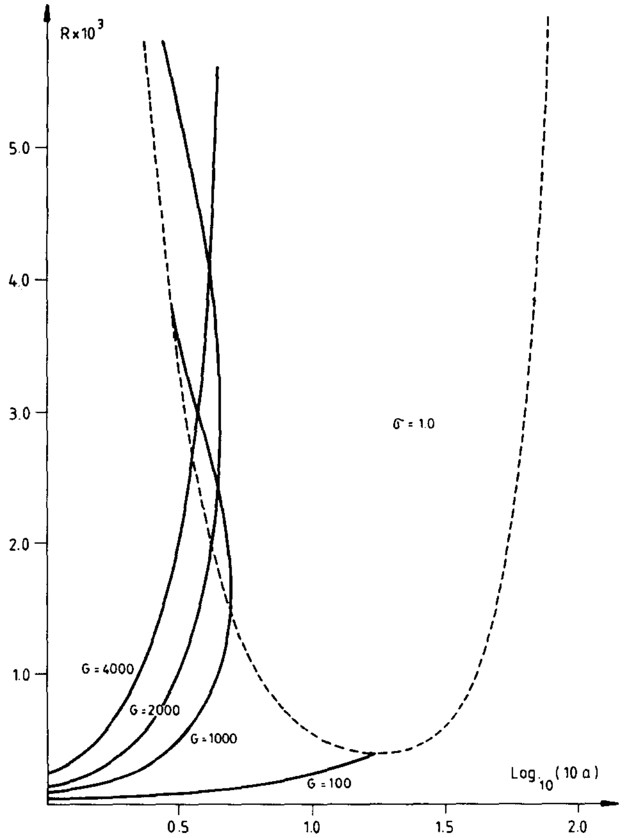

FIG. 2. Marginal curves for overstability for different values of G. The dashed line corresponds to the exchange of stabilities ($R_c=384.7$; $a_c=1.76$ is the critical point on the dashed line). The lower surface is free and isothermic and the upper surface is free and thermally insulated.

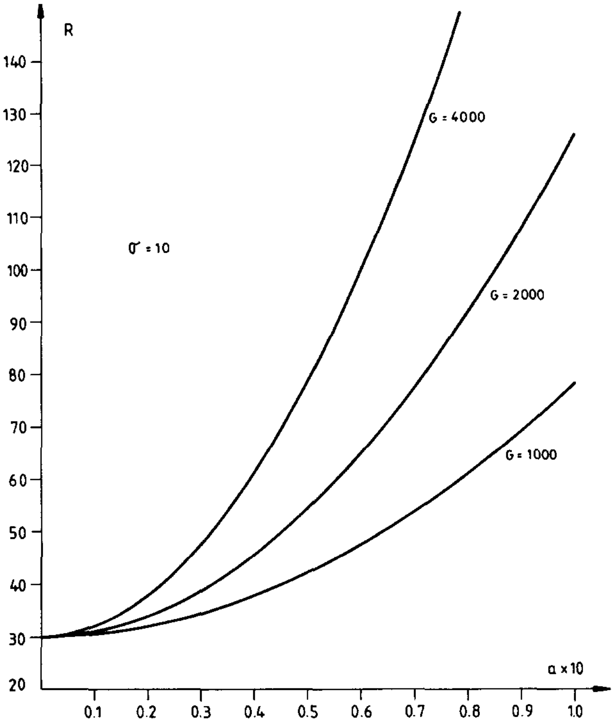

FIG. 3. Small wavenumber region of the graph shown in Fig. 2.

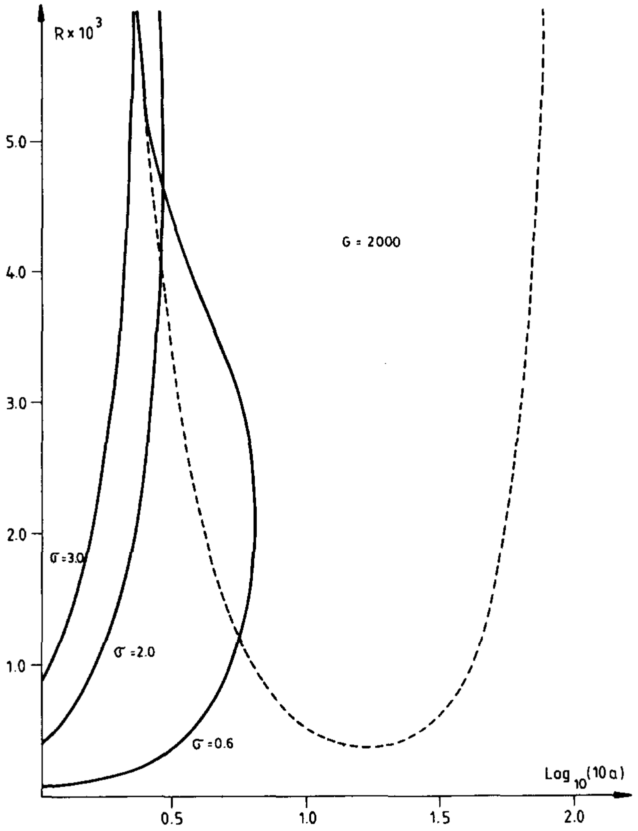

FIG. 4. For the same boundary conditions as in Fig. 2, marginal curves for overstability for different values of the Prandtl number.

their minimum at a finite value of the wavenumber and merge with the time-independent marginal curve. The critical Rayleigh number for overstability is an increasing function of G. The behavior of R_c as a function of G is, in this case, similar to that found in a recent experiment.[13] For small values of G, the oscillatory instability occurs at lower values of R than the time-independent instability. As G increases, the situation reverses. We must recall, however, that we have assumed that the fluid is described by the Boussinesq approximation which is valid when density variations across the layer are small. From Eq. (15) we see that this corresponds to $R \ll \sigma G$. Therefore, within the range of validity of the Boussinesq approximation, overstability occurs at values of R larger than the critical value for the onset of steady convection. With the same thermal boundary condition, when the lower surface obeys the idealized stress-free boundary condition, the wavenumber that first becomes un-

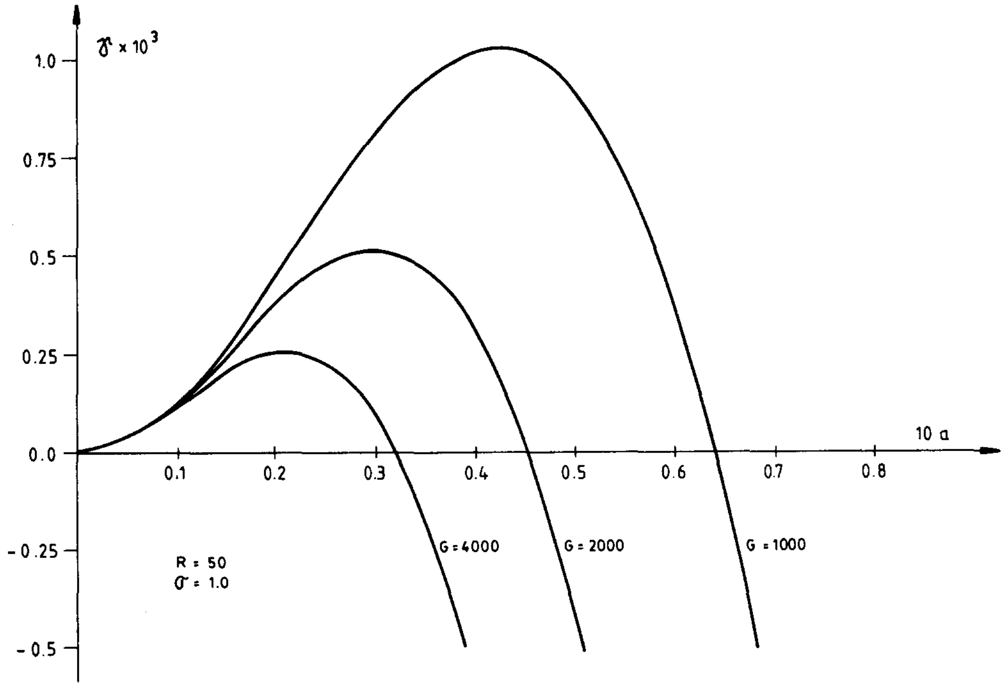

FIG. 5. Growth rates for the overstable curves shown in Fig. 3.

stable is $a=0$, and the critical Rayleigh number is $R_c = 30$ for all values of σ and G. The value R_c is lower than the critical value for the onset of steady convection. In Fig. 2, curves for the onset of overstability are shown for different values of G. They all merge with the marginal curve (dashed line). The value of R at which they merge increases with G. In Fig. 3 we show, for fixed σ, the curves for different values of G; they all converge to $R_c = 30$. As mentioned above, this is true for all values of σ. In Fig. 4, the marginal curves for overstability are shown for fixed G and different values of σ. We can see that they merge with the marginal curve for the exchange of stabilities at a value of R that increases with σ. Although not shown in Fig. 4, they all tend to $R_c = 30$ as a goes to zero. Growth rates for the oscillatory instability are shown in Fig. 5. The maximum growth rate occurs at a finite value of a, a value which depends on G.

V. SUMMARY AND CONCLUSION

We have solved the linear stability theory of a Boussinesq fluid with an upper free surface. The linear equations were solved numerically for different thermal boundary conditions and for rigid and slip conditions on the lower surface. Our purpose was to examine the validity of idealized stress-free boundary. We find that oscillatory instabilities are present, but only when the temperature is fixed on the lower surface and the heat flux is fixed on the upper free surface does the marginal curve for overstability lie below the marginal curve for the onset of steady convection. In addition, we must constrain our results to values of $R \ll \sigma G$, a restriction imposed by the Boussinesq approximation, in which case only when the lower surface is free does overstability occur before steady convection. We have not included surface tension since our main interest was to study the effect of the nonideal stress-free boundary condition alone; for the more interesting case we do not expect significant alterations because of the fact that the longest wavelengths are the most unstable ones.

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APPENDIX: FORMULAS FOR THE DETERMINANTS

The 6×6 determinants of Sec. III can be reduced to simple three-dimensional vector products. Let us define the quantities

$$\begin{aligned} D_1 &= [unf_1][\alpha gr] + [snf_1][\alpha hg] + [vnf_1][\alpha rh],\\ D_2 &= [nsu][\alpha f_2 g] + [nuv][\alpha f_2 r] + [nvs][\alpha f_2 h],\\ D_3 &= [f_3 rh][vn\gamma] + [f_3 hg][sn\gamma] + [f_3 gr][un\gamma],\\ D_4 &= [ghr], \end{aligned} \tag{A1}$$

where $[abc]$ denotes the box product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ for vectors in $\mathbb{R}^3$. Here n denotes the vector $(1,1,1)$, $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, and $\gamma = (\alpha_1^2,\alpha_2^2,\alpha_3^2)$, where α_1, α_2, and α_3 are the three different roots of $(\alpha^2 - a^2)(\alpha^2 - a^2 - \lambda/\sigma)(\alpha^2 - a^2 - \lambda) + a^2 R = 0$. The functions $f_1, f_2, f_3, g, h, u, v, r$, and s are understood as the vectors $f_1 = [f_1(\alpha_1), f_1(\alpha_2), f_1(\alpha_3)]$, etc; their explicit form is given by

$$\begin{aligned} f_1(x) &= (x^2 - a^2 - \lambda)^{-1},\\ f_2(x) &= x(x^2 - a^2)(x^2 - a^2 - \lambda/\sigma),\\ f_3(x) &= x/(x^2 - a^2 - \lambda),\\ g(x) &= (a^2 + x^2)\sinh(x),\\ h(x) &= \lambda x(x^2 - 3a^2 - \lambda/\sigma)\cosh(x) - Ga^2\sigma\sinh(x),\\ u(x) &= \lambda x(x^2 - 3a^2 - \lambda/\sigma)\sinh(x) - Ga^2\sigma\cosh(x),\\ v(x) &= (a^2 + x^2)\cosh(x). \end{aligned} \tag{A2}$$

As for the functions $s(x)$ and $r(x)$, they may adopt two different values depending on the boundary condition:

$$s_1(x) = \cosh(x)/(x^2 - a^2 - \lambda)^2(x^2 - a^2 - \lambda/\sigma) \tag{A3}$$

or

$$s_2(x) = x(x^2 - a^2)(x^2 - a^2 - \lambda/\sigma)\sinh(x), \tag{A4}$$

and

$$r_1(x) = \sinh(x)/(x^2 - a^2 - \lambda)^2(x^2 - a^2 - \lambda/\sigma) \tag{A5}$$

or

$$r_2(x) = x(x^2 - a^2)(x^2 - a^2 - \lambda/\sigma)\cosh(x). \tag{A6}$$

Then, according to the thermal boundary conditions and the mechanical boundary condition on the lower surface we must solve

	Case (i)	Case (ii)	Case (iii)	Case (iv)
rigid lower surface	$D_1 = 0$	$D_2 = 0$	$D_1 = 0$	$D_2 = 0$
free lower surface	$D_3 = 0$	$D_4 = 0$	$D_3 = 0$	$D_4 = 0$,

where $s = s_1$ and $r = r_1$ for cases (i) and (iv) and $s = s_2$ and $r = r_2$ for cases (ii) and (iii).

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