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Citation: Physics of Fluids A: Fluid Dynamics (1989-1993) **1**, 1123 (1989); doi: 10.1063/1.857336
View online: http://dx.doi.org/10.1063/1.857336
View Table of Contents: http://scitation.aip.org/content/aip/journal/pofa/1/7?ver=pdfcov
Published by the AIP Publishing

On the linear stability theory of Bénard–Marangoni convection

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(Received 28 July 1988; accepted 27 February 1989)

The linear stability of a fluid bounded above by a free deformable surface is studied numerically. When the heat flux is fixed on the free surface and the lower surface is plane and isothermic, oscillatory instabilities, which may occur at lower values of the Rayleigh number than the critical value for the onset of steady convection, are found.

I. INTRODUCTION

In recent work,[1–5] the convective stability of fluids bounded by a free deformable surface has been studied, in search of new instabilities. The most remarkable effect of allowing a more realistic boundary condition, as opposed to the simpler ideal stress-free boundaries, is that new oscillatory instabilities have been found. These instabilities are due exclusively to the correct free surface boundary condition and do not occur if the free surface is assumed to be plane. Previous work has dealt with the Bénard problem of two nearly identical superimposed fluids,[1–3] with surface tension driven convection,[4] which we shall refer to as Marangoni convection, and with the usual Rayleigh–Bénard problem.[5] In this paper we shall study the existence of oscillatory instabilities in the combined Bénard–Marangoni problem for a Boussinesq fluid bounded above by a deformable free surface.

The linear problem of Bénard–Marangoni convection without the inclusion of surface deflection was studied initially by Nield[6]; the nonlinear problem was treated by Cloot and Lebon.[7] The effect of the surface deflection was first considered by Scriven and Sterling,[8] only for the case of surface tension driven convection. Davis and Homsy[9] later studied the effect of surface deflection on the combined Bénard–Marangoni problem and concluded that surface deformation stabilizes buoyancy driven convection and destabilizes surface tension driven convection. They did not attempt to look for oscillatory instabilities. Their study was extended by Castillo and Velarde,[10] who generalized their results to two fluid layers. Experimental[11] and numerical[12] studies have been performed in order to relate the sign of surface deflection to the main source of convection.

In this article we study numerically the linear stability theory and show that there are oscillatory instabilities that may occur at values of the Rayleigh number lower than the critical value for the onset of steady convection. Oscillatory instabilities are found when the fluid is heated from below and for a positive Marangoni number. Their occurrence is more common than was to be expected from the analysis of pure Marangoni convection in a semi-infinite layer of fluid, where oscillatory instability is predicted only for negative values of the Marangoni number.[4]

II. MATHEMATICAL FORMULATION

Let us consider a layer of fluid, which, at rest, lies between $z=0$ and $z=d$. Upon it acts a gravitational field $\mathbf{g} = -g\hat{z}$. The fluid is described by the Boussinesq equations

$$\nabla\cdot\mathbf{v} = 0, \tag{1}$$

$$\rho_0 \frac{d\mathbf{v}}{dt} = -\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{g}\rho, \tag{2}$$

$$\frac{dT}{dt} = \kappa\nabla^2 T, \tag{3}$$

$$\rho = \rho_0[1-\alpha(T-T_0)], \tag{4}$$

where $d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla$ is the convective derivative; p, T, ρ, and $\mathbf{v}$ denote the pressure, temperature, density, and fluid velocity, respectively. The quantities ρ_0 and T_0 are reference values. The fluid properties, that is, its viscosity μ, thermal diffusivity κ, and coefficient of thermal expansion α, are constant. Furthermore, we restrict ourselves to two-dimensional motion, so that $\mathbf{v} = (u,0,w)$.

The fluid is bounded above by a free surface in contact with a passive gas, which exerts upon it a constant pressure p_a, and below, by a plane surface. As motion sets in, the free surface is deformed; we shall write its position as $z = d + \eta(x,t)$. The boundary conditions on the upper surface are[7,9]

$$\eta_t + u\eta_x = w, \tag{5}$$

$$p - p_a - (2\mu/N^2)[w_z + u_x\eta_x^2 - \eta_x(u_z + w_x)] = -\tau(\eta_{xx}/N^3), \tag{6}$$

$$\mu(1-\eta_x^2)(u_z + w_x) + 2\mu\eta_x(w_z - u_x) = N(\tau_x + \eta_x\tau_z), \tag{7}$$

on $z = d + \eta$. Here subscripts denote derivatives $N = \sqrt{1+\eta_x^2}$ and τ is the surface tension, for which we adopt the simple linear form

$$\tau = \tau_0 - \gamma(T - T_0), \tag{8}$$

where τ_0 is a constant reference value and γ is its rate of change with temperature. For normal fluids, γ is positive. We shall assume that the fluid is bounded below by a perfect thermal conductor and above by a perfect thermal insulator. The thermal boundary conditions are then $T =$ const on

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$z=0$ and $\hat{n}\cdot\nabla T = -F/k$ on $z=1+\eta(x,t)$. Here F is the normal heat flux and k the thermal conductivity. The unit vector $\hat{n} = (-n_x,0,1)/N$ is normal to the free surface.

Next the equations are linearized around the static solution. We shall adopt d as unit of length, d^2/κ as the unit of time, $\rho_0 d^3$ as the unit of mass, and $\Delta T = Fd/k$ as the unit of temperature. Assuming that all perturbations evolve in time as $e^{\lambda t}$ and in the horizontal variable as e^{iax}, the equations for the perturbations reduce to

$$(D^2-a^2)(D^2-a^2-\lambda/\sigma)\psi(z) = iaR\theta(z)\,, \qquad (9)$$

$$(D^2-a^2-\lambda)\theta(z) = ia\psi(z), \qquad (10)$$

where $\theta(z)$ is the amplitude of the temperature perturbation and $\psi(z)$ is the amplitude of the streamfunction $\psi(x,z,t)$ in terms of which the velocity is given by $\mathbf{v} = (\psi_z,0,-\psi_x)$. Here D denotes the derivative with respect to the vertical variable z. The dimensionless numbers that have appeared are the Prandtl number $\sigma = \mu/\rho_0\kappa$ and the Rayleigh number $R = \rho_0 g\alpha d^3\Delta T/\kappa\mu$. The linearized boundary conditions are, on $z=1$,

$$\lambda(D^2-3a^2-\lambda/\sigma)D\psi - a^2(\sigma G + a^2/C)\psi = 0, \qquad (11)$$

$$(D^2+a^2)\psi - [R\Gamma/(\sigma G + a^2/C)](D^2-3a^2-\lambda/\sigma)D\psi + iaR\Gamma\theta = 0\,, \qquad (12)$$

$$D\theta = 0. \qquad (13)$$

The dimensionless numbers that have appeared are the Capillary number $C = \mu\kappa/\tau_0 d$, $\Gamma = \gamma/\rho_0 g\alpha d^2$, and the Galileo number $G = gd^3\rho_0^2/\mu^2$. The Marangoni number is given by $M = \Gamma R$; we have chosen to use Γ as an independent parameter instead of M.

The boundary conditions on $z=0$ are

$$\psi = D\psi = \theta = 0, \qquad (14)$$

if it is rigid and

$$\psi = D^2\psi = \theta = 0, \qquad (15)$$

if it is stress-free.

Equations (9) and (10), subject to the boundary conditions (11)–(15), constitute the problem to be solved.

III. NUMERICAL RESULTS

The problem of finding the eigenvalues R and λ of Eqs. (9) and (10), with the boundary conditions (11)–(15), can be reduced to finding the roots of a 6×6 determinant, which may be simplified to yield a combination of three-dimensional vector products. Details are given in the Appendix.

The eigenvalues λ and R depend on a, σ, Γ, and on the group $(\sigma G + a^2/C)$. This term measures the degree of surface deformation. For infinite surface tension $C=0$, the surface cannot be deformed; from Eqs. (11) and (12), we see that in this limit the boundary conditions reduce to the usual stress-free case. The same occurs when G goes to infinity. For numerical purposes, what is important is the group mentioned above. For simplicity we have set $1/C = 0$ and $\sigma = 1$; the effect of surface deformation will be measured only by the value of G. We shall restrict ourselves to positive values of Γ that correspond to normal fluids.

Let us first recall the main features of the marginal curve for the onset of steady convection. The critical Rayleigh number is a decreasing function of Γ, both when the upper surface is flat[6] or deformable.[9] In the case when it is allowed to be deformed, two regimes may be distinguished. When convection is mainly buoyancy driven, that is, for small Γ, surface deformation has a stabilizing effect, in the opposite case of surface tension driven convection (large Γ), surface deformation is destabilizing.[9] As a first step and to check our numerical procedure, we reobtained the marginal curves for the onset of steady convection and found good agreement with known results.

Now let us describe the results for the search of oscillatory instabilities. When the bottom surface is rigid, overstability was found for sufficiently large surface deformation and for large values of Γ (Figs. 1 and 2). In the parameter range explored the critical Rayleigh number for the onset of oscillations is higher than the corresponding value for the onset of steady convection. The critical Rayleigh number for overstability decreases with Γ, which indicates that the driving mechanism is surface tension. It is possible to decrease arbitrarily the Rayleigh number by decreasing G, but this violates the condition for the validity of the Boussinesq approximation $R < \sigma G$. To sum up, when the bottom surface is rigid, steady convection occurs at a lower temperature gradient than oscillatory convection. When the bottom surface is stress-free, oscillatory instability is found for both small and large values of Γ. As in the previous case of a rigid bottom, an oscillatory instability with the same qualitative features described above is present (Figs. 3 and 4). Here too, if one remains within the range of validity of the Boussinesq approximation, the critical Rayleigh number for the onset of oscillations is higher that the corresponding value for the onset of steady convection. When convection is mainly buoyancy driven, that is, for small values of Γ, an additional oscillatory instability is found that has zero critical wavenumber. The critical Rayleigh number for this instability depends only on Γ, and for sufficiently low values of it is lower than the critical value for the onset of steady convection (Fig. 5). Surface tension is stabilizing (Fig. 6), which shows that the driving mechanism for this instability is buoyancy alone, with a result expected, since it is found even when $\Gamma = 0$.[5] The critical Rayleigh number for overstability

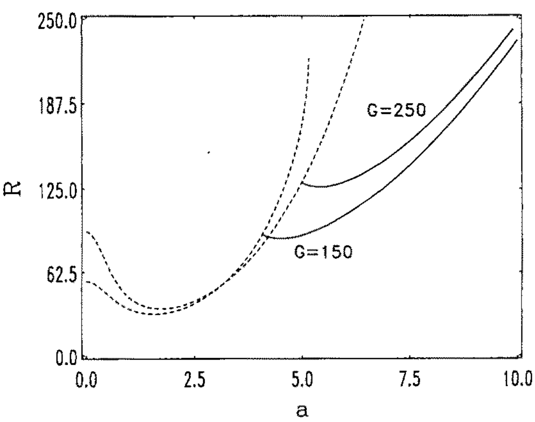

FIG. 1. Marginal curves, for overstability, for $\Gamma = 1.8$ and different values of G. The lower surface is rigid. The dashed lines in all figures correspond to the exchange of stabilities.

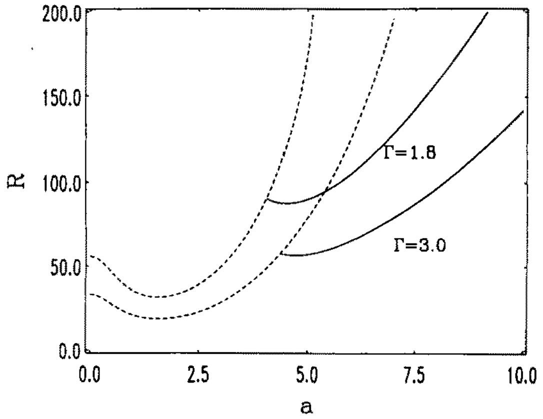

FIG. 2. Marginal curves, for overstability, for $G = 150$. The lower surface is rigid.

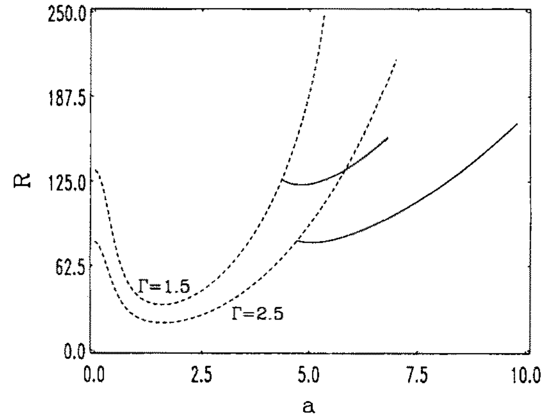

FIG. 4. Marginal curves, for overstability, for $G = 200$. The lower surface is free.

is lower than its value for steady convection for values of Γ up to about 0.3, the precise value depends on G, since the critical Rayleigh number for steady convection depends on G. Here, the critical value of R, found numerically, is in agreement with what the asymptotic calculation, which we give next, yields.

IV. LONG WAVELENGTH ASYMPTOTIC SOLUTION

A long wavelength asymptotic solution to Eqs. (9)–(13) can be obtained performing an expansion in the modulus of the wavenumber, $|a|$, which we shall consider a small parameter. For simplicity we shall write $|a| = \epsilon$, where ϵ is a small positive parameter. It is convenient to eliminate the temperature variable $\theta(z)$, in which case the problem reduces to the solution of a sixth-order differential equation for $\psi(z)$ with two eigenvalues. The eigenvalues will be determined as functions of the parameters Γ, σ, G, and C, which we take to be of order 1. Next, let us introduce the expansions

$$\psi(z) = \psi_0(z) + \epsilon\psi_1(z) + \epsilon^2\psi_2(z) + \cdots,$$
$$R = R_0 + \epsilon^2 R_1 + \epsilon^4 R_2 + \cdots,$$
$$\lambda = \epsilon(\lambda_0 + \epsilon^2\lambda_1 + \epsilon^4\lambda_2 + \cdots),$$

and solve the equations to each order in ϵ.

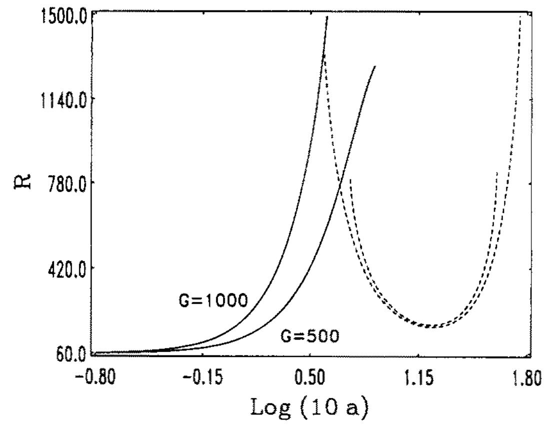

FIG. 5. The marginal curve, for the long wavelength oscillatory instability, for $\Gamma = 0.2$. The lower surface is free.

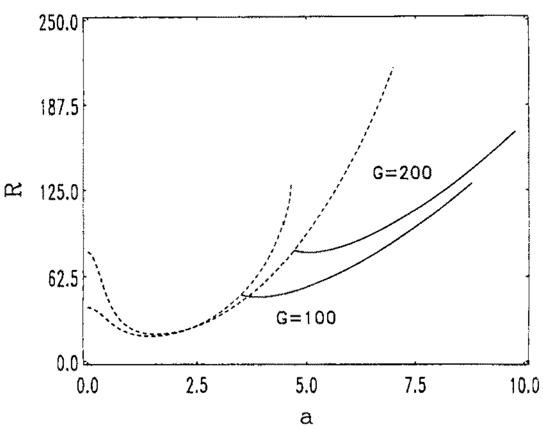

FIG. 3. Marginal curves, for overstability, for $\Gamma = 2.5$. The lower surface is free.

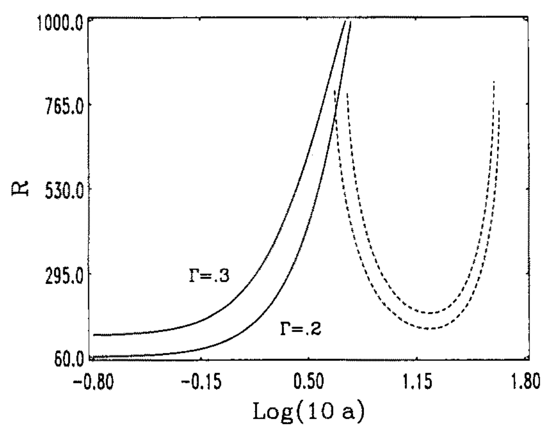

FIG. 6. The marginal stability curve for $G = 500$. The critical value, when $\Gamma = 0.2$, is 60 and, when $\Gamma = 0.3$, it is 120. The lower surface is free.

In leading order, we find that ψ_0 is the solution of

$$D^6\psi_0 = 0, \tag{16}$$

subject to

$$D^2\psi_0 + \Gamma D^4\psi_0 = 0, \tag{17a}$$

$$D^3\psi_0 = 0, \tag{17b}$$

$$D^5\psi_0 = 0, \tag{17c}$$

on $z = 1$ and $\psi_0 = D^2\psi_0 = D^4\psi_0 = 0$ on $z = 0$. The solution is $\psi_0 = Az$, where A is an arbitrary constant. In the next order we find that

$$D^6\psi_1 = 0 \tag{18}$$

and the boundary conditions on $z = 1$ are now

$$D^2\psi_1 + \Gamma D^4\psi_1 - (\Gamma R_0/\lambda_0)\psi_0 - (\lambda_0\Gamma/\sigma)D^2\psi_0 = 0, \tag{19a}$$

$$\lambda_0 D^3\psi_1 - (\lambda_0^2/\sigma)D\psi_0 - \sigma G\psi_0 = 0, \tag{19b}$$

$$D^5\psi_1 = 0, \tag{19c}$$

and, on $z = 0$, $\psi_1 = D^2\psi_1 = D^4\psi_1 = 0$. This system has solution only if $\lambda_0 = \pm i\omega_0$, where

$$\omega_0 = \sqrt{\sigma^2 G - \sigma\Gamma R_0}\,, \tag{20}$$

and the solution for ψ_1 is $\psi_1(z) = (A\Gamma R_0/6\lambda_0)z^3$. In the next order, we must solve

$$D^6\psi_2 = -R_0\psi_0, \tag{21}$$

subject to

$$D^2\psi_2 + \Gamma D^4\psi_2 - (\Gamma R_0/\lambda_0)\psi_1 - (\lambda_0\Gamma/\sigma)D^2\psi_1 = 0\,, \tag{22a}$$

$$\lambda_0 D^3\psi_2 - 3\lambda_0 D\psi_0 - (\lambda_0^2/\sigma)D\psi_1 - \sigma G\psi_1 = 0, \tag{22b}$$

$$D^5\psi_2 - (\lambda_0/\sigma)D^3\psi_1 = 0, \tag{22c}$$

on $z = 1$, and $\psi_2 = D^2\psi_2 = D^4\psi_2 = 0$ on $z = 0$. The solvability condition in this order determines the critical value of the Rayleigh number, which is

$$R_0 = 30/(1 - \tfrac{5}{2}\Gamma). \tag{23}$$

In the next two orders, corrections to the critical frequency and Rayleigh number are found; we do not give them as they have no qualitative effect. The frequency of oscillation $\omega = \sqrt{\sigma(\sigma G - \Gamma R_0)}$ must be real; this implies a constraint on the admissable values of R; for normal fluids this is $R < \bar{R}$, where $\bar{R} = \sigma G/\Gamma$. If we demand $R_0 < \bar{R}$, we find that this instability is present in fluids with a value of $\Gamma \leqslant \Gamma_{\max} = \sigma G/(30 + 5\sigma G/2)$ and R_0 will always be positive. The asymptotic and numerical results given above are in good agreement. We recall that when $\Gamma = 0$, the critical Rayleigh number for the onset of steady convection is 384.7,[13] and its value decreases continuously as Γ increases. Here, when $\Gamma = 0$, the critical value for the onset of oscillations is $R_0 = 30$ and it increases continuously with Γ. Thus, for sufficiently low values of Γ, the oscillatory instability occurs at values of the Rayleigh number lower than the critical value for the exchange of stabilities.

In order to obtain a clear picture of the fluid motion, consider the eigenfunctions. The streamfunction ψ is given in leading order by $\psi(x,z,t) = Aze^{iax}e^{\lambda t}$ with $\lambda = \pm i\omega$. The simplest solution consists of standing waves that are constructed by superposition;

$$\psi = \tfrac{1}{4}Az(e^{i(ax-\omega t)} + e^{i(ax+\omega t)} + \text{c.c.})$$
$$= Az\cos ax\cos\omega t\,.$$

The velocity of the fluid is then given by $\mathbf{v} = A\cos\omega t(\cos ax, 0, az\sin ax)$. The surface displacement obtained from the linearized boundary condition given in Eq. (5) is in leading order,

$$\eta(x,t) = (A/\omega)\sin ax\sin\omega t\,.$$

At any instant the horizontal velocity in each cell is independent of z, its direction oscillates with time. Therefore the motion of the fluid does not consist of closed loops, but rather of up and down oscillations. The surface is deformed following the motion of the fluid. This shows that the long wave instability corresponds to gravity waves. The frequency, when $\Gamma = 0$, in terms of dimensional quantities, is given by $k\sqrt{gd}$, where k is the wavenumber; this corresponds to the usual shallow water wave frequency.

V. SUMMARY AND CONCLUSIONS

We have studied numerically the linear stability theory of a layer of fluid bounded above by a free deformable surface, on which the normal heat flux is fixed, and below by a plane surface, on which the temperature is constant. Our main purpose was to determine the effect of surface deformation on the linear stability of normal fluids, that is, fluids for which surface tension is a decreasing function of temperature. We found oscillatory instabilities that do not occur when the upper surface is free but undeformable. Two types of instabilities occur; the first one corresponds to a finite wavenumber instability, which is mainly driven by surface tension. This instability was not predicted by an analytic treatment of surface tension driven convection in a semi-infinite fluid in which oscillatory instability was found only for values of $\Gamma < 0$ (see Ref. 4); we attribute this to the fact that the thermal boundary condition in the lower surface is an important factor for the existence of oscillatory instabilities in the presence of a free surface[5] and its effect is not taken into account, assuming a semi-infinite layer of fluid. The second instability is found only when the bottom surface is free but plane, the critical wavenumber for this instability is zero and the critical Rayleigh number depends only on Γ. This instability, which may be found at lower values of R than steady convection, is driven by buoyancy alone; an asymptotic analysis permits us to identify this mode with gravity waves; surface tension has a stabilizing action on it. Finally, we conclude that the use of the full boundary conditions leads to new instabilities, which do not exist when the idealized stress-free boundary conditions are employed.

ACKNOWLEDGMENTS

This work was financed by FONDECYT, DIUC Universidad Católica, and DIB Universidad de Chile.

APPENDIX: FORMULAS FOR THE DETERMINANTS

The solution to Eqs. (9) and (10), subject to boundary conditions (11)–(15), is obtained as follows. The temperature variable is eliminated to yield an equation for ψ,

$$(D^2-a^2-\lambda)(D^2-a^2-\lambda/\sigma)(D^2-a^2)\psi=0, \tag{A1}$$

subject to the boundary conditions

$$(D^2-a^2)(D^2-a^2-\lambda/\sigma)D\psi=0, \tag{A2}$$

$$\lambda(D^2-3a^2-\lambda/\sigma)D\psi-a^2(\sigma G+a^2/C)\psi=0, \tag{A3}$$

and

$$(D^2+a^2)\psi-[\Gamma R/(\sigma G+a^2/C)](D^2-3a^2-\lambda/\sigma)D\psi + \Gamma(D^2-a^2)(D^2-a^2-\lambda/\sigma)\psi=0, \tag{A4}$$

on $z=1$. On the bottom surface, at $z=0$, the boundary conditions are

$$(D^2-a^2)(D^2-a^2-\lambda/\sigma)\psi=0, \tag{A5}$$

and either

$$\psi=D\psi=0, \tag{A6}$$

for a rigid surface or

$$\psi=D^2\psi=0, \tag{A7}$$

for a free plane surface.

The solution for ψ may be written as

$$\psi=\sum_{i=1}^{3}[A_i\sinh(\alpha_i z)+B_i\cosh(\alpha_i z)],$$

where the α_i's are the three different roots of $(\alpha^2-a^2-\lambda)(\alpha^2-a^2-\lambda/\sigma)(\alpha^2-a^2)+Ra^2=0$. The six boundary conditions for ψ provide six homogeneous equations to determine the coefficients A_i and B_i. The system has a nontrivial solution if the determinant of the coefficients vanish; this is the characteristic equation from which the eigenvalues λ and R are obtained.

If the bottom surface is free, the determinant can be written as

$$D_f=[grh],$$

where $[grh]$ denotes the box product $\mathbf{g}\cdot(\mathbf{r}\times\mathbf{h})$. Here $\mathbf{g}=[g(\alpha_1),g(\alpha_2),g(\alpha_3)]$; the same notation is used for other vectors. If the lower bottom is rigid, then the determinant may be reduced to

$$D_r=[pn\gamma][\alpha hr]+[sn\gamma][\alpha gh]+[l\,n\,\gamma][\alpha rg],$$

where the following definitions have been made:

$$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3),\quad \mathbf{n}=(1,1,1),$$

$$\gamma(\alpha)=\alpha^2(\alpha^2-2a^2-\lambda/\sigma),$$

$$g(\alpha)=[(\alpha^2+a^2)+\Gamma(\alpha^2-a^2)(\alpha^2-a^2-\lambda/\sigma)]\times\sinh(\alpha)-\Gamma R\alpha[(\alpha^2-3a^2-\lambda/\sigma)/(\sigma G+a^2/C)]\cosh(\alpha),$$

$$h(\alpha)=\lambda(\alpha^2-3a^2-\lambda/\sigma)\alpha\cosh(\alpha)-a^2(\sigma G+a^2/C)\sinh(\alpha),$$

$$l(\alpha)=\lambda(\alpha^2-3a^2-\lambda/\sigma)\alpha\sinh(\alpha)-a^2(\sigma G+a^2/C)\cosh(\alpha),$$

$$p(\alpha)=[(\alpha^2+a^2)+\Gamma(\alpha^2-a^2)(\alpha^2-a^2-\lambda/\sigma)]\times\cosh(\alpha)-\Gamma R\alpha[(\alpha^2-3a^2-\lambda/\sigma)/(\sigma G+a^2/C)]\sinh(\alpha),$$

$$r(\alpha)=\alpha(\alpha^2-a^2)(\alpha^2-a^2-\lambda/\sigma)\cosh(\alpha),$$

$$s(\alpha)=\alpha(\alpha^2-a^2)(\alpha^2-a^2-\lambda/\sigma)\sinh(\alpha).$$

The eigenvalues λ and R are obtained by either solving $D_f=0$ or $D_r=0$.

[1] Y. Renardy and D. D. Joseph, Phys. Fluids **28**, 788 (1985).
[2] Y. Renardy and M. Renardy, Phys. Fluids **28**, 2699 (1985).
[3] Y. Renardy, Phys. Fluids **29**, 1769 (1986).
[4] P. L. García-Ybarra and M. G. Velarde, Phys. Fluids **30**, 1649 (1987).
[5] R. D. Benguria and M. C. Depassier, Phys. Fluids **30**, 1678 (1987).
[6] D. A. Nield, J. Fluid Mech. **19**, 341 (1964).
[7] A. Cloot and G. Lebon, J. Fluid Mech. **145**, 447 (1984).
[8] L. E. Scriven and C. V. Sterling, J. Fluid Mech. **19**, 321 (1964).
[9] S. H. Davis and G. M. Homsy, J. Fluid Mech. **98**, 527 (1980).
[10] J. L. Castillo and M. G. Velarde, J. Fluid Mech. **125**, 463 (1982).
[11] P. Cerisier, C. Jamond, J. Pantaloni, and J. C. Charmet, J. Phys. (Paris) **45**, 405 (1984).
[12] C. Pérez-García, J. Pantaloni, R. Occelli, and P. Cerisier, J. Phys. (Paris) **46**, 2047 (1985).
[13] G. Z. Gershuni and E. M. Zhukhovitskiĭ, *Convective Stability of Incompressible Fluids* (Keter, Jerusalem, 1976).