Validity of the Linear Speed Selection Mechanism for Fronts of the Nonlinear Diffusion Equation

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We consider the problem of the speed selection mechanism for the one-dimensional nonlinear diffusion equation $u_t = u_{xx} + f(u)$. It has been rigorously shown by Aronson and Weinberger that for a wide class of functions f, sufficiently localized initial conditions evolve in time into a monotonic front which propagates with speed c^* such that $2\sqrt{f'(0)} \leq c^* < 2\sqrt{\sup(f(u)/u)}$. The lower value $c_L = 2\sqrt{f'(0)}$ is that predicted by the linear marginal stability speed selection mechanism. We derive a new lower bound on the speed of the selected front, this bound depends on f and thus enables us to assess the extent to which the linear marginal selection mechanism is valid.

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In several problems arising in biology, population dynamics, pulse propagation in nerves, crystal growth, fluid flow, and others, it is found that if the system is suddenly made unstable, the subsequent dynamics is characterized by the propagation of fronts. The systems for which this phenomenon occurs have received much attention recently, especially related to the problem of pattern selection. A small perturbation at a localized point grows to eventually cover the whole space. An important problem to be solved has been the determination of the speed at which the front of the pattern moves into the undisturbed regions of the system and the wavelength of the pattern left behind. (For a recent and extensive review of this subject we refer to [1] and references therein.) Several authors [2–5] have formulated criteria that provide an answer to these questions. These criteria are heuristic extensions to higher order equations of rigorous results and heuristic arguments which have been developed for the nonlinear diffusion equation

$$u_t = u_{xx} + f(u), \tag{1}$$

where $f(u) \in C^1[0,1]$, $f(0) = f(1) = 0$. In what follows we assume that f is positive in (0,1). In this case $u = 0$ is the unstable fixed point and $u = 1$ is a stable fixed point. Aronson and Weinberger [6] have shown that any positive initial condition $u_0(x) < 1$ for all x, which decays exponentially or faster at infinity, will evolve into a front propagating with speed c^*. This asymptotic speed is the lower speed for which Eq. (1) has a monotonic front joining the stable state $u = 1$ to the unstable state $u = 0$. Moreover,

$$2\sqrt{f'(0)} \leq c^* < 2\sqrt{\sup(f(u)/u)}. \tag{2}$$

For the special case of the Fisher-Kolmogorov equation $f(u) = u - u^3$, $f'(0) = 1$, and $\sup(f(u)/u) = 1$ so that $c^* = 2$. In general [7], for any concave $f(u)$, $\sup(f(u)/u) = f'(0)$, and $c^* = 2\sqrt{f'(0)}$. The value $c^* = 2$ is the value which had been derived by Kolmogorov, Petrovsky, and Piskunov [8] using an heuristic argument (the linear marginal stability mechanism) which is equivalent to the conjecture that the asymptotic speed of the front is that for which a perturbation to the front is marginally stable in the frame moving with the front speed. Based on the applicability of this argument for the Fisher-Kolmogorov equation and more generally for concave functions f, several authors have developed extensions of this argument to higher order equations. These generalizations are purely heuristic, the only rigorous results available being those of Aronson and Weinberger. In general, however, $\sup(f(u)/u)$ is not $f'(0)$ and Eq. (2) gives a bound on the selected speed. It is known that for some choices of f, and explicit examples have been given, c^* is greater than 2. These cases, referred to as those in which a nonlinear marginal stability mechanism operates, have been generalized [5] for higher order equations based on the observation that for the nonlinear diffusion equation the selected front is that with the steepest decay to zero. The exact point of transition from the linear marginal stability to the nonlinear regime has been determined for functions f of the form $f(u) = \mu u = u^n - u^{2n-1}$ for which an exact solution for a monotonic front can be given. It has been shown that for μ smaller than a critical value, the solution corresponds to a nonlinear marginal stability solution [9]. To the best of our knowledge, the only lower bound on the speed that, for general f, shows that the linear speed is not always preferred, has been given recently by Berestycki and Nirenberg [10]. They show that

$$c^2 \geq 2 \int_0^1 f(u)\, du \tag{3}$$

from which it is evident that for sufficiently large f the speed exceeds the marginal value c_L. The purpose of this work is to give a new bound that enables one to evaluate the regime of validity of the linear marginal stability mechanism with increased accuracy. As shown by Aronson and Weinberger, the asymptotic speed of the front is the low-

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est for which there is a monotonic traveling wave solution $u = q(x - c^*t)$ of Eq. (1). The selected speed satisfies $q_{zz} + c^*q_z + f(q) = 0$, $\lim u_{z\to-\infty} = 1$, $\lim u_{z\to-\infty} = 0$, where $z = x - c^*t$. We find it convenient to work in phase space, where monotonic fronts obey an equation of an order less than the original equation. Since the selected speed corresponds to that of a decreasing monotonic front, we may consider the dependence of its derivative dq/dz on q. Calling $p(q) = -dq/dz$, where the minus sign is included so that p is positive, we find that the monotonic fronts are solutions of

$$p(q)\frac{dp}{dq} - c^*p(q) + f(q) = 0, \tag{4a}$$

with

$$p(0) = 0, \quad p(1) = 0, \quad p > 0 \quad \text{in } (0,1). \tag{4b}$$

The bound follows in a simple way from Eq. (4a). Let g be any positive function in (0,1) such that $h = -dg/dq > 0$. Multiplying Eq. (4a) by g/p and integrating with respect to q we find that

$$\int_0^1 \left(hp + \frac{f(q)}{p}g\right) dq = c^* \int_0^1 g\, dq, \tag{5}$$

where the first term is obtained after integration by parts. However, since p, h, f, and g are positive, we have that for every fixed q

$$hp + \frac{f(q)g}{p} \geq 2\sqrt{fgh};$$

hence we obtain our main result,

$$c^* \geq 2\int_0^1 \sqrt{fgh}\, dq \bigg/ \int_0^1 g\, dq, \tag{6a}$$

where

$$g \geq 0 \quad \text{and} \quad h = -g' \geq 0 \quad \text{in } (0,1). \tag{6b}$$

That this result yields a better bound than that given by Eq. (3) can be seen by choosing g so that $gh = f$ and $g(1) = 0$ [11].

Next we illustrate the use of this bound by applying it to two explicit forms of f. Since here we wish only to illustrate the use of this bound, we take three simple trial functions. As a first trial choose g so that $f = h = -g_1'$ and $g_1(1) = 0$. That is,

$$g_1(q) = \int_q^1 f(x)\, dx,$$

then

$$c \geq \frac{4}{3}\left[\int_0^1 f(q)\, dq\right]^{3/2} \bigg/ \int_0^1 qf(q)\, dq. \tag{7}$$

As a second trial function we choose $g_2(q) = 1 - q^s$, and the last trial function $g_3(q) = \exp(-sx)$. Consider first the example given in Ref. [3], $f(u) = u(1-u)(1+au)$, with $a > 0$. This form falls in the category given above for which an exact solution may be found. The transition from the regime of validity of the linear marginal stability mechanism to the regime of nonlinear behavior occurs at $a = 2$. The results obtained for this function are shown in Fig. 1. The dotted line labeled AW corresponds to the upper bound $2\sqrt{\sup(f(u)/u)}$, and the dotted line labeled BN is the bound obtained from Eq. (3). It crosses $c^* = 2$

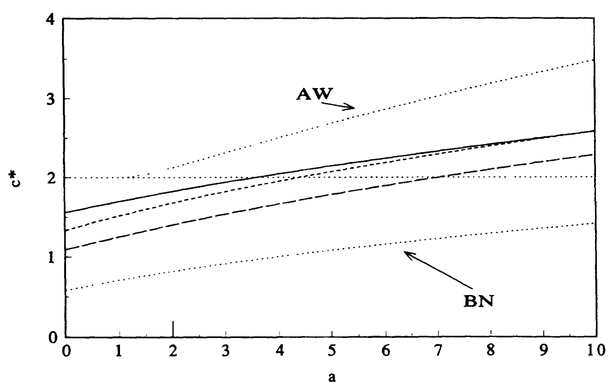

FIG. 1. Bounds on the speed of the monotonic fronts for the exactly solvable case $f(u) = u(1-u)(1+au)$. The dotted lines are the upper and lower bounds on the speed obtained from Eqs. (2) and (3). The solid and dashed lines show bounds obtained from Eq. (6) with different trail functions. The bound obtained with the simple trail function $\exp(-7x)$, shown with a solid line, indicates that linear marginal stability is not valid for $a > 3.6$. The exact value for the transition is $a = 2$.

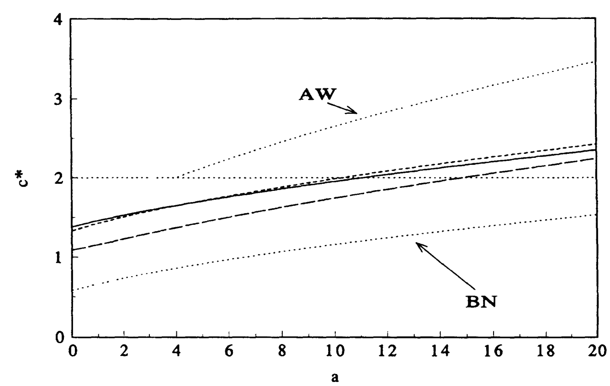

FIG. 2. Bounds on the speed of the monotonic fronts for the quartic polynomial $f(u) = u(1-u)(1+au^2)$ for which the exact solution is not known. The labeling of curves is as in Fig. 1. The bound obtained with the trial function $1 - u^s$ with $s = 0.1$, shown with the short-dashed line, indicates that linear marginal stability is not valid for $a > 10.3$.

at larger a. The solid line corresponds to the bound with the trial function g_3 with $s = 7$, the short dashed line corresponds to the bound obtained using g_2 with $s = 0.5$, and the long dashed line is the bound using g_1 calculated from Eq. (7). Aronson and Weinberger's criterion shows that linear marginal stability is valid for $0 < a < 1$, and our bound indicates that it is not valid for $a > 3.6$. As we said above, the exact solution for this case is known; the transition value from linear to nonlinear marginal stability occurs at $a = 2$. Next we apply the bound to the quartic polynomial $f = x(1 - x)(1 + ax^2)$. In this case the exact solution is not known and neither is the transition value from the linear to the nonlinear regime. The results are shown in Fig. 2, where we have used the same labeling and type of line as in Fig. 1. Aronson and Weinberger's criterion guarantees that linear marginal stability is valid for $0 \leq a \leq 4$, and of the simple bounds calculated here the best shown in the picture corresponds to that obtained with g_2 for $s = 0.1$ which shows that linear marginal stability is not valid for $a \geq 10.3$. One could, of course, attempt to obtain a sharper estimate by choosing better trial functions, but this is not our purpose here.

In conclusion, it is evident from the present results that for all nonconcave functions $f(u)$ the linear speed c_L is the asymptotic speed in a rather limited region. There is no substantial difference in the behavior of arbitrary polynomials, for which no exact solutions are known, with those already analyzed in the literature for which the exact solution and point of transition can be calculated. Once the function f becomes sufficiently large, the selected speed will be that of the so-called nonlinear front. Given the limited validity of the linear selection mechanism for the nonlinear diffusion equation a similar situation can be expected for higher order equations. Moreover, since the lower bound on the speed depends on the integral properties of f, it is not difficult to imagine a situation where two functions are identical near the origin and differ significantly near $u = 1$. In that case it is possible that the asymptotic speed for one of them be the linear value and for the other the nonlinear value. No local analysis of the approach to $u = 0$ can then predict the transition from the linear to the nonlinear marginal stability regime. Finally we wish to point out that the analysis of monotonic fronts in phase space is useful not only in the case presented here but for generalized diffusion equations and in higher order equations as well. For the porous media equation

$$u_t = (u^m)_{xx} + f(u), \qquad m \geq 1,$$

with $f(0) = f(1) = 0$, and $f > 0$ in (0,1) monotonic fronts may exist for

$$c \geq 2 \int_0^1 \sqrt{f \sigma h}\, dq \Big/ \int_0^1 \sigma\, dq,$$

where $\sigma(u) > 0$ must be chosen so that

$$h(u) \equiv -m u^{m-1} \sigma'(u) > 0 \quad \text{in } (0,1).$$

For a more general equation

$$u_t = (\phi(u))_{xx} + f(u),$$
$$\text{with } \phi' > 0 \in (0,1), \quad \phi(0) = 0,$$

with the same conditions on f, monotonic fronts may exist for

$$c \geq 2 \int_0^1 \sqrt{f \sigma h}\, dq \Big/ \int_0^1 \sigma\, dq,$$

where $\sigma(u) > 0$ must be chosen so that

$$h(u) \equiv -\sigma'(u)\, \phi'(u) > 0.$$

The details will be given elsewhere. It has been applied by us to obtain bounds on the speed of certain third order nonlinear differential equations of the type which arise in crystal growth problems [12] and for the dispersive Kuramoto-Sivashinsky equation [11]. It also enables one to characterize the type of functions $f(u)$ for which the exact point of transition to the nonlinear regime in the nonlinear diffusion Eq. (1) can be calculated without solving the equation explicitly [11]

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