Phase Space Derivation of a Variational Principle for One Dimensional Hamiltonian Systems

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We consider the bifurcation problem $u'' + \lambda u = N(u)$ with two point boundary conditions where $N(u)$ is a general nonlinear term which may also depend on the eigenvalue λ. A new derivation of a variational principle for the lowest eigenvalue λ is given. This derivation makes use only of simple algebraic inequalities and leads directly to a more explicit expression for the eigenvalue than what had been given previously.

2.30.Hq, 3.20+i, 2.30.Wd

In recent work a variational principle for the lowest eigenvalue λ of the one dimensional problem

$$u'' + \lambda u = N(u) \qquad \text{subject to} \qquad u'(0) = 0, \qquad u(1) = 0 \tag{1}$$

was given [1]. This equation, with different nonlinear terms $N(u)$ arises in the study of many elementary mechanics problems [2–6]. The derivation made use of an auxiliary variational principle whose Euler-Lagrange equation had to be solved. The variational principle is

$$\lambda = \max \frac{\int_0^{u_m} N(u)g(u)du + \frac{1}{2}K(u_m)}{\int_0^{u_m} ug(u)\,du} \tag{2}$$

where the maximum is taken over all positive functions g such that $g(0) = 0$ and $g'(u) > 0$. Here u_m is the amplitude $u(0)$ of the solution and $K(u_m)$ is obtained from the minimization of the functional

$$J_g[v] = -\int_0^1 (v')^2 g'(v)v'dx \quad \text{with} \quad v(0) = u_m, \quad v(1) = 0, \quad \text{and} \quad v' < 0 \quad \text{in} \quad (0, 1). \tag{3}$$

The purpose of the present work is to give a new derivation of the variational principle for λ, which only makes use of simple inequalities. This method is the generalization of the phase space method used to obtain the asymptotic speed of fronts of the reaction diffusion equation [7,8] suitably modified to include the consideration of a finite domain in the independent variable.

From Eq.(1) together with the boundary conditions it follows that the positive solution of (1) satisfies $u' < 0$ in $(0, 1)$. We may introduce then the phase variable $p(u) = -u'$. The equation for the trajectory in phase space is

$$p(u)\frac{dp}{du} + \lambda u = N(u), \qquad \text{with} \qquad p(u_m) = 0. \tag{4}$$

Energy is conserved, from Eq.(4) we have $\frac{1}{2}p^2 + V(u) = E$, with $V(u) = \frac{1}{2}\lambda u^2 - \int_0^u N(u')d\,u'$. Let $g(u)$ be an arbitrary positive function such that $g(0) = 0$ and $g'(u) > 0$. Multiplying Eq.(4) by $g(u)$ and integrating between $u = 0$ and $u = u_m$ we obtain, after integrating by parts,

$$\lambda \int_0^{u_m} ug(u)du = \int_0^{u_m} N(u)g(u)du + \frac{1}{2}\int_0^{u_m} p^2 g'(u)du, \tag{5}$$

where the surface terms vanish because $p(u_m) = 0$ and $g(0) = 0$.

Now observe that p has to satisfy the integral constraint

$$\int_0^{u_m} \frac{du}{p} = 1, \tag{6}$$

which follows from the fact that $\int_0^1 dx = 1$. Writing $dx = (dx/du)du$ and using the definition of p the constraint is obtained. This leads us to consider the integral

$$I = \int_0^{u_m} \left(\frac{1}{2}p^2 g' + \frac{K}{p}\right) du$$

where K is an arbitrary positive parameter. At each value of u the integrand, seen as a function of p, satisfies

$$\frac{1}{2}p^2 g' + \frac{K}{p} \geq \frac{3}{2}K^{2/3}(g')^{1/3}$$

where the equal sign holds if $p^3 = K/g'$. We have then,

$$I = \int_0^{u_m} \left(\frac{1}{2}p^2 g' + \frac{K}{p}\right) du \geq \int_0^{u_m} \frac{3}{2}K^{2/3}(g')^{1/3} du$$

which, after using the constraint Eq.(6) can be written as

$$\int_0^{u_m} \frac{1}{2}p^2 g' \geq \frac{3}{2}K^{2/3} \int_0^{u_m} (g')^{1/3} du - K.$$

This inequality holds for any positive value of K, in particular it is valid for the value of K which maximizes the right hand side. The maximizing K is given by $K^{1/3} = \int_0^{u_m} (g')^{1/3} du$ and we finally have

$$\int_0^{u_m} \frac{1}{2}p^2 g' \geq \frac{1}{2}\left[\int_0^{u_m} (g')^{1/3} du\right]^3.$$

Going back to Eq.(5) we have the final expression

$$\lambda \geq \frac{\int_0^{u_m} N(u)g(u)du + \frac{1}{2}[\int_0^{u_m} [g'(u)]^{1/3} du]^3}{\int_0^{u_m} ug(u)\, du} \tag{7}$$

where the equal sign holds for $g = \hat{g}$ which satisfies $\hat{g}' = (1/p^3) = 1/[E - V(u)]^{3/2}$.

This final expression is equivalent to Eq.(2) after noticing that the Euler equation for the functional Eq.(3) is [1] $(v')^3 g'(v) = -K$ from where it follows that $-K^{1/3} = -K^{1/3}\int_0^1 dx = \int_0^1 v'(x)[g'(v)]^{1/3} dx = -\int_0^{u_m} [g'(v)]^{1/3} dv$.

The present derivation in phase space makes use only of integration by parts and of the two simple inequalities $ax^2/2 + b/x \geq (3/2)b^{2/3}a^{1/3}$ and $aK^{2/3} - K \leq 4a^3/27$. This derivation avoids consideration of the variational principle Eq.(3) for which the existence of a single minimum had to be proved. It also provides a unified treatment for the present problem and that of the speed of fronts of the reaction diffusion equation.

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