Resistive ballooning modes in helical axis stellarators with longitudinal mass flow

W. A. Cooper
Centre de Recherches en Physique des Plasmas, Association Euratom–Confédération Suisse, Ecole Polytechnique Fédérale de Lausanne, 21, Avenue des Bains, CH-1007 Lausanne, Switzerland

M. C. Depassier
Facultad de Física, Pontificia Universidad Católica de Chile, Casilla 114-D, Santiago, Chile

(Received 22 May 1986; accepted 5 September 1986)

The magnetohydrodynamic equilibrium equation for magnetic confinement systems with helical symmetry and longitudinal plasma flow is derived. The incompressible resistive ballooning mode equation for systems with a coordinate of symmetry and rigid longitudinal flow is also derived. A reduced equation is then obtained by expanding the ballooning mode equation for a system with a large field period length compared with the plasma cross section. An analytic solution is obtained for a model equilibrium with circular flux surfaces. In the limit of small longitudinal velocity, the static mode width and growth rate scaling as the resistivity to the one-third power are recovered. For small growth rates and large longitudinal velocity, these growth rates are driven by the interaction of the magnetic curvature and the centrifugal force with the radial pressure and mass density gradient, respectively, and scale linearly with the resistivity at a reduced level.

I. INTRODUCTION

Resistive ballooning mode activity can have a significant impact on the confinement properties of plasma containment systems at high values of beta (β),[1] which is the ratio of the plasma pressure to the energy density of the magnetic field. To reach high values of β, auxiliary heating techniques such as radio frequency waves or neutral beams must be employed. These schemes can alter the distribution function of one or more of the species that make up the plasma, resulting in a net mass flow. Experimentally, toroidal plasma rotation has been measured in the ISX-B[2] and PDX[3] tokamaks. In the PDX device, poloidal rotation velocities have also been measured (with on-axis and off-axis neutral beam injection), but these are about two orders of magnitude smaller than the toroidal rotation velocities with the data points displaying a wide scatter about a mean value of zero.[3] The small poloidal velocities detected are attributable to the strong damping caused by the magnetic pumping that results when plasma moves across a spatially varying magnetic field. Plasma motion across planes of symmetry is unaffected by this damping mechanism, thus the toroidal rotation is significant in an axisymmetric system. Similarly, we can expect plasma rotation across symmetric planes in a system with helical symmetry.

The magnetohydrodynamic (MHD) equilibrium problem in tokamaks with toroidal mass flow has been investigated both analytically[4] and numerically.[5,6] In this paper we shall extend the axisymmetric formulation presented in Ref. 4 for isothermal flux surfaces to a system with helical symmetry. Unlike static plasma configurations with a coordinate of symmetry, the ideal and resistive ballooning stability properties of rotating plasmas have not been extensively analyzed. Hamieri and Laurence have investigated the ballooning mode spectrum and have derived a ballooning mode equation for an axisymmetric tokamak with rigid toroidal flow in the ideal limit.[7] Following the approach of Bateman and Nelson,[8] and incompressible resistive ballooning mode equation in such a configuration (with rigid flow) was subsequently derived and analyzed.[9]

In this paper we concentrate on stellarator configurations, particularly those having a significant spatial magnetic axis, so that the curvature caused by the helical component of the magnetic field dominates over the curvature caused by the toroidal component of that field. We thus treat the limit in which the configuration possesses helical symmetry. We consider the case of longitudinal mass flow, which in contravariant representation has only a component in the ignorable coordinate. This corresponds to mass flow across planes of symmetry. This work constitutes, therefore, an extension to helical symmetry of the axisymmetric calculation presented in Ref. 9.

In Sec. II we derive the MHD equilibrium equation for a helically symmetric configuration with purely longitudinal mass flow in which the plasma temperature is considered to be a constant on each flux surface (isothermal model). In Sec. III we derive the incompressible resistive ballooning mode equation for arbitrary symmetry from the linearized MHD equations for the case of rigid longitudinal flow. This equation is also applicable if we ignore velocity shear as a source of energy for instabilities. Thus Kelvin–Helmholtz instabilities are excluded from the dynamics of the problem. For simplicity, we have assumed that the plasma pressure and mass density evolve only by convection as in Ref. 8, and we therefore have neglected the coupling to sound waves. This coupling has been shown to have a net stabilizing effect on resistive ballooning modes.[10,11] In Sec. IV we obtain the reduced incompressible resistive ballooning mode equation in systems with helical symmetry and rigid longitudinal flow by expanding in the smallness of the parameter $\epsilon = ah$,

 0031-9171/86/124112-07$01.90

where a is the minor radius and h is the helical pitch. This limit corresponds to the minor cross section of the plasma being small compared with the length of a field period. In Sec. V we obtain an analytic solution of the reduced incompressible resistive ballooning mode equation in a model equilibrium of a helical magnetic axis stellarator that has circular flux surfaces. Finally, in Sec. VI we discuss the summary and the conclusions.

II. MHD EQUILIBRIUM WITH LONGITUDINAL FLOW

The relevant equations required to determine MHD equilibrium with longitudinal mass flow are Maxwell's equation

$$\nabla\cdot\mathbf{B}=0, \tag{1}$$

the combination of Ohm's with Faraday's law

$$\nabla\times(\mathbf{V}\times\mathbf{B})=0, \tag{2}$$

and the MHD force balance equation

$$\nabla P+\rho_M(\mathbf{V}\cdot\nabla)\mathbf{V}=(\nabla\times\mathbf{B})\times\mathbf{B}. \tag{3}$$

In a system with a coordinate of symmetry, the condition $\nabla\cdot\mathbf{B}=0$ implies that the magnetic field $\mathbf{B}$ in contravariant representation is

$$\mathbf{B}=\nabla\phi\times\nabla\psi+\sqrt{g}(\mathbf{B}\cdot\nabla\phi)\nabla\rho\times\nabla\theta, \tag{4}$$

where $2\pi\psi(\rho)$ is the helical magnetic flux, ρ is the radial coordinate, θ is the poloidal angle, and $\phi=hZ$ is the anglelike ignorable coordinate. The Jacobian of the transformation from rotating Cartesian coordinates (X,Y,ϕ) to flux coordinates (ρ,θ,ϕ) is $\sqrt{g}$. For the case that the velocity field $\mathbf{V}$ has only a longitudinal component in the contravariant representation, the scalar product of $\nabla\phi$ with Eq. (2) shows that

$$\mathbf{V}\cdot\nabla\phi=\Omega(\rho) \tag{5}$$

is a flux surface constant. From Ohm's law, $\nabla\Phi_E=\mathbf{V}\times\mathbf{B}$, one finds also that $\Omega(\rho)=-\Phi_E'/\psi'$, where $\Phi_E(\rho)$ is the electrostatic potential, and the primes indicate derivatives with respect to ρ. For this case of purely longitudinal flow in helical symmetry (or axisymmetry), the force balance equation becomes

$$\nabla P-\tfrac{1}{2}\rho_M\Omega^2(\rho)\nabla g_{\phi\phi}=(\nabla\times\mathbf{B})\times\mathbf{B}, \tag{6}$$

where $g_{\phi\phi}$ is the lower metric element associated with the coordinate of symmetry. Specifically for helical symmetry, $g_{\phi\phi}=(1+h^2r^2)/h^2$, where r is the distance of any point from the geometric axis. For axisymmetry $g_{\phi\phi}=R^2$, where R is the distance from the geometric axis.

The $\nabla\rho\times\nabla\theta$ component of the force balance equation yields the condition that the longitudinal magnetic field in the covariant representation is a constant on a flux surface, namely, that

$$\sqrt{g}\mathbf{B}\cdot\nabla\rho\times\nabla\theta=F(\rho). \tag{7}$$

We consider a two-species plasma and express the pressure $P=(N_i+N_e)T$, where the electrons with density N_e and the ions with density N_i have the same temperature T. We further assume that $N_i=N_e$, so that we can express the mass density $\rho_M=M_iN_i$ ($M_i=$ ion mass) in terms of P.

$$\rho_M=(M_i/2T)\,P. \tag{8}$$

Invoking the condition of rapid heat flow along the magnetic field lines, the temperature is taken as a constant on a flux surface. Then the component of force balance along the magnetic field lines yields the relation

$$P=\Pi(\rho)\exp\{[M_i\Omega^2(\rho)/4T(\rho)]g_{\phi\phi}\}. \tag{9}$$

In helical symmetry, $g_{\phi\phi}$ depends only on r, which implies $P=P(\rho,r)$ and $\rho_M=\rho_M(\rho,r)$. In axisymmetry, $g_{\phi\phi}$ depends only on R, thus $P=P(\rho,R)$ and $\rho_M=\rho_M(\rho,R)$. Two useful relations can be obtained from Eqs. (8) and (9):

$$\left.\frac{\partial P}{\partial r}\right|_\rho=\rho_M(\rho,r)\,\Omega^2(\rho)r, \tag{10}$$

$$\left.\frac{\partial \rho_M}{\partial r}\right|_\rho=\frac{\rho_M^2(\rho,r)\Omega^2(\rho)r}{P(\rho,r)}. \tag{11}$$

The $\nabla\theta\times\nabla\phi$ (radial) component of force balance yields an expression for the longitudinal current density in contravariant representation given by

$$\mathbf{j}\cdot\nabla\phi=-\left.\frac{\partial P}{\partial\psi}\right|_r-(\mathbf{B}\cdot\nabla\phi)\,\frac{dF}{d\psi}. \tag{12}$$

Expanding $(\mathbf{j}\cdot\nabla\phi)$ we obtain the MHD equilibrium equation

$$\nabla\cdot(K\nabla\psi)=-\frac{1}{h^2}\left.\frac{\partial P}{\partial\psi}\right|_r-KF(\psi)\,\frac{dF}{d\psi}-2hK^2F(\psi), \tag{13}$$

where $K\equiv(1+h^2r^2)^{-1}$. This equation describes the MHD equilibrium in helically symmetric systems with longitudinal mass flow in which the plasma temperature is a flux surface quantity. The axisymmetric version of this equation was derived earlier by Maschke and Perrin.[4]

III. INCOMPRESSIBLE RESISTIVE BALLOONING EQUATION

The system of linearized MHD equations required to determine the stability properties of a magnetic confinement device are Maxwell's equation

$$\nabla\cdot\mathbf{b}=0, \tag{14}$$

the combination of Ohm's law with Faraday's law

$$\frac{\partial\mathbf{b}}{\partial t}=\nabla\times(\mathbf{V}\times\mathbf{b})+\nabla\times(\mathbf{v}\times\mathbf{B})-\eta\nabla\times(\nabla\times\mathbf{b}), \tag{15}$$

the MHD equation of motion

$$\begin{aligned}\rho_M\frac{\partial\mathbf{v}}{\partial t}=&-\rho_M\nabla(\mathbf{v}\cdot\mathbf{V})-\rho_M(\nabla\times\mathbf{v})\times\mathbf{V}\\&-\rho_M(\nabla\times\mathbf{V})\times\mathbf{v}-\tilde{\rho}(\mathbf{V}\cdot\nabla)\mathbf{V}-\nabla p\\&+(\nabla\times\mathbf{b})\times\mathbf{B}+(\nabla\times\mathbf{B})\times\mathbf{b},\end{aligned} \tag{16}$$

the convective pressure evolution equation

$$\frac{\partial p}{\partial t}=-(\mathbf{V}\cdot\nabla)p-(\mathbf{v}\cdot\nabla)P, \tag{17}$$

and the convective mass density evolution equation

$$\frac{\partial \tilde{\rho}}{\partial t} = -(\mathbf{V}\cdot\nabla)\tilde{\rho} - (\mathbf{v}\cdot\nabla)\rho_M. \tag{18}$$

The equilibrium magnetic and velocity fields are $\mathbf{B}$ and $\mathbf{V}$, respectively. The equilibrium pressure and mass density are P and ρ_M, respectively. The perturbed magnetic and velocity fields are $\mathbf{b}$ and $\mathbf{v}$, respectively. The perturbed pressure and mass density are p and $\tilde{\rho}$, respectively.

We invoke the ballooning mode representation[12,13] in a flux coordinate system in which the magnetic field lines are straight for configurations that have a coordinate of symmetry. Then any perturbation ξ is expressed as

$$\xi(\rho,\theta,\phi,t) = \hat{\xi}(\rho,\theta,t)\exp[inS(\rho,\theta,\phi,t)], \tag{19}$$

where the amplitude $\hat{\xi}$ is a slowly varying function, the exponential contains a rapidly varying phase, and $n \gg 1$ is the mode number associated with the ignorable anglelike coordinate ϕ. The eikonal function S is assumed to satisfy the condition $\mathbf{B}\cdot\nabla S = 0$, which guarantees that the perturbations vary slowly along the magnetic field lines. Analysis of any one of the set of equations (15)–(18) also shows that the eikonal phase factor must satisfy the condition

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \mathbf{V}\cdot\nabla S = 0. \tag{20}$$

These two criteria for the eikonal S can be satisfied by choosing

$$S(\rho,\theta,\phi,t) = \phi - q(\rho)\theta + k_v(\rho) - \Omega(\rho)t, \tag{21}$$

where $k_v(\rho)$ is the radial wavenumber and $q(\rho)$ is the derivative of the longitudinal magnetic flux with respect to the helical magnetic flux within the flux surface labeled with ρ in a coordinate system in which the field lines are straight.[14] The last term in Eq. (21) actually corresponds to a Doppler shift of the mode frequency as observed in a frame of reference rotating with the fluid. This Doppler shift is introduced because in the laboratory frame of reference, a ballooning instability of mode number n appears to rotate at a frequency $n\Omega$ past the observer, a rate that is much faster than the characteristic growth rate of that mode. To resolve this type of local instability on a given flux surface, it is necessary, then, to make the transformation to the frame of reference moving with the fluid at the angular frequency $\Omega(\rho)$ that corresponds to the plasma rotation on that particular surface. The salient feature of this transformation is the Doppler shift, which we find convenient, however, to include as a modification of the eikonal because it is multiplied by the mode number n, the inverse of which is an expansion parameter in the determination of ballooning instabilities. The problem is simplified further by neglecting the velocity shear Ω' as a source of free energy for instabilities. We thus ignore the Kelvin–Helmholtz class of instabilities and concentrate on those associated with a rigid longitudinal plasma mass flow. The potential implications of velocity shear on the ideal ballooning stability of a rotating plasma fluid are discussed further in Ref. 7.

We express the perturbed magnetic field as

$$\mathbf{b} = b_\parallel(\mathbf{B}/B^2) + b_s\nabla S + b_\perp(\mathbf{B}\times\nabla S)/|\nabla S|^2, \tag{22}$$

and the perturbed velocity field as

$$\mathbf{v} = v_s\nabla S + v_\perp(\mathbf{B}\times\nabla S)/B^2, \tag{23}$$

which ignores the effects of parallel dynamics thus excluding the propagation of sound waves along the magnetic field lines. Then from Maxwell's equation [Eq. (14)] we find that b_s is a quantity of $O(1/n)$ in the ballooning expansion. Similarly, the component of Ohm's law [Eq. (15)] along the magnetic field lines demonstrates that $\nabla\cdot\mathbf{v} \sim O(1)$ in the $1/n$ expansion, which implies that $v_s \sim O(1/n)$. The ∇S component of the equation of motion [Eq. (16)] demonstrates that $b_\parallel = -p + O(1/n)$. Then, assuming that the amplitudes of the perturbations evolve as $\exp(\omega t)$, we find that the $\mathbf{B}\times\nabla S$ component of Ohm's law reduces to

$$(\omega + n^2\eta|\nabla S|^2)b_\perp = (|\nabla S|^2/B^2)(\mathbf{B}\cdot\nabla)v_\perp + O(1/n), \tag{24}$$

with $\eta \sim O(1/n^2)$, the $\mathbf{B}\times\nabla S$ component of the equation of motion reduces to

$$\rho_M|\nabla S|^2\omega v_\perp = -[\mathbf{B}\times\nabla S\cdot(\mathbf{V}\cdot\nabla)\mathbf{V}]\,\tilde{\rho} - 2(\mathbf{B}\times\nabla S\cdot\boldsymbol{\kappa})p + B^2(\mathbf{B}\cdot\nabla)b_\perp + O(1/n), \tag{25}$$

the convective pressure equation reduces to

$$\omega p = -[(\mathbf{B}\times\nabla S\cdot\nabla P)/B^2]v_\perp + O(1/n), \tag{26}$$

and the convective density equation reduces to

$$\omega\tilde{\rho} = -[(\mathbf{B}\times\nabla S\cdot\nabla\rho_M)/B^2]v_\perp + O(1/n). \tag{27}$$

Substituting for b, p, and $\tilde{\rho}$ in the $\mathbf{B}\times\nabla S$ component of the equation of motion, we obtain the incompressible resistive ballooning mode equation for systems with a coordinate of symmetry and mass flow in the ignorable coordinate

$$(\mathbf{B}\cdot\nabla)\left(\frac{|\nabla S|^2/B^2}{[1 + (n^2\eta/\omega)|\nabla S|^2]}(\mathbf{B}\cdot\nabla)v_\perp\right) - \rho_M\omega^2\frac{|\nabla S|^2}{B^2}v_\perp + 2\left(\frac{\mathbf{B}\times\nabla S\cdot\nabla P}{B^2}\right)\left(\frac{\mathbf{B}\times\nabla S\cdot\boldsymbol{\kappa}}{B^2}\right)v_\perp + \left(\frac{\mathbf{B}\times\nabla S\cdot\nabla\rho_M}{B^2}\right)\left(\frac{\mathbf{B}\times\nabla S\cdot(\mathbf{V}\cdot\nabla)\mathbf{V}}{B^2}\right)v_\perp = 0. \tag{28}$$

The first and second terms correspond to the usual field line bending stabilization weakened by resistivity and inertia, respectively. The third term represents the interaction of the pressure gradient with the magnetic field line curvature (denoted by $\boldsymbol{\kappa}$), which drives ballooning and interchange modes. The fourth term constitutes the effect of the mass flow and corresponds to the interaction of the mass density gradient with the centrifugal force associated with that flow.

IV. REDUCED RESISTIVE BALLOONING EQUATION

The reduced incompressible resistive ballooning mode equation for helically symmetric systems with rigid longitudinal mass flow is obtained by expanding the corresponding full equation in the smallness of the parameter $\epsilon = ah$. The expansion is carried out in a magnetic flux coordinate system (ρ,θ,ϕ), where the poloidal angle θ is such that the magnetic field lines are straight. As a result we have $\sqrt{g}(\mathbf{B}\cdot\nabla\phi) = q\psi'$, from which we derive that the Jacobian is

$$\sqrt{g}=\frac{q(\rho)\psi'}{h^2F(\rho)}\left\{1+\epsilon^2\left[X^2+Y^2+\frac{1}{q}\times\left(X\frac{\partial Y}{\partial\theta}-Y\frac{\partial X}{\partial\theta}\right)\right]\right\}, \tag{29}$$

after expanding the contravariant longitudinal magnetic field. Here X represents the distance from the geometric axis to the projection of some point in the plasma onto the midplane of the configuration and Y represents the distance from that point to the midplane. Both X and Y are normalized to the minor radius a in Eq. (29). Then the $\mathbf{B}\cdot\nabla$ operator reduces to

$$\mathbf{B}\cdot\nabla=\frac{\psi'}{\sqrt{g}}\frac{\partial}{\partial\theta}=\frac{h^2F(\rho)}{q(\rho)}\,[1+O(\epsilon^2)]\,\frac{\partial}{\partial\theta}, \tag{30}$$

while B^2 can be written as

$$B^2=h^2F^2(\rho)[1+O(\epsilon^2)]. \tag{31}$$

The derivatives of B^2 with respect to θ and ρ require retaining the $O(\epsilon^2)$ terms to yield

$$\frac{\partial B^2}{\partial\theta}=-2\epsilon^2h^2F^2(\rho)\left[X\frac{\partial X}{\partial\theta}+Y\frac{\partial Y}{\partial\theta}-\frac{1}{q^2}\left(\frac{\partial X}{\partial\theta}\frac{\partial^2X}{\partial\theta^2}+\frac{\partial Y}{\partial\theta}\frac{\partial^2Y}{\partial\theta^2}\right)+O(\epsilon^2)\right] \tag{32}$$

and

$$\frac{\partial B^2}{\partial\rho}=2h^2FF'-2\epsilon^2h^2F^2(\rho)\left\{X\frac{\partial X}{\partial\rho}+Y\frac{\partial Y}{\partial\rho}+\frac{F'}{F}(X^2+Y^2)-\frac{1}{q^2}\left(\frac{\partial X}{\partial\theta}\frac{\partial^2X}{\partial\rho\,\partial\theta}+\frac{\partial Y}{\partial\theta}\frac{\partial^2Y}{\partial\rho\,\partial\theta}\right)+\frac{1}{q^2}\left(\frac{q'}{q}-\frac{F'}{F}\right)\left[\left(\frac{\partial X}{\partial\theta}\right)^2+\left(\frac{\partial Y}{\partial\theta}\right)^2\right]+O(\epsilon^2)\right\}. \tag{33}$$

The expression for the magnetic field line curvature ($\mathbf{B}\times\nabla S\cdot\boldsymbol{\kappa}$) is

$$\mathbf{B}\times\nabla S\cdot\boldsymbol{\kappa}=\frac{1}{2\psi'}\left\{2\frac{\partial P}{\partial\rho}\bigg|_r+\frac{\partial B^2}{\partial\rho}-\left[\frac{B_\rho\psi'}{\sqrt{g}B^2}+\frac{\psi'F(\rho)q'}{\sqrt{g}B^2}(\theta-\theta_k)\right]\frac{\partial B^2}{\partial\theta}\right\}, \tag{34}$$

where B_ρ is the radial magnetic field in the covariant representation and $\theta_k\equiv k'_v/q'$ represents the radial wavenumber. Expansion of the MHD equilibrium equation [Eq. (13)] for $\epsilon\ll1$ yields

$$\frac{\partial P}{\partial\rho}\bigg|_r+h^2FF'=-\frac{\epsilon^2h^2F^2(\rho)}{q^2(\rho)}\left\{\left(\frac{F'}{F}-\frac{q'}{q}\right)\left[\left(\frac{\partial X}{\partial\theta}\right)^2+\left(\frac{\partial Y}{\partial\theta}\right)^2\right]-q^2\frac{F'}{F}(X^2+Y^2)+\frac{2\psi'q^2}{a\epsilon F(\rho)}+\frac{\partial X}{\partial\theta}\frac{\partial^2X}{\partial\rho\,\partial\theta}+\frac{\partial Y}{\partial\theta}\frac{\partial^2Y}{\partial\rho\,\partial\theta}-\frac{\partial X}{\partial\rho}\frac{\partial^2X}{\partial\theta^2}-\frac{\partial Y}{\partial\rho}\frac{\partial^2Y}{\partial\theta^2}+O(\epsilon^2)\right\}. \tag{35}$$

Equations (33) and (35) allow us to obtain an expression for $[2(\partial P/\partial\rho)|_r+\partial B^2/\partial\rho]$, and noting that when we expand $B_\rho\psi'/(\sqrt{g}B^2)$, it is $O(\epsilon^2)$ compared with the leading-order terms, the magnetic field line curvature in the ballooning mode equation reduces to

$$\frac{\mathbf{B}\times\nabla S\cdot\boldsymbol{\kappa}}{B^2}=-\frac{\epsilon^2}{\psi'}K_c(\rho,\theta)[1+O(\epsilon^2)], \tag{36}$$

the centrifugal force term reduces to

$$\frac{\mathbf{B}\times\nabla S\cdot(\mathbf{V}\cdot\nabla)\mathbf{V}}{B^2}=-\frac{\epsilon^2}{\psi'}\left(\frac{\Omega^2}{h^2}\right)K_r(\rho,\theta)[1+O(\epsilon^2)], \tag{37}$$

the term $(\mathbf{B}\times\nabla S\cdot\nabla P)/B^2$ reduces to

$$\frac{\mathbf{B}\times\nabla S\cdot\nabla P}{B^2}=\frac{1}{\psi'}\left(\frac{\partial P}{\partial\rho}\bigg|_r+\epsilon^2\frac{\rho_M\Omega^2}{h^2}K_r(\rho,\theta)\right)\times[1+O(\epsilon^2)], \tag{38}$$

and the term $(\mathbf{B}\times\nabla S\cdot\nabla\rho_M)/B^2$ reduces to

$$\frac{\mathbf{B}\times\nabla S\cdot\nabla\rho_M}{B^2}=\frac{1}{\psi'}\left(\frac{\partial\rho_M}{\partial\rho}\bigg|_r+\epsilon^2\frac{\rho_M^2\Omega^2}{h^2P}K_r(\rho,\theta)\right)\times[1+O(\epsilon^2)]. \tag{39}$$

The notation we have employed here is such that

$$K_r(\rho,\theta)\equiv X\frac{\partial X}{\partial\rho}+Y\frac{\partial Y}{\partial\rho}-\left(X\frac{\partial X}{\partial\theta}+Y\frac{\partial Y}{\partial\theta}\right)\frac{q'}{q}\times(\theta-\theta_k) \tag{40}$$

and

$$K_c(\rho,\theta)\equiv K_r(\rho,\theta)+\frac{2\psi'}{a\epsilon F(\rho)}-\frac{1}{q^2}\left(\frac{\partial X}{\partial\rho}\frac{\partial^2X}{\partial\theta^2}+\frac{\partial Y}{\partial\rho}\frac{\partial^2Y}{\partial\theta^2}\right)+\frac{1}{q^2}\left(\frac{\partial X}{\partial\theta}\frac{\partial^2X}{\partial\theta^2}+\frac{\partial Y}{\partial\theta}\frac{\partial^2Y}{\partial\theta^2}\right)\frac{q'}{q}(\theta-\theta_k). \tag{41}$$

We normalize $F(\rho)$ to F_e, its value at the edge of the plasma, and ψ to $a\epsilon F_e$. The pressure P, the mass density ρ_M, the resistivity η, and the longitudinal velocity function Ω are normalized to P_0,ρ_{M0},η_0, and Ω_0, their respective values at the magnetic axis. The growth rate ω is expressed in helical Alfvén units $\omega_H\equiv h^2F_e/\sqrt{\rho_{M0}}$. We define $\beta_p\equiv2P_0/(h^2F_e^2)$ as the beta on axis caused by the pressure and $\beta_r=\rho_{M0}\Omega_0^2/h^4F_e^2$ as the beta on axis caused by the rotation. The magnetic Reynolds number is $S_R\equiv a^2\omega_H/\eta_0$. With these normalizations, we find that the expression

$$|\nabla S|^2=(\nabla\phi\cdot\nabla\phi)-2q(\nabla\theta\cdot\nabla\phi)+q^2(\nabla\theta\cdot\nabla\theta)-2q'[(\nabla\rho\cdot\nabla\phi)-q(\nabla\rho\cdot\nabla\theta)](\theta-\theta_k)+(q')^2(\nabla\rho\cdot\nabla\rho)(\theta-\theta_k)^2 \tag{42}$$

reduces to

$$|\nabla S|^2=[F^2/a^2(\psi')^2]\alpha_\perp(\rho,\theta)[1+O(\epsilon^2)], \tag{43}$$

where

$$\alpha_\perp(\rho,\theta)=\left(\frac{\partial X}{\partial\rho}\right)^2+\left(\frac{\partial Y}{\partial\rho}\right)^2-2\frac{q'}{q}\left(\frac{\partial X}{\partial\rho}\frac{\partial X}{\partial\theta}+\frac{\partial Y}{\partial\rho}\frac{\partial Y}{\partial\theta}\right)\times(\theta-\theta_k)+\left(\frac{q'}{q}\right)^2\left[\left(\frac{\partial X}{\partial\theta}\right)^2+\left(\frac{\partial Y}{\partial\theta}\right)^2\right]\times(\theta-\theta_k)^2, \quad (44)$$

and the reduced incompressible resistive ballooning mode equation becomes

$$\frac{\partial}{\partial\theta}\left\{\frac{[F^2(\rho)/(\psi')^2]\alpha_\perp(\rho,\theta)}{[1+(n^2\eta/\omega S_R)[F^2/(\psi')^2]\alpha_\perp(\rho,\theta)]}\frac{\partial v_\perp}{\partial\theta}\right\}-\frac{q^2(\rho)}{(\psi')^2}D(\rho,\theta)v_\perp-\frac{\rho_M\omega^2q^2}{(\psi')^2}\alpha_\perp(\rho,\theta)v_\perp=0, \quad (45)$$

with $D(\rho,\theta)$ the driving term, which is given by

$$D(\rho,\theta)=\beta_p\frac{\partial P}{\partial\rho}\bigg|_r K_c(\rho,\theta)+\beta_r\Omega^2\frac{\partial\rho_M}{\partial\rho}\bigg|_r K_r(\rho,\theta)+2\epsilon^2\beta_r\rho_M\Omega^2K_r(\rho,\theta)\left[K_c(\rho,\theta)+\frac{\beta_r}{\beta_p}\left(\frac{\rho_M\Omega^2}{P}\right)K_r(\rho,\theta)\right]. \quad (46)$$

The first term corresponds to the usual interaction between the radial pressure gradient and the magnetic field line curvature. The second term corresponds to the interaction between the radial mass density gradient with the centrifugal force. The third term represents the interaction between the gradients of the pressure and the mass density that lie on the flux surface with the magnetic field line curvature and with the centrifugal force, respectively. This last term in the expression for D contains elements that are constant and linear in $(\theta-\theta_k)$, which are $O(\epsilon^2)$ smaller than the corresponding elements in the first and second terms. For consistency, these should be neglected. However, from the last term we are interested in retaining the coefficient that is quadratic in $(\theta-\theta_k)$, which though formally small can have a significant impact on the mode structure and growth rate scaling at large $(\theta-\theta_k)$. This would be more obvious if D were written as $D=a_0+a_1(\theta-\theta_k)+\epsilon^2a_2(\theta-\theta_k)^2$, where the coefficients a_j are periodic in θ and a_2 vanishes in the absence of plasma flow.

V. ANALYTIC SOLUTION

We consider a model equilibrium, with circular flux surfaces, which corresponds to $X=X_M+\rho\cos\theta$ and $Y=\rho\sin\theta$, where X_M is the distance between the magnetic axis and the geometric axis. For this model, the functions $\alpha_\perp(\rho,\theta)$, $K_r(\rho,\theta)$, and $K_c(\rho,\theta)$ reduce to

$$\alpha_\perp(\rho,\theta)=1+s^2(\theta-\theta_k)^2, \quad (47)$$

$$K_r(\rho,\theta)=\rho+X_M\cos\theta+X_Ms(\theta-\theta_k)\sin\theta, \quad (48)$$

and

$$K_c(\rho,\theta)=(\rho/q^2)(1+q^2)+2(\psi'/F)+X_M\cos\theta+X_Ms(\theta-\theta_k)\sin\theta, \quad (49)$$

respectively, where we have defined the magnetic shear as $s\equiv\rho q'/q$. We invoke the electrostatic approximation $a_R\omega(\psi')^2/F^2\ll 1$, where $a_R\equiv S_R/n^2\eta$ to simplify the field bending stabilization term to $a_R\omega(\partial^2v_\perp/\partial\theta^2)$. We apply the two-length-scale expansion to express $v_\perp$ as

$$v_\perp=v_0+v_c\cos\theta+v_s\sin\theta, \quad (50)$$

where v_0, v_c, and v_s are slowly varying amplitude functions compared with the trigonometric functions, and solve the ballooning mode equation order by order to obtain

$$v_c=\frac{2u}{a_R\omega+w+\rho_M(\omega^2+\lambda^2)s^2(\theta-\theta_k)^2}v_0 \quad (51)$$

and

$$v_s=\frac{2us(\theta-\theta_k)}{a_R\omega+w+\rho_M(\omega^2+\lambda^2)s^2(\theta-\theta_k)^2}v_0, \quad (52)$$

where we have defined

$$u\equiv-\frac{1}{2}\frac{q^2}{(\psi')^2}\left(\beta_p\frac{\partial P}{\partial\rho}\bigg|_r+\beta_r\Omega^2\frac{\partial\rho_M}{\partial\rho}\bigg|_r\right)X_M, \quad (53)$$

$$\lambda^2\equiv\beta_r\Omega^2[1+(\beta_r/\beta_p)(\rho_M\Omega^2/P)](\epsilon X_M)^2, \quad (54)$$

and

$$w\equiv\frac{q^2}{(\psi')^2}\left[\beta_p\frac{\partial P}{\partial\rho}\bigg|_r\left(\frac{\rho}{q^2}(1+q^2)+\frac{2\psi'}{F}\right)+\beta_r\rho\Omega^2\frac{\partial\rho_M}{\partial\rho}\bigg|_r+\rho_M\omega^2\right]. \quad (55)$$

Using Eqs. (51) and (52), the equation for the envelope of the resistive ballooning mode becomes

$$a_R\omega\frac{\partial^2v_0}{\partial\theta^2}-\frac{(wd_0-2u^2)+c_1s^2(\theta-\theta_k)^2+d_1^2s^4(\theta-\theta_k)^4}{d_0+d_1s^2(\theta-\theta_k)^2}\times v_0=0, \quad (56)$$

where

$$d_0\equiv a_R\omega+w, \quad (57)$$

$$d_1\equiv\rho_M[q^2/(\psi')^2](\omega^2+\lambda^2), \quad (58)$$

and

$$c_1\equiv\rho_M[q^2/(\psi')^2](\omega^2+\lambda^2)(2w+a_R\omega)-2u^2. \quad (59)$$

Asymptotic analysis of the envelope equation for resistive ballooning modes yields expressions for the width L of the mode and a scaling for the growth rate ω.[11] The width of the mode is

$$L=\left(\frac{a_R\omega(\psi')^2}{\rho_Mq^2(\omega^2+\lambda^2)s^2}\right)^{1/4}, \quad (60)$$

and the approximate dispersion relation for the regime dominated by resistive ballooning activity is

$$\rho_M[q^2/(\psi')^2](\omega^2+\lambda^2)a_R\omega=2u^2. \quad (61)$$

In the small rotation velocity regime $\omega^2\gg\beta_r\Omega^2(\epsilon X_M)^2$, we recover the static results[15] that the mode width is limited by plasma inertia

$$L=\left(\frac{S_R}{n^2\eta}\frac{(\psi')^2}{\rho_M\omega q^2s^2}\right)^{1/4}, \quad (62)$$

and the growth rate

$$\omega = \left[\frac{n^2\eta}{2S_R} \frac{q^2}{\rho_M(\psi')^2} \left(\frac{\beta_p}{\epsilon} \frac{\partial P}{\partial \rho}\bigg|_r + \frac{\beta_r}{\epsilon} \Omega^2 \frac{\partial \rho_M}{\partial \rho}\bigg|_r \right)^2 (\epsilon X_M)^2 \right]^{1/3} \quad (63)$$

is determined by the interaction of the radial pressure gradient with the geodesic curvature plus a correction caused by the interaction of the mass density gradient with the geodesic component of the centrifugal force. The mode scales with the resistivity to the one-third power. In the small growth rate, large velocity regime $\omega^2 \ll \beta_r \Omega^2 (\epsilon X_M)^2$, the width of the mode is determined by the interaction of the pressure and mass density gradients that lie on the flux surface with the geodesic components of the curvature and centrifugal force, respectively,

$$L = \left(\frac{S_R}{n^2\eta} \frac{(\psi')^2}{\rho_M q^2 s^2} \times \frac{\omega}{\beta_r \Omega^2 [1 + (\beta_r/\beta_p)(\rho_M \Omega^2/P)] (\epsilon X_M)^2} \right)^{1/4} \quad (64)$$

and the growth rate ω scales linearly with the resistivity at a reduced level

$$\omega = \left(\frac{n^2\eta}{2S_R} \frac{q^2}{\rho_M(\psi')^2} \times \frac{[(\beta_p/\epsilon)(\partial P/\partial \rho)|_r + (\beta_r/\epsilon)\Omega^2(\partial \rho_M/\partial \rho)|_r]^2}{\beta_r \Omega^2 [1 + (\beta_r/\beta_p)(\rho_M \Omega^2/P)]} \right), \quad (65)$$

where the interaction of the radial pressure and mass density gradients with the geodesic curvature and centrifugal force components, respectively, drive the modes, while the pressure and mass density gradients on the flux surfaces interact with geodesic components of the curvature and the centrifugal force, respectively, to stabilize the modes. It is of interest to note that the axisymmetric growth rate scaling and mode width are recovered in either of the limits we have investigated by taking $\epsilon X_M = 1$ (see Ref. 9).

As an application, we consider a model equilibrium that has $P = 0.5$, $P' = 1.0$, $\rho_M = 0.5$, $\rho'_M = 1.0$, $q^2 = 0.25$, $(\psi')^2 = 0.25$, $\epsilon = \frac{1}{3}$, $(\epsilon X_M)^2 = 0.1$, $\eta = 1.0$, $S_R = 10^6$, $\Omega = 1.0$, and $\beta_p = 0.2$. We investigate first a case with $\beta_r = 0.01$. We find that a mode with $n = 10$ is in the rotationally dominant regime and has $\omega = 3.8 \times 10^{-3}$. For $n = 50$, however, the mode is in the resistively ($\eta^{1/3}$) dominant regime and has $\omega = 4.6 \times 10^{-2}$. The transition phase between the two regimes occurs when $\omega^2 \simeq \beta_r \Omega^2 (\epsilon X_M)^2$, which corresponds to $\omega = 3.2 \times 10^{-2}$ in this example. If we increase the plasma rotation so that $\beta_r = 0.05$, we obtain that the $n = 50$ mode is now in the rotationally dominant regime and has a reduced growth rate $\omega = 2.3 \times 10^{-2}$. For higher values of n, finite Larmor radius effects that have been neglected become increasingly more important.

VI. SUMMARY AND CONCLUSIONS

We have derived the MHD equilibrium equation for systems with helical symmetry with mass flow in the longitudinal direction in which the plasma temperature is assumed to be a constant on a flux surface. We have also derived the incompressible resistive ballooning mode equation for helically symmetric systems with longitudinal mass flow that is either rigid or in which we neglect velocity shear as a source of free energy for instabilities. This equation is simplified by expansion in the smallness of the parameter $\epsilon = ah$. An analytic solution of this equation is then obtained for a model equilibrium of a helical axis stellarator with circular flux surfaces. For the small velocity limit, we recover the static results that the mode width is determined by the plasma inertia and that the growth rate scales as the resistivity to the one-third power. However, in the small growth rate—large plasma velocity limit—the width of the mode is determined by the interaction of the pressure and mass density gradients that lie on the flux surface with the geodesic components of the magnetic curvature and the centrifugal force, respectively. Also in this limit, the growth rate scales linearly with the resistivity at a reduced level and is driven by the interaction of the radial pressure and mass density gradients with the geodesic components of the magnetic curvature and centrifugal force, respectively.

In stellarator configurations, which are basically current-free devices, it is the resistive pressure-driven MHD activity that may have the most profound impact on the confinement properties of the plasma at high β. We have demonstrated that a relatively modest amount of plasma rotation can significantly reduce the resistive ballooning mode growth rates and as a result diminish the potentially deleterious effects that these types of instabilities may induce on the plasma transport.

ACKNOWLEDGMENTS

This work has been supported by the Ecole Polytechnique Fédérale de Lausanne, by the Fonds National Suisse de la Recherche Scientifique, by Euratom, and by the Dirección de Investigación, Universidad Católica de Chile through DIUC Project No. 19/85. Early portions of this work were sponsored by the Office of Fusion Energy, United States Department of Energy, under Contract No. DE-AC05-84OR21400 with Martin Marietta Energy Systems, Inc.

[1]B. A. Carreras, P. H. Diamond, M. Murakami, J. L. Dunlap, J. D. Bell, H. R. Hicks, J. A. Holmes, E. A. Lazarus, V. K. Paré, P. Similon, C. E. Thomas, and R. M. Wieland, Phys. Rev. Lett. **50**, 503 (1983).
[2]R. C. Isler, L. E. Murray, E. C. Crume, C. E. Bush, J. L. Dunlap, P. H. Edmonds, S. Kasai, E. A. Lazarus, M. Murakami, G. H. Neilson, V. K. Paré, S. D. Scott, C. E. Thomas, and A. J. Wootton, Nucl. Fusion **23**, 1017 (1983).
[3]K. Brau, M. Bitter, R. J. Goldston, D. Manos, K. McGuire, and S. Suckewer, Nucl. Fusion **23**, 1643 (1983).
[4]E. K. Maschke and H. Perrin, Plasma Phys. **22**, 579 (1980).

[5]S. Semenzato, R. Gruber, and H. P. Zehrfeld, Comput. Phys. Rep. **1**, 389 (1984).
[6]W. Kerner and O. Jandl, Comput. Phys. Commun. **31**, 269 (1984).
[7]E. Hamieri and P. Laurence, J. Math. Phys. **25**, 396 (1984).
[8]G. Bateman and D. B. Nelson, Phys. Rev. Lett. **41**, 1804 (1978).
[9]W. A. Cooper, Plasma Phys. Controlled Fusion **26**, 1351 (1984).
[10]M. S. Chance, R. L. Dewar, E. A. Frieman, A. H. Glasser, J. M. Greene, R. C. Grimm, S. C. Jardin, J. L. Johnson, J. Manickam, M. Okabayashi, and A. M. M. Todd, in *Plasma Physics and Controlled Nuclear Fusion Research* (IAEA, Vienna, 1979), Vol. 1, p. 677.
[11]T. C. Hender, B. A. Carreras, W. A. Cooper, J. A. Holmes, P. H. Diamond, and P. L. Similon, Phys. Fluids **27**, 1439 (1984).
[12]J. W. Connor, R. J. Hastie, and J. B. Taylor, Phys. Rev. Lett. **40**, 396 (1978).
[13]R. L. Dewar and A. H. Glasser, Phys. Fluids **26**, 3038 (1983).
[14]R. L. Dewar, D. A. Monticello, and W. N.-C. Sy, Phys. Fluids **27**, 1723 (1984).
[15]W. A. Cooper and M. C. Depassier, Phys. Rev. A **32**, 3124 (1985).