Variational principle for the asymptotic speed of fronts of the density-dependent diffusion-reaction equation

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We show that the minimal speed for the existence of monotonic fronts of the equation $u_t=(u^m)_{xx}+f(u)$ with $f(0)=f(1)=0$, $m>1$ and $f>0$ in (0,1), is derived from a variational principle. The variational principle allows us to calculate, in principle, the exact speed for general f. The case $m=1$ when $f'(0)=0$ is included as an extension of the results.

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Several problems arising in population growth [1,2] combustion theory [3,4], chemical kinetics [5], and others [6], lead to an equation of the form

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{j}=F(\rho) ,$$

where the source term $F(\rho)$ represents net growth and saturation processes. The flux $\vec{j}$ is given by Fick's law

$$\vec{j}=-D(\rho)\vec{\nabla}\rho ,$$

where the diffusion coefficient $D(\rho)$ may depend on the density or in simple cases be taken as a constant. In one dimension this leads to the equation

$$\frac{\partial\rho}{\partial t}=\frac{\partial}{\partial x}\left[D(\rho)\frac{\partial\rho}{\partial x}\right]+F(\rho) . \tag{1a}$$

In what follows we shall assume that

$$F(\rho)>0 \text{ in } (0,1), \quad F(0)=F(1)=0 , \tag{1b}$$

restrictions which are satisfied by several models. When the diffusion coefficient is constant and the additional requirement $F'(0)>0$ is satisfied, the asymptotic speed of propagation of localized small perturbations to the unstable state $u=0$ is bounded below and in some cases coincides [7] with the value $c_L=2\sqrt{F'(0)}$, which is obtained from considerations on the linearized equation [8]. However, when either $F'(0)=0$ or $D(\rho)$ is not a constant, no hint for the speed of propagation of disturbances can be obtained from linear theory alone. A common choice for the diffusion coefficient is a power law, a case with which we shall be concerned here. Therefore the equation that we study is

$$\frac{\partial\rho}{\partial t}=(\rho^m)_{xx}+F(\rho) , \tag{2a}$$

with

$$F(0)=F(1)=0, \quad F>0 \text{ in } (0,1). \tag{2b}$$

Aronson [2] and Aronson and Weinberger [7] have shown that the asymptotic speed of propagation of disturbances from rest is the minimal speed $c^*(m)$ for which there exist monotonic traveling fronts $\rho(x,t)=q(x-ct)$ joining $q=1$ to $q=0$. The equation satisfied by the traveling fronts is

$$(q^m)_{zz}+cq_z+F(q)=0 , \tag{3a}$$

with

$$q(-\infty)=1, \quad q>0, \quad q'<0 \text{ in } (-\infty,\omega), \quad q(\omega)=0 , \tag{3b}$$

where $z=x-ct$. The wave of minimal speed is sharp; that is, $\omega<\infty$ when $m>1$ [2].

An explicit solution is known [1,2] for the case $F(q)=q(1-q)$ and $m=2$; the wave form is given by

$$q(z)=(1-\tfrac{1}{2}e^{z/2})_+ ,$$

and it travels with speed $c^*(2)=1$ {here $[x]_+\equiv\max(x,0)$}. Recently the derivative dc/dm at $m=2$ has been calculated by two different methods. Its value is $-\frac{7}{24}$ [9,10]. Other exact solutions for different choices for m and F have been given in [11].

The purpose of this work is to give a variational characterization of the minimal speed $c^*(m)$ for Eq. (3) when $m>1$, and as a by-product for the case $m=1$ when $F'(0)=0$, both cases for which no information is obtained from linear theory. The case $m=1$ with $F'(0)>0$ has been studied elsewhere [13]. Lower bounds have been obtained on the minimal speed $c^*(m)$ [12]; the present results allow its exact calculation for arbitrary f.

Since the selected speed corresponds to that of a decreasing monotonic front, we may consider the dependence of its derivative dq/dz on q. Calling $p(q)=-q^{m-1}dq/dz$, where the minus sign is included so that p is positive, we find that the monotonic fronts are solutions of

$$p\frac{dp}{dq}-\frac{c^*}{m}p+\frac{1}{m}q^{m-1}F(q)=0 , \tag{4a}$$

with

$$p(0)=p(1)=0, \quad p>0 \text{ in } (0,1) . \tag{4b}$$

Although the wave of minimal speed is sharp and therefore $q'(0)<0$, by the definition of p, $p(0)=0$. We now show that the minimal speed $c^*(m)$ follows from a varia-

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tional principle whose Euler equation is Eq. (4a).

Let g be a positive function such that $h=-g'>0$. Multiplying Eq. (4a) by g/p and integrating we obtain after integration by parts,

$$\frac{c^*}{m}=\frac{\int_0^1\left[\frac{1}{m}q^{m-1}F(q)\frac{g(q)}{p(q)}+h(q)p(q)\right]}{\int_0^1 g(q)dq} . \tag{5}$$

By Schwarz's inequality, since q,F,g, and h are positive, we know

$$\frac{1}{m}q^{m-1}F\frac{g}{p}+hp\geq 2\left[\frac{1}{m}q^{m-1}Fgh\right]^{1/2} , \tag{6}$$

and therefore, replacing in Eq. (5) we have

$$c^*\geq 2\frac{\int_0^1\sqrt{mq^{m-1}Fgh}\,dq}{\int_0^1 gdq} \tag{7}$$

This bound has been already given by us [12]. We now show that it is always possible to find a $g(q)$ such that the equality in Eq. (6) and therefore also in Eq. (7) holds. We do so by explicit construction of such a function g. The equality in Eq. (6) holds if

$$\frac{1}{m}q^{m-1}F\frac{g}{p}=hp . \tag{8}$$

Let $v(q)$ be the positive solution of

$$\frac{v'}{v}=\frac{c^*}{mp} \tag{9a}$$

and choose

$$g=\frac{1}{v'} . \tag{9b}$$

We have then

$$\frac{v''}{v}=\frac{(v')^2}{v^2}-\frac{c^*}{mp^2}p'=\frac{c^*}{m^2p^3}q^{m-1}F(q) ,$$

where we have used Eq. (9a) to eliminate v' and Eq. (4a) to eliminate p'. Therefore,

$$h=-g'=\frac{v''}{(v')^2}=\frac{1}{mp^2}q^{m-1}Fg>0 , \tag{9c}$$

where we have made use of Eqs. (9a) and (9b). With this expression for h, we can see that Eq. (8) is satisfied. In addition we must check that g as we have defined it is such that its integral exists. In fact it exists and, moreover, one can always normalize g so that $g(0)=1$ and $g(1)=0$. From the definition of g we obtain

$$g(q)=\frac{mp(q)}{c^*}\exp\left[-\int_{q0}^{q}\frac{c^*}{mp}dq'\right] , \tag{9d}$$

where $0<q_0<1$. Since $p(1)=0$ and p is positive between 0 and 1 it follows that $g(1)=0$. At zero no divergence occurs, as we now show. Call $\hat{c}=c^*/m$ and $f(q)=q^{m-1}F(q)/m$. Then Eq. (4a) reads

$$pp'-\hat{c}p+f=0 , \tag{10a}$$

with

$$f(0)=f(1)=0 \text{ and } f'(0)=0 . \tag{10b}$$

For this case Aronson and Weinberger [7] have shown that $p(q)$ approaches the fixed point $q=0$ as $p=\hat{c}q=c^*q/m$. Then, from (9a) it follows that $v'/v\approx 1/q$ near zero, i.e., $v\approx Aq$ near zero. Choosing $A=1$ and using (9b) we have that $g(0)=1$. That $g(0)=1$ can also be derived from the explicit formula for $g(q)$ given in (9d) (through a careful analysis of the divergence of the integral near zero). Then the integral of g exists. We have shown then

$$c^*(m)=\max\left[2\frac{\int_0^1\sqrt{mq^{m-1}Fgh}\,dq}{\int_0^1 gdq}\right] , \tag{11}$$

where the maximum is taken over all functions g such that

$$g(0)=1,\quad g(1)=0 \text{ and } h=-g'>0 .$$

It is perhaps of some interest to verify explicitly that the Euler equation for the maximizing g is indeed Eq. (4a). Let us study the maximization of the functional

$$J_m(g)=2\int_0^1\sqrt{mq^{m-1}Fgh}\,dq$$

where $h=-g'>0$ subject to

$$\int_0^1 g(q)dq=1 .$$

The Euler equation for this problem is

$$\lambda+\left[\frac{mq^{m-1}Fh}{g}\right]^{1/2}+\frac{d}{dq}\left[\frac{mq^{m-1}Fg}{h}\right]=0 ,$$

where λ is the Lagrange multiplier. Using the expression given in Eq. (9c) for h we see that this is exactly Eq. (4a) with the Lagrange multiplier $\lambda=-c^*$.

As an application we shall consider the case $F(q)=q(1-q)$ and $m=2$ for which the exact solution is known. Take as the trial function $g(q)=(1-q)^2$. Then we obtain

$$c^*\geq 4\frac{\int_0^1 q(1-q)^2dq}{\int_0^1(1-q)^2dq}=1 ,$$

the exact value, which shows that this is the function g for which the maximum is attained. In addition, due to the existence of the variational principle we may use the Feynman-Hellman formula to calculate the dependence of $c^*(m)$ on parameters of F. We illustrate this by applying it to the calculation of dc^*/dm at $m=2$. Taking the derivative of Eq. (10) with respect to m we obtain

$$\frac{dc^*}{dm}=\frac{1}{\int_0^1 gdq}\int_0^1\frac{ghF}{\sqrt{mFq^{m-1}gh}}[q^{m-1}(1+m\log q)]dq .$$

Evaluating at $m=2$, with $g(q)=(1-q)^2$, we obtain

$$\frac{dc^*}{dm}(2)=3\int_0^1 q(1-q)^2(1+2\log q)dq=-\tfrac{7}{24} ,$$

the value previously obtained by other methods [9,10].

A fast estimation of the speed for other values of m can be obtained with simple trial functions. In Fig. 1 we show lower bounds for $F=q(1-q)$ using as trial functions $g_1=(1-q)^2$ and $g_2=(1-q)$. With the first trial function we have the exact value at $m=2$. The dotted line is the line of slope $-\frac{7}{24}$ that coincides with the tangent at $m=2$. For larger m a better estimate is obtained using g_2. The dashed line is the curve $\sqrt{2/m}$, which has been suggested by Newman [1] as the best fit to his numerical results. With better choice of trial functions the exact value can be approached arbitrarily close.

Finally we observe that the case $m=1$ when $F'(0)=0$ follows directly here. Repeating the procedure starting now from Eq. (10), one obtains

$$c^*=\max 2\frac{\int_0^1 \sqrt{Fgh}\,dq}{\int_0^1 g\,dq},$$

where the maximum is taken over all functions g such that

$$g(0)=1,\ g(1)=0,\ \text{and}\ h=-g'>0\ .$$

To show this we have used $v'/v=c^*/p$ and $g=1/v'$ and the asymptotic behavior described above.

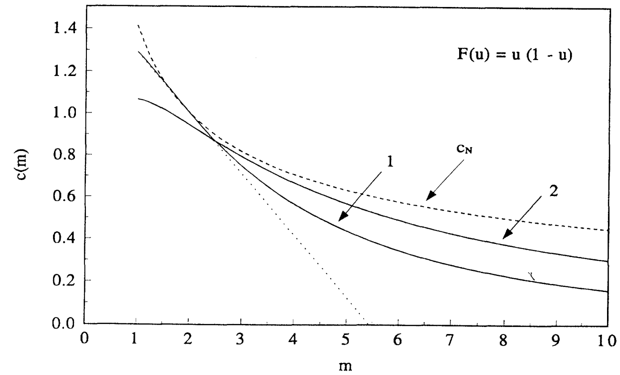

FIG. 1. Estimation of the asymptotic speed $c^*(m)$ for $F(u)=u(1-u)$. The dashed line corresponds to the fit $\sqrt{2/m}$ obtained previously from numerical integrations of the initial value problem. The solid lines are approximate values obtained using as trial functions $g_1=(1-u)^2$ and $g_2=1-u$. The dotted line is the calculated slope at $m=2$.

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