THE ASTRONOMICAL JOURNAL VOLUME 86, NUMBER 3 MARCH 1981

THE LARGE-SCALE STRUCTURE OF COMPRESSIBLE CONVECTION

M. C. DEPASSIER AND E. A. SPIEGEL

Astronomy Department, Columbia University, New York, New York 10027

Received 21 July 1980; revised 11 December 1980

ABSTRACT

We study convection of small but finite amplitude in a plane-parallel layer of perfect gas. The conductivity is assumed constant, hence the static state is polytropic. The heat flux on the boundaries is held fixed. When the polytropic index is not too large, the critical horizontal wave number for the onset of convection is zero and there is finite-amplitude instability. The finite-amplitude instability persists into the thin-layer limit provided that there are no geometrical limits to the horizontal scale of the convection. This result contradicts conclusions drawn from the strict Boussinesq approximation and it suggests that results based on that approximation are not generic for convection with flux prescribed on the boundaries. For all the Rayleigh numbers and layer thicknesses accessible to our amplitude expansions, convection with very large horizontal extent is expected to be prevalent at significant amplitudes. We suggest ways in which the nonlinear solutions found here may be useful in the interpretation of large-scale solar convection.

I. INTRODUCTION

While the occurrence of large-scale order in intensely turbulent flows is a familiar phenomenon, it still seems to require explanation in explicit cases. Hence several authors have speculated on the possible reasons for the appearance of large cell-like structures in the solar surface. The commonly accepted qualitative rationalization for the fact that these are not wholly disordered is that the eddy coefficients produced by small-scale turbulence lead to values for the effective stability parameters not greatly in excess of the critical values for instability. From this argument some authors have drawn scientific license to apply, however indirectly, results on small-amplitude convection to the sun. In the present work we likewise presume that there is vigorous small-scale turbulence throughout the solar convection zone and that the larger scales of motion are, on this account, only mildly unstable.

In this picture we would consider the solar convective zone, when viewed in the large, as having much higher thermal conductivity than the layers adjacent to it. It follows that from among the usual idealized thermal boundary conditions used in convection theory, we should prefer those in which heat flux rather than temperature is prescribed on the boundaries of the layer under study. A possible objection to this view has been mentioned by one of the referees, J.-P. Zahn. He points out that in the basic thermodynamics of a compressible fluid it is entropy that is advected. But we here follow the usual procedure of replacing the entropy advection by a combination of pressure and enthalpy advection. The pressure boundary conditions are replaced by the requirement that the upper and lower boundaries of the fluid are fixed in place, and boundary conditions on only the enthalpy are needed. Those, we suggest, correspond to fixed heat flux for the case of large-scale motion in the convection zone.

If this argument is accepted and we may be guided in our approach by the Boussinesq results of Hurle, Jakeman, and Pike (1967), then we may anticipate that motions with large horizontal scales should be prevalent in convection at large Rayleigh number. For the sun, the implied scales should greatly exceed the depth of the convective layer, which is approximately 200 000 km. By this standard, not only are the scales of the traditional granules (~2000 km) considered quite small, but even the supergranules (whose horizontal scale ~30 000 km) should be regarded as small. Therefore, if the calculations reported here apply at all to convection in the sun, they probably bear most directly on the giant cells, or on even larger structures that have not yet been recognized as convection in the usual sense.

To focus on the largest scales of motion, we go to the asymptotic extreme of zero horizontal wave number, ignoring the natural lower bound implied by the finite solar circumference. This simplification, which we hope is only mildly false, compensates for its deficiencies with enormous computational advantages, which permit us to perform finite-amplitude calculations. The idea is based on the fact that the linear results of Hurle, Jakeman, and Pike may be simply recovered by an asymptotic expansion in horizontal wave number (e.g., Childress, Levandowsky, and Spiegel 1975). Moreover, it is then possible to extend the procedure to the nonlinear problem by assuming an appropriate relation between amplitude and horizontal scale, as in shallow water theory. The particular scales chosen depend to some extent on the kind of solution sought and we here begin with one that works well in an example of bioconvection (Childress and

Spiegel 1978). This example provides one of the few laboratory simulations of highly stratified convection, and is one to which the flux boundary conditions used here apply. Other nonlinear treatments of convection with fixed flux are available, including a mean-field study by Van der Borght (1974) and a finite-amplitude study for the Boussinesq case by Busse and Riahi (1980). Since the inertial terms are more important in our scaling than the mean-field terms, no direct comparison with the first case is made here; the Boussinesq limit is discussed in Sec. VI, where the work reported by Chapman and Proctor (1980) provides a useful guide. We shall see how the Boussinesq results are not generic for real convection at small amplitude if the geometry of the problem is compatible with sufficiently large horizontal scales.

We should recall that the first complete nonlinear compressible convection calculations were Graham's (1975, 1977) numerical solutions, whose most striking feature was a lively time dependence in three dimensions. Here we are restricting ourselves to two-dimensional motion and are able to find only steady solutions. Further comparison is difficult since Graham considered only fixed temperatures on the boundaries. This choice of boundary conditions was preferred also by Massaguer and Zahn (1980; Zahn 1979), who made a truncated modal analysis of anelastic convection (and to whom we refer the reader for a wider survey of the literature of compressible convection theory). Here too we make no direct comparison to their work, but we do make use of Massaguer and Zahn's observations concerning the structure of the linear solutions to interpret qualitatively the results described in Sec. III.

II. FORMULATION

Consider a layer of ideal gas confined between the two planes $z = 0$ and $z = d$. A uniform gravitational field with constant acceleration g acts in the positive z direction. We assume, as in previous studies, that d is fixed. The thermal conductivity K, viscosity μ, specific heat C_p, and gas constant $\mathcal{R}$ are each taken as constant. Under these assumptions the static atmosphere has a simple structure, as we shall recall presently. Its description includes two integration constants which we treat as parameters. We shall take them to be $\tilde{\rho}_1$, the static density at $z = d$, and $\tilde{T}_0$, the static temperature at $z = 0$. We have $\tilde{\rho}_1$ in mind as the parameter that varies as we sweep through a family of static models to see the effect of varying the convective instability, while $\tilde{T}_0$ is, as we shall see, a measure of the deviation of the conditions from those of normal laboratory convection.

Let us adopt d as the unit of length, $\tilde{\rho}_1 C_p d^2/\mathrm{K}$ as the unit of time, $\tilde{\rho}_1$ as the unit of density, and Fd/K as the unit of temperature, where F is the constant flux of heat through the layer. Let $\tilde{\rho}$, $\tilde{p}$, and $\tilde{T}$ be the density, pressure, and temperature and let $\mathbf{u}$ be the velocity. The equations of motion are written in these units as

$$\tilde{\rho}_t + \nabla \cdot (\tilde{\rho}\mathbf{u}) = 0, \tag{2.1}$$

$$\tilde{\rho}[\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}] + \nabla\tilde{p} = \Lambda\tilde{\rho}\hat{\mathbf{k}} + \sigma\nabla^2\mathbf{u} + \frac{1}{3}\sigma\nabla(\nabla\cdot\mathbf{u}), \tag{2.2}$$

$$\Lambda\tilde{\rho}[\tilde{T}_t + (\mathbf{u}\cdot\nabla)\tilde{T}] - \frac{(\gamma-1)(m+1)}{\gamma} \times [\tilde{p}_t + (\mathbf{u}\cdot\nabla)\tilde{p}] = \Lambda\nabla^2\tilde{T} + \frac{(\gamma-1)(m+1)}{\gamma}\sigma\Phi, \tag{2.3}$$

and

$$\tilde{p} = \Lambda\tilde{\rho}\tilde{T}/(m+1), \tag{2.4}$$

where

$$\Phi = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}\right)\frac{\partial u_i}{\partial x_j}, \tag{2.5}$$

and where differentiation with respect to time is denoted by the subscript t. The four following nondimensional parameters have appeared explicitly:

$$\Lambda = gd^3(\tilde{\rho}_1 C_p)^2/\mathrm{K}^2, \tag{2.6}$$

$$\gamma = C_p/(C_p - \mathcal{R}), \tag{2.7}$$

$$\sigma = C_p\mu/\mathrm{K}, \tag{2.8}$$

$$m + 1 = g\mathrm{K}/F\mathcal{R}, \tag{2.9}$$

and $\hat{\mathbf{k}}$ is a unit vector. Three of the nondimensional parameters, the Prandtl number (σ), the polytropic index (m), and the ratio of specific heats (γ), are the usual ones used in compressible convection. The fourth (Λ) is related to a parameter known as the Galileo number (Λ/σ^2). It is fixed by the choice of the density at the bottom of the layer, which is one of the integration constants. We shall seek the value of Λ for which steady convection of given amplitude and scale occurs. However, we shall also introduce the Rayleigh number as the more interesting parameter for discussion of results.

We may write at once the static solution

$$T_s = z + z_0, \tag{2.10a}$$

$$\rho_s = \left(\frac{z+z_0}{1+z_0}\right)^m, \tag{2.10b}$$

$$p_s = \frac{\Lambda(1+z_0)}{m+1}\left(\frac{z+z_0}{1+z_0}\right)^{m+1}, \tag{2.10c}$$

where the integration constant z_0 is just the surface temperature $\tilde{T}_0$ in nondimensional form. To the extent that this layer is like an outer convective zone of a star, z_0 is approximately the effective temperature, and should be controlled by radiative boundary conditions. Here we regard z_0 as a specified parameter of the layer.

Now we introduce the Rayleigh number:

$$R = \frac{\Lambda(1 + m - \gamma m)}{\sigma\gamma(1+z_0)}. \tag{2.11}$$

In this expression for R the state variables that enter are evaluated at the bottom of the layer; another choice is used in Sec. VI.

The Rayleigh number is conveniently used in laboratory convection because one may vary it by changing one of its factors and yet keep the other important nondimensional quantities fixed. This is less true in compressible and anelastic convection. For the polytrope, for example, R depends on m. If we change m we alter the static structure of the model. But we want to see what happens when we vary the Rayleigh number for a given static model. Hence we want to vary R while keeping m fixed, and we may do this by varying the factor Λ. We shall temporarily treat Λ as a governing parameter in the formal development since that simplifies the appearance of some formulas.

We consider only two-dimensional motion with velocity components u and w. On the boundaries these satisfy the conditions

$$\tilde{\rho} w = 0 \tag{2.12}$$

and

$$\frac{\partial u}{\partial z} = 0. \tag{2.13}$$

(Most of the calculations reported here have been done with the so-called rigid boundaries as well, with no changes in the qualitative results.) The condition that flux is fixed on the boundaries is

$$\frac{\partial \tilde{T}}{\partial z} = 1. \tag{2.14}$$

Conditions (2.12)–(2.14) apply at $z = 0$ and 1.

Together m and z_0 measure the degree to which a given model deviates from the conditions in which the Boussinesq approximation holds good. The former enters in the part of R that directly reflects the static structure through the quantity

$$\Delta = (1 + m - \gamma m)/\gamma, \tag{2.15}$$

which tells us the static entropy gradient. The other parameter, z_0, is simply expressed on recalling that it is the surface temperature $\tilde{T}_0$ divided by the unit of temperature, Fd/K. Then from (2.9) we obtain

$$z_0 = (m+1)H_p/d; \tag{2.16a}$$

$$H_p = \mathcal{R}\tilde{T}_0/gd. \tag{2.16b}$$

We shall see in Sec. VI how these parameters figure in the thin-layer and Boussinesq limits.

Finally, for the deviations of thermodynamic quantities from their hydrostatic values, we introduce the notation

$$\rho = \tilde{\rho} - \rho_s,$$

$$p = \tilde{p} - p_s,$$

and

$$T = \tilde{T} - T_s. \tag{2.17}$$

III. LINEAR THEORY

The stability of the static configuration has been considered in some detail for the case when the boundary temperatures are specified, but there does not seem to have been any extensive examination of the case with the fluxes fixed on the boundaries. Yet this problem may be treated quite easily by numerical methods. In particular, with the help of N. H. Baker's GNR1, a program for solving two-point boundary value problems of ordinary differential equations, we have constructed steady numerical solutions of the linear stability problem. We digress briefly now to indicate some features of these results by way of introduction of the general problem.

For the moment, we assume that $\mathbf{u}$, T/T_s, ρ/ρ_s, and p/p_s all have small magnitude and we neglect nonlinear terms in these quantities. The resulting linear equations may be reduced to equations for w and T alone and, for steady convection, we may seek solutions of the form

$$w = \mathcal{A}\mathcal{W}(z)\cos ax, \quad T = \mathcal{A}\Delta\Theta(z)\cos ax, \tag{3.1}$$

where $\mathcal{A}$ is a (small) constant amplitude and $\mathcal{W}$ and Θ are the solutions of the linear equations (e.g., Spiegel 1965)

$$(D^2 - a^2)^2\,\Theta = \rho_s\mathcal{W} \tag{3.2}$$

and

$$\begin{aligned}(z+z_0)^4(D^2-a^2)^2\mathcal{W} &- (z+z_0)^3D(D^2-a^2)\mathcal{W} \\ &+ m(m+4)\,[(z+z_0)^2D^2 + 2(z+z_0)D - 2]\mathcal{W} \\ &+ \frac{1}{3}ma^2(4m+7)(z+z_0)^2\mathcal{W} \\ &= -a^2R(1+z_0)(z+z_0)^4\rho_s\Theta/T_s,\end{aligned} \tag{3.3}$$

and where the operator D indicates differentiation with respect to z. From the boundary conditions prescribed in Sec. II, we obtain

$$\mathcal{W} = 0 \tag{3.4a}$$

$$D\Theta = 0, \tag{3.4b}$$

$$D^2\mathcal{W} + \frac{m}{z+z_0}D\mathcal{W} = 0, \tag{3.4c}$$

on $z = 0$ and 1. To obtain (3.4c) we used the condition $z_0 \neq 0$.

As in Boussinesq theory neither σ nor γ appears explicitly and we have simply to find R, $\mathcal{W}$, and Θ for various values of m, z_0, and a. We have done the relevant calculations for values of m up to about 40 and for values of z_0 ranging from 0.5 to 25. For all cases considered, R as a function of a is stationary at $a = 0$. There is a curve in the m-z_0 plane that is well approximated by

$$m = 9 + 13z_0, \tag{3.5}$$

and which divides the plane into two regions: on the side of large m, R has a relative maximum at $a = 0$, and on the other side it has a relative minimum there. Astro-

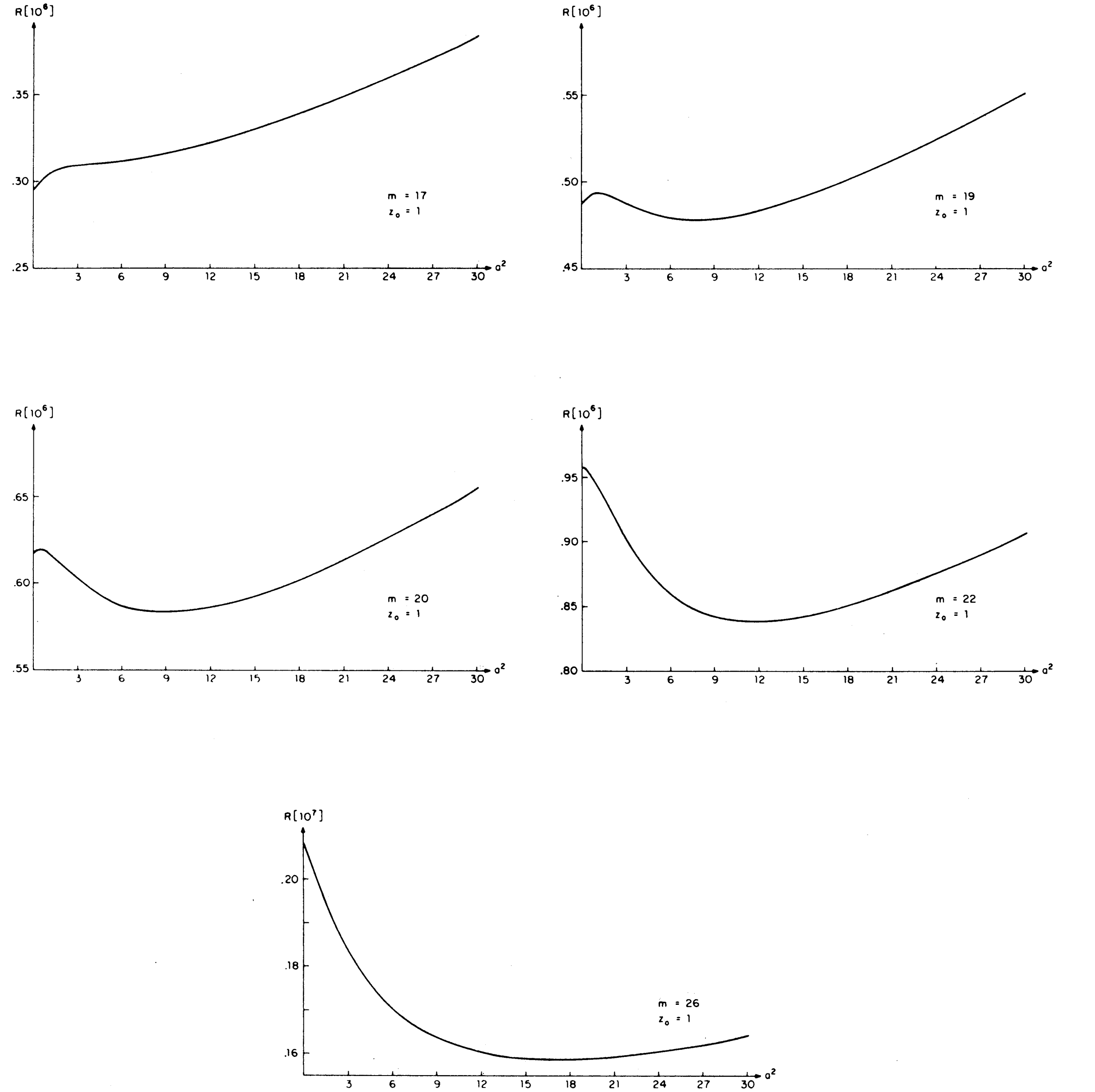

FIG. 1. Values of the Rayleigh number for which linear theory yields steady solutions, plotted in units of 10^6 vs the square of the horizontal wave number for various values of m and $z_0 = 1$.

physically, the latter is more relevant, and the pertinent results are very close to those obtained analytically as described below. However, the results for large m may interest those trying to understand the general theory.

For fixed z_0 and m not large, R has a minimum at $a = 0$ and increases monotonically with a. For a larger value of m, yet a bit below the value indicated by the curve defined by (3.5), a second relative minimum appears at a finite value of a. As m increases further, this minimum deepens and it typically becomes an absolute minimum before the value of m indicated by (3.5) is reached. For the values of z_0 we have studied numerically, the critical wave number becomes discontinuously nonzero when a certain value of m is exceeded. This behavior is illustrated in Fig. 1, which shows R as a function of a for $z_0 = 1$ for several values of m, beginning at $m = 17$, where R first ceases to be monotonic.

Such a transition from zero to finite critical wave number was found in a problem in Boussinesq convection by Poyet (1980). He considered the case where the layer of convecting fluid was bounded by plates of finite thickness. The flux was given on the outer boundaries of these plates. The variation of a_c as function of plate thickness reported by Poyet is very similar to the de-

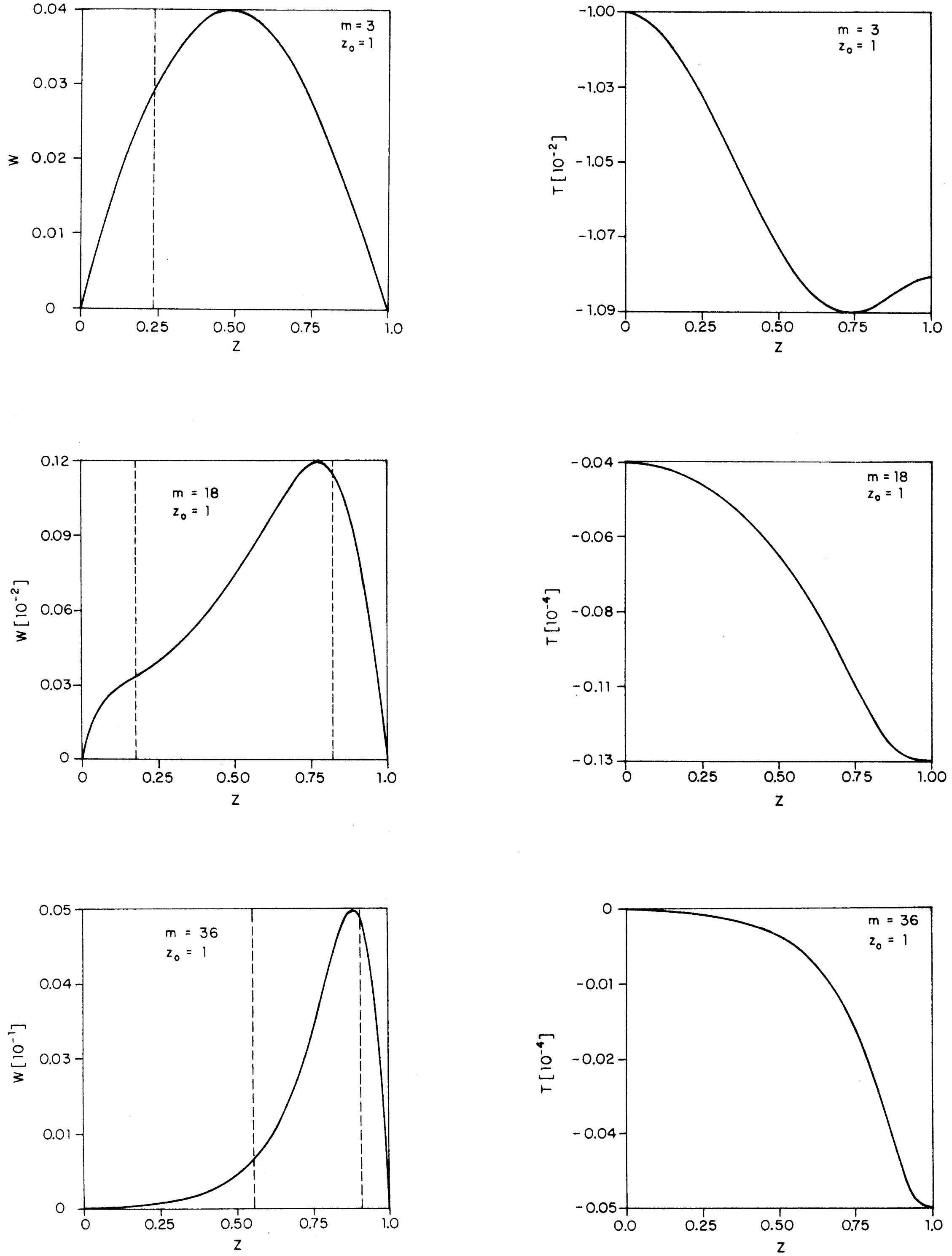

FIG. 2. Variation of vertical velocity and temperature perturbation with depth for solutions of (3.2) and (3.3). The dashed vertical lines border the sublayer in which buoyancy reversal occurs. It is in the upper layer for $m = 3$ and in the central layer for the higher values.

pendence on m seen here. The similarity points to a qualitative explanation.

Massaguer and Zahn (1980; Zahn 1979) have found that for compressible convection the sign of the density perturbation may change with height in a convecting fluid even though the sign of the vertical velocity remains independent of height. In the case they studied, with temperature fixed on the boundaries, the sublayer in which buoyancy reversal occurs is at the top of the fluid. In the present case, a similar phenomenon occurs, for m near the value given by (3.5), but, as m is raised further, the region in which buoyancy reversal takes place moves ever more deeply into the body of the fluid. For the highest values of m that we have studied, the reversed layer separates two normal layers. Eigenfunctions are shown in Fig. 2 for the case $z_0 = 1$ and for various values of m. The dashed vertical lines separate out the sublayers in which buoyancy reversal occurs, as explained in the caption. The transition to finite critical wave number is associated with the development of significant buoyancy reversal. The effect is loosely analogous to that found by Poyet (which in fact was studied for that reason). The sublayer in which buoyancy reversal is found is like a layer in which penetrative convection is occurring and it is modeled by the solid plates of Poyet's discussion.

As m increases above the value indicated by (3.5), the critical wave number increases monotonically with m. It appears that the preferred horizontal scale is comparable to the thickness of that portion of the layer not affected by buoyancy reversal. Indeed, if we multiply the fractional thickness of this part of the layer into a_c, we find that the result tends to a constant as m increases.

The values of m for which the transitions to finite a_c occur for polytropes are larger than those normally encountered in stellar models, for the values of z_0 which we have studied carefully. Since none of these considerations apply unless $\Delta > 0$, they generally can become interesting only when γ is rather close to unity. It is more usual to expect the physically interesting values of m to be of the order of unity. Then the results for the lower portion of the m-z_0 plane apply. For this case the expansions developed here are applicable and we rely on them for our discussion. In particular, they reveal finite-amplitude instability for m and z_0 both of the order of unity. Hence detailed numerical results for the linear problem are perhaps not so vital and we have refrained from presenting many here. In this spirit we close this section by illustrating, in Fig. 3, the way that the critical Rayleigh number depends on m and z_0. For all the values shown, the critical horizontal wave number is zero.

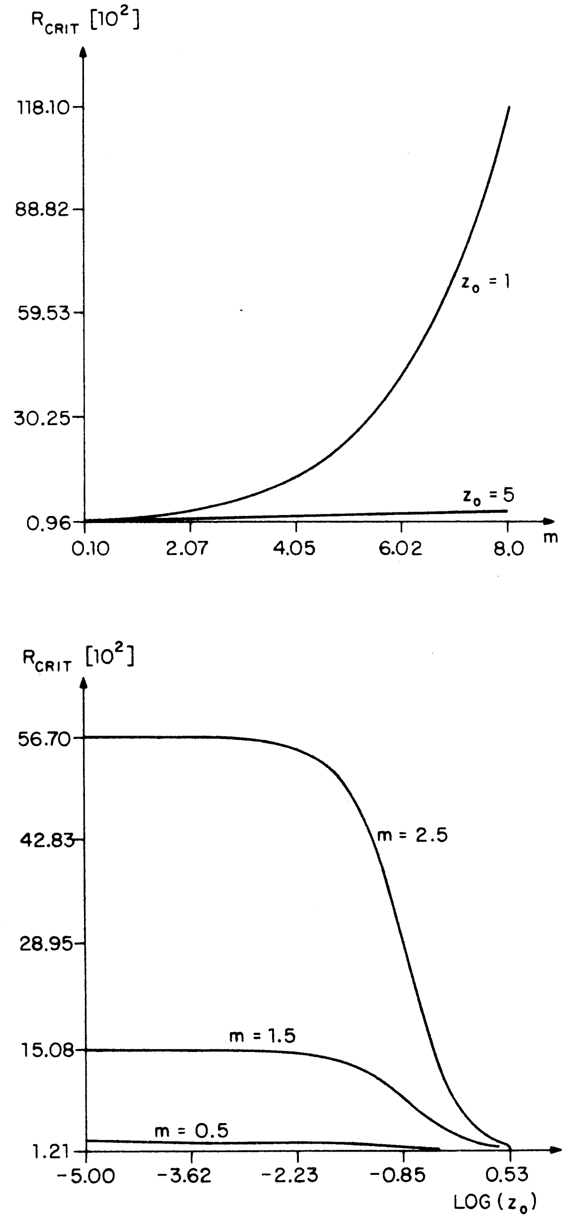

FIG. 3. The critical Rayleigh number for a range of values of m and z_0. For very low values of z_0, the results of the analytical calculations were used; where both analytical and numerical results are available, they agree to 0.2%.

IV. ASYMPTOTIC DEVELOPMENT

a) Scaling

We restrict our attention now to large horizontal scales and seek to rescale the equations much as in the theory of long waves. Let the ratio of the layer thickness to the horizontal scale of motion be the small parameter $\epsilon^{1/2}$. From the continuity equation we conclude that the vertical velocity is about $\epsilon^{1/2}$ times the horizontal velocity, hence that the motion is nearly hydrostatic. Deviations from radiative (conductive) equilibrium are due mainly to vertical advection of the mean temperature and horizontal diffusion. If these are to balance, the vertical velocity must be smaller than the temperature perturbation by a factor of ϵ, since the horizontal diffusion has two x derivatives. The problems that remain open are the choices of time scale and of the amplitude of the convective disturbance. In each case we choose these so as to bring nonlinear terms and terms with time derivatives into the development at the lowest order at which they can be handled, but we have no compelling physical reasons for such a procedure. It does, however, suggest the following scalings which lead to a self-consistent asymptotic development for $\epsilon \to 0$.

We introduce the new independent variables

$$\xi = \epsilon^{1/2} x, \tau = \epsilon^2 t, \quad (4.1)$$

where x is the unscaled horizontal coordinate, and the velocity components are then scaled according to the rules

$$w = \epsilon W,\ u = \epsilon^{1/2} U. \qquad (4.2)$$

We introduce (4.1) and (4.2) into the equations of Sec. II, let the subscripts z, τ, and ξ denote differentiation, and obtain

$$\epsilon\rho_\tau + [(\rho_s+\rho)U]_\xi + [(\rho_s+\rho)W]_z = 0, \qquad (4.3)$$

$$(\rho_s+\rho)(\epsilon^2 U_\tau + \epsilon U U_\xi + \epsilon W U_z) + p_\xi = \sigma\left(U_{zz} + \frac{4}{3}\epsilon U_{\xi\xi} + \frac{1}{3}\epsilon W_{z\xi}\right), \qquad (4.4)$$

$$(\rho_s+\rho)(\epsilon^3 W_\tau + \epsilon^2 U W_\xi + \epsilon^2 W W_z) + p_z = \Lambda\rho + \sigma\left(\frac{4}{3}\epsilon W_{zz} + \frac{1}{3}\epsilon U_{\xi z} + \epsilon^2 W_{\xi\xi}\right), \qquad (4.5)$$

$$\Lambda(\rho_s+\rho)(\epsilon^2 T_\tau + \epsilon U T_\xi + \epsilon W T_z + \epsilon W) - \frac{(m+1)(\gamma-1)}{\gamma}(\epsilon^2 p_\tau + \epsilon U p_\xi + \epsilon W p_z + \epsilon\Lambda\rho_s W) - \frac{(m+1)(\gamma-1)}{\gamma}\sigma\left[\epsilon^2 U_\xi\left(\frac{4}{3}U_\xi - \frac{2}{3}W_z\right) + \epsilon^2 W_z\left(\frac{4}{3}W_z - \frac{2}{3}U_\xi\right) + \epsilon(U_z + \epsilon W_\xi)^2\right] = \Lambda(T_{zz} + \epsilon T_{\xi\xi}), \qquad (4.6)$$

and

$$p = \Lambda(\rho_s T + T_s\rho + \rho T)/(m+1). \qquad (4.7)$$

This much is standard in shallow water theory: the situation is nearly hydrostatic and the motion is slow. But when we proceed to carry out expansions in ϵ, considerations involving the thermodynamic quantities, which do not occur in conventional shallow water theory, must enter. For example, the amplitude of the relative temperature variation $T/\tilde{T}$ is measured by ϵ. That is, the perturbation temperature has already been scaled like the static temperature, hence by z_0. The scale of T is therefore ϵz_0. We must decide on the order of ϵz_0 and we consider first the case $z_0 = O(1)$. However, when we come to examine the limit of thin layers, we shall treat z_0 as large and we shall attach special interest to the case where ϵz_0 is of the order of unity.

For the standard compressible case with $z_0 = O(1)$, we seek the following asymptotic sequences:

$$U = \epsilon U_1 + \epsilon^2 U_2 + \cdots,\quad W = \epsilon W_1 + \epsilon^2 W_2 + \cdots,$$
$$\rho = \epsilon\rho_1 + \epsilon^2\rho_2 + \cdots,\quad p = \epsilon p_1 + \epsilon^2 p_2 + \cdots,$$
$$T = \epsilon T_1 + \epsilon^2 T_2 + \cdots,$$
$$\Lambda = \Lambda_0 + \epsilon\Lambda_1 + \epsilon^2\Lambda_2 + \cdots. \qquad (4.8)$$

b) $O(\epsilon)$

To first order in ϵ we find

$$(\rho_s U_1)_\xi + (\rho_s W_1)_z = 0, \qquad (4.9a)$$
$$p_{1\xi} = \sigma U_{1zz}, \qquad (4.9b)$$
$$p_{1z} = \Lambda_0\rho_1, \qquad (4.9c)$$
$$T_{1zz} = 0, \qquad (4.9d)$$

and

$$p_1 = \Lambda_0(\rho_s T_1 + T_s\rho_1)/(m+1). \qquad (4.9e)$$

The boundary conditions become

$$W_1 = 0,\quad U_{1z} = 0,\quad T_{1z} = 0 \qquad (4.10)$$

on both $z = 0$ and $z = 1$.

The solution of this system is of the form

$$T_1 = f(\xi,\tau), \qquad (4.11a)$$
$$p_1 = \Lambda_0 f P_1(z), \qquad (4.11b)$$
$$\rho_1 = f P_1'(z), \qquad (4.11c)$$
$$U_1 = \frac{\Lambda_0}{\sigma} f_\xi \mathcal{U}_1(z), \qquad (4.11d)$$

and

$$W_1 = \frac{\Lambda_0}{\sigma} f_{\xi\xi}\mathcal{W}_1(z), \qquad (4.11e)$$

where the arbitrary function f and the constant Λ_0 are yet to be determined, while P_1, $\mathcal{U}_1$, and $\mathcal{W}_1$ are polynomials in z that are given in Appendix A.

c) $O(\epsilon^2)$

In second order the equations become

$$(\rho_s U_2)_\xi + (\rho_s W_2)_z = -\rho_{1\tau} - (\rho_1 W_1)_z - (\rho_1 U_1)_\xi, \qquad (4.12a)$$

$$p_{2\xi} = \sigma U_{2zz} + \frac{1}{3}\sigma(4U_{1\xi\xi} + W_{1z\xi}), \qquad (4.12b)$$

$$p_{2z} = +\frac{1}{3}\sigma(4W_{1zz} + U_{1z\xi}) + \Lambda_0\rho_2 + \Lambda_1\rho_1, \qquad (4.12c)$$

$$T_{2zz} = -T_{1\xi\xi} + (m + 1 - m\gamma)\rho_s W_1/\gamma, \qquad (4.12d)$$

and

$$p_2 = \Lambda_0(\rho_s T_2 + T_s\rho_2 + \rho_1 T_1)/(m+1) + \Lambda_1(\rho_s T_1 + T_s\rho_1)/(m+1). \qquad (4.12e)$$

The boundary conditions in this order are

$$\rho_s W_2 + \rho_1 W_1 = 0,\quad U_{2z} = 0,\quad T_{2z} = 0, \qquad (4.13)$$

on both $z = 0$ and $z = 1$.

The solvability condition in this procedure is straightforward on account of the fixed-flux boundary conditions. We integrate (4.12d) over z and observe that because of the third of conditions (4.13) the left side mush vanish. If we substitute on the right from (4.11a) and (4.11e), we obtain the condition

$$(1 + z_0)R^{(0)} \int_0^1 \rho_s(z)\mathcal{W}_1(z)dz = 1, \quad (4.14)$$

where

$$R^{(n)} = \Delta\Lambda_n/[\sigma(1 + z_0)]. \quad (4.15)$$

In this way we find $R^{(0)}$ as a function of z_0 and m. Unfortunately this dependence is too complicated to be included in the text and has had to be displaced to Appendix A, as are other details of the solution in this order. It may be observed here though that as $z_0 \to \infty$ for fixed m, $R^{(0)} \to 120$, in agreement with the Boussinesq result of Hurle, Jakeman, and Pike (1967).

d) $O(\epsilon^3)$

To complete the solution to first order, all we need to do in third order is to evaluate the solvability condition. This leads to a partial differential equation for the function f, which has thus far remained arbitrary. The calculation need not be spelled out here. It consists in integrating the third-order contributions from (4.6) over z, and making use of the fact that T_{3z} must vanish on the boundaries. This is the same procedure that led us to (4.14) in second order. Accordingly, we may stop the expansion at this point and turn to the study of the evolution equation for f, the planform function. However, we should interject at this point that the equation for f, except for the values of certain numerical coefficients, is precisely the same as one obtained earlier in the study of a model of bioconvection (Childress and Spiegel 1978).

V. THE EVOLUTION EQUATION

a) *The Equation*

The function $f(\xi,\tau)$ is the temperature fluctuation to first order and it characterizes the horizontal structure of the convection. It is more analogous to the waveforms of long water waves than to the planform function of convection theory with fixed boundary temperatures in that it is governed by a nonlinear partial differential equation, namely

$$\Omega f_\tau + rf_{\xi\xi} - \nu(f^2)_{\xi\xi} + \kappa f_{\xi\xi\xi\xi} = 0, \quad (5.1)$$

where

$$r = \Lambda_1/\Lambda_0 = R^{(1)}/R^{(0)}, \quad (5.2)$$

and the positive constants Ω, ν, and κ are known functions of m, z_0, and γ as defined in Appendix B. Values for z_0 equal to zero and infinity can be found in Appendix B and Sec. VI, respectively.

b) *Linear Theory*

Considering first the linearized version of (5.1), we examine solutions of the form

$$f = e^{\eta\tau} \cos\alpha\xi. \quad (5.3)$$

Then we find the dispersion relation

$$\eta = (r - \alpha^2\kappa)\alpha^2/\Omega. \quad (5.4)$$

The physical content of (5.4) depends on the sign of κ, and is seen most clearly in an examination of the condition for marginal stability, $\eta = 0$. This leads to the following value of the Rayleigh number:

$$R = R_0 \equiv R^{(0)} (1 + a^2\kappa), \quad (5.5)$$

where a is the usual (unscaled) horizontal wave number,

$$a = \alpha\epsilon^{1/2}. \quad (5.6)$$

For $\kappa > 0$ the value of the Rayleigh number for marginal stability increases with a, much as in Boussinesq convection theory. Then $R^{(0)}$ is the critical value of the Rayleigh number and the critical value of the wave number is zero. For $\kappa < 0$, R_0 has a relative maximum at $a = 0$. The critical wave number is not determined in this order and it is apparently no longer zero. Hence the condition

$$\kappa = 0 \quad (5.7)$$

plays a special role in the theory.

The curve in the m-z_0 plane described by condition (5.7) corresponds to (3.5) in the numerical results and the agreement is very good. As before, in the upper region of the m-z_0 plane, the critical wave number is finite. In the lower portion, $R_0(a)$ has a minimum at $a = 0$, but there may be a second relative minimum not revealed by the present calculations. In any case, for $\kappa > 0$ and $R > R_0$, instability occurs in a band of wave numbers with a^2 between 0 and $(R - R_0)/\kappa R_0$. The wave number of maximum growth rate is given by $\alpha^2 = r/2\kappa$.

For negative κ we would need to go to higher order to determine the critical wave and Rayleigh numbers. We therefore do not explore the linear theory further here. However, before turning to the nonlinear solutions, we should observe that for the calculations on linear stability theory the expansion procedure seems qualitatively good. The values obtained for condition (5.7) and for $R^{(0)}$ agree well with those from the numerical solutions.

c) *Steady Nonlinear Solutions*

We consider next the steady nonlinear solutions of (5.1). An important difference with the time-dependent case, such as we just considered in linear theory, is that steady motion can occur for only certain values of R;

hence, even in the nonlinear case, the steady problem is an eigenvalue problem for r. We treat this problem in the domain $0 \leq \xi \leq 2\pi/\alpha$ and assume that at the edges of this domain there are no tangential stresses or normal heat fluxes. Then we are led to the boundary conditions

$$f_\xi = 0, \qquad f_{\xi\xi\xi} = 0 \tag{5.8}$$

on $\xi = 0, 2\pi/\alpha$.

If we average (5.1) over the domain of ξ, we find that

$$\overline{f_\tau} = 0, \tag{5.9}$$

where the overbar denotes horizontal average. Hence $\overline{f}$ is a constant whose value must be zero if we are to conserve mass to order ϵ. To see why, we integrate (4.11c) over the volume. Since the volume is kept constant, this integral must vanish. As we may see from Appendix A, the z integral of ρ_1 does not vanish, and we must require that $\overline{f}$ be zero.

We may immediately perform two integrations of the steady-state form of (5.1) to obtain

$$\kappa f'' - \nu(f^2 - \overline{f^2}) + rf = 0. \tag{5.10}$$

The solution of this equation may be expressed in terms of an elliptic function and a parameter k $(1 \geq k \geq 0)$ as

$$f = \frac{6k^2\alpha^2\kappa \mathrm{K}^2}{\pi^2\nu}\left(\mathrm{sn}^2\,(\alpha \mathrm{K}\xi/\pi, k) - \frac{\mathrm{K} - E}{k^2\mathrm{K}}\right), \tag{5.11}$$

where K and E are complete elliptic integrals and

$$r = \frac{4\alpha^2\kappa\mathrm{K}}{\pi^2}\,[3E - (2 - k^2)\mathrm{K}]. \tag{5.12}$$

Though this provides a closed analytic expression for the finite-amplitude steady solutions for compressible convection, its form is deceptively simple. The nature of the asymptotic development is such that the effects of nonzero wave number and finite amplitude are intertwined, whereas we normally think of them as separate. But they can be separated when $k \to 0$. In that limit

$$\mathrm{K} = \frac{1}{2}\pi\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \cdots\right),$$

$$E = \frac{1}{2}\pi\left(1 - \frac{1}{4}k^2 + \frac{3}{64}k^4 + \cdots\right).$$

In addition, $\mathrm{sn}(u,0) = \sin u$. Hence, for small k, we have

$$\epsilon f = \mathcal{A}\cos ax, \tag{5.13}$$

where the amplitude is given by

$$\mathcal{A} = -3\kappa k^2 a^2/2\nu. \tag{5.14}$$

Now the last of expansions (4.8), when combined with (4.15), (5.5), and (5.6), becomes

$$R = R_0 + \epsilon(r - \kappa\alpha^2)R^{(0)} + \cdots. \tag{5.15}$$

For k near unity, amplitudes are large and it is clear that $r < 0$. For small amplitudes, k is small and we have the approximation

$$r = \alpha^2\kappa\left(1 - \frac{3}{32}k^4 + \cdots\right). \tag{5.16}$$

Thus we find

$$R = R_0 - \frac{3}{32}k^4\kappa a^2 R^{(0)} + \cdots. \tag{5.17}$$

We may rewrite this in the form of a conventional amplitude expansion for fixed a as

$$R = R_0 + \mathcal{A}^2 R_2 + \cdots, \tag{5.18}$$

where

$$R_2 = -\frac{\nu^2}{24a^2\kappa}R^{(0)}. \tag{5.19}$$

The expressions (5.13), (5.18), and (5.19) are like those arising in the conventional finite-amplitude treatment at fixed horizontal wave number, with the first finite-amplitude correction to R being of the order of the square of the amplitude. However, as we go toward the critical value of the wave number, $a = 0$, we see that R_2 diverges. It was this difficulty (or, at least, its analog in bioconvection) that led to the present approach. Though R_2 diverges at the critical wave number, we have been able to find a relation between amplitude and horizontal scale that makes (5.18) convergent.

From (5.19) we see that κ determines the sign of R_2. In particular, for positive κ the bifurcation to convection is subcritical; that is, for $\kappa > 0$ we have finite-amplitude instability. This is not without precedent since we know that even weakly non-Boussinesq systems may bifurcate subcritically for the case of fixed temperatures on the boundaries (Busse 1978). But in the fixed-temperature case the degree of subcriticality depends mainly on the size of the deviation from the strict Boussinesq approximation, whereas here it is extremely sensitive to the horizontal scale of motion. In the limit of thin layers, $z_0 \to \infty$, nothing qualitative about (5.19) is changed; κ simply takes a specific value of about 1/5. This suggests that there is subcritical bifurcation for all layer thickness at the largest horizontal scales. Moreover, this result is not a special feature of polytropes or compressibility since it occurs also in an incompressible model for bioconvection (Childress and Spiegel 1978).

On the other hand, if one tries to do the same kind of calculation starting from the strict Boussinesq equations, one discovers different results: the bifurcation to convection is always supercritical and the solutions appear in the form of a conventional amplitude expansion. The second of these differences between the two approaches was resolved by Chapman and Proctor (1980), who made the temperature perturbation of the order of unity in the Boussinesq case. They obtained a nonlinear evolution equation as in the non-Boussinesq examples, but the bifurcation remains supercritical in the strict

Boussinesq case. Equation (5.1) does not reduce to the evolution equation found by Chapman and Proctor when $z_0 \to \infty$. We conclude that the limits $\epsilon \to 0$ (the small-amplitude limit) and $z_0 \to \infty$ (the thin-layer limit) do not commute.

These considerations suggest that a uniformly valid theory of small-amplitude motions for a thin layer with fixed flux ought to be based on a two-parameter expansion. But it is best at this stage to try to consider a particular relation between the parameters so as to isolate a case of special significance in some sense yet to be decided. In Sec. VI, we seek such a distinguished limit and we are led to an evolution equation which reproduces qualitative features of both Eq. (5.1) and the Boussinesq case.

VI. THIN LAYERS

For fixed m, z_0 tends to infinity as the layer thickness tends to zero. This is the limit of normal laboratory convection and it has naturally figured importantly in the development of the subject. For a perfect gas, the expansion coefficient is the inverse of the temperature, hence it is inversely proportional to $1/z_0$. Other things being equal, the fluid becomes increasingly incompressible as z_0 becomes larger. But the layer thickness is not the only important length. Therefore, we should not expect that the limit of thin layers will necessarily lead to the Boussinesq approximation.

From (2.10a) we may expect T to be $O(z_0)$, hence, before making expansions in ϵ, we must decide on the relation between ϵ and z_0 if we are proposing also to let $z_0 \to \infty$. It is simple and convenient to consider only the special cases summarized in the assumption

$$z_0 = \lambda \epsilon^c. \tag{6.1}$$

In Secs. III–V, we have dealt with $c = 0$. For the study of thin layers the case $c = -1$ is the decisive one. The static configuration then simplifies to

$$\begin{aligned} T_s &= \lambda/\epsilon + z, \\ \rho_s &= 1 + \epsilon m(z-1)/\lambda + \cdots. \end{aligned} \tag{6.2}$$

At this point it is worth making two notational alterations that will simplify the appearance of the thin-layer results. We introduce the new vertical coordinate

$$\zeta = \frac{1}{2} - z, \tag{6.3}$$

which varies from $-1/2$ to $1/2$ as z goes from 0 to 1. Then we introduce a Rayleigh number evaluated at $\zeta = 0$, which choice is indicated by stability theory (e.g., Spiegel 1965):

$$\hat{R} = \left(\frac{1/2 + z_0}{1 + z_0}\right)^{2m-1} R. \tag{6.4}$$

Now we seek asymptotic expansions of the form

$$\begin{aligned} &U = U_0 + \epsilon U_1 + \cdots, \quad W = W_0 + \epsilon W_1 + \cdots, \\ &\rho = \epsilon\rho_1 + \epsilon^2\rho_2 + \cdots, \quad p = p_0 + \epsilon p_1 + \cdots, \\ &T = T_0 + \epsilon T_1 + \cdots, \\ &\hat{R} = \hat{R}^{(0)} + \epsilon\hat{R}^{(1)} + \epsilon^2\hat{R}^{(2)} + \cdots. \end{aligned} \tag{6.5}$$

We see that the temperature perturbation is of the order of unity as a natural consequence of our scaling.

In leading order we obtain the system

$$U_{0\xi} + 2W_{0\zeta} = 0, \tag{6.6a}$$

$$4\sigma U_{0\zeta\zeta} = p_{0\xi}, \tag{6.6b}$$

$$2\Delta p_{0\zeta} = \lambda\sigma\hat{R}^{(0)}\rho_1, \tag{6.6c}$$

$$T_{0\zeta\zeta} = 0, \tag{6.6d}$$

$$\lambda\rho_1 + T_0 = 0. \tag{6.6e}$$

The boundary conditions in this order are

$$\begin{aligned} T_{0\zeta} &= 0, \\ W_0 &= 0, \\ U_{0\zeta} &= 0 \end{aligned} \tag{6.7}$$

on $\zeta = \pm\frac{1}{2}$.

We obtain the solution

$$T_0 = f(\xi,\tau), \tag{6.8a}$$

$$\rho_1 = -f/\lambda, \tag{6.8b}$$

$$p_0 = -\frac{\sigma\hat{R}^{(0)}}{\Delta}\left(\frac{\zeta}{2}\right)f, \tag{6.8c}$$

$$U_0 = \frac{\hat{R}^{(0)}}{16\Delta}\left(\zeta - \frac{\zeta^3}{3}\right)f_\xi, \tag{6.8d}$$

$$W_0 = \frac{\hat{R}^{(0)}}{32\Delta}\left(\frac{\zeta^4}{12} - \frac{\zeta^2}{2} + \frac{5}{12}\right)f_{\xi\xi}. \tag{6.8e}$$

Thus we are already in possession of the general form of the zero-order solution and need only to find the horizontal structure function and the zero-order Rayleigh number. These come, as before, from the solvability conditions in the higher orders, which also bring in corrections to the Rayleigh number. We need not repeat this process and shall simply quote the results.

In first order the solvability condition gives

$$\hat{R}^{(0)} = R_c, \tag{6.9}$$

where R_c is the critical Rayleigh number of Boussinesq theory. With the present boundary conditions,

$$R_c = 5!, \tag{6.10}$$

in agreement with the Boussinesq result (Hurle, Jakeman, and Pike 1967). In second order the solvability condition yields the equation

$$f_\tau - \frac{\mu}{\Delta^2}[(f_\xi)^3]_\xi - \frac{3}{2\lambda}(f^2)_{\xi\xi} + \hat{r}f_{\xi\xi} + \kappa f_{\xi\xi\xi\xi} = 0, \tag{6.11}$$

where the complicated object κ of Sec. V here has the constant value 0.196 789, $\mu = 1.230$, and

$$\hat{r} = \hat{R}^{(1)}/R_c. \tag{6.12}$$

As before, the linear theory is straightforward. Marginal stability occurs when

$$\hat{R} = \hat{R}_0 \equiv R_c(1 + \kappa a^2). \tag{6.13}$$

Now when working with thin layers it is usual to take the difference of potential temperature across the static layer as the unit of temperature. The appropriate change in (6.11) is made with the transformation

$$f(\xi,\tau) = \Delta\theta(x,t), \tag{6.14}$$

by which we also return to the original independent variables. Then the temperature perturbation evolves according to the equation

$$\theta_t + \left(\frac{\hat{R}}{R_c} - 1\right)\theta_{xx} - \frac{3}{2}\delta(\theta^2)_{xx} - \mu[(\theta_x)^3]_x + \kappa\theta_{xxxx} = 0, \tag{6.15}$$

where

$$\delta = \Delta/z_0. \tag{6.16}$$

Equation (6.15) has two nonlinear terms: one like that in (5.1) and one like that in the Boussinesq theory. Since δ is small for thin layers, it might seem appropriate to omit the former in thin-layer theory. But no matter how small δ may be, it has a significant qualitative effect on the nature of nonlinear solutions. To see this, consider first solutions of (6.15) with small amplitude and for δ and the horizontal scale held fixed.

We introduce a parameter, $\hat{\epsilon}$, whose magnitude is small and make conventional amplitude expansions:

$$\begin{aligned} \theta &= \hat{\epsilon}\theta_1 + \hat{\epsilon}^2\theta_2 + \cdots, \\ \hat{R} &= \hat{R}_0 + \hat{\epsilon}^2\hat{R}_2 + \cdots. \end{aligned} \tag{6.17}$$

As in the previous calculation, we assume that θ_x and θ_{xxx} vanish on $x = 0$ and $2\pi/a$. Then, on using well-known procedures (Kogelman and Keller 1971), we find that

$$\theta_1 = A(s)\cos ax, \tag{6.18}$$

where $s = \hat{\epsilon}^2 t$ and A satisfies the Landau equation

$$\dot{A} = \eta A + \upsilon A^3, \tag{6.19}$$

with

$$\eta = a^2\hat{R}_2/\hat{R}^{(0)}, \tag{6.20}$$

and

$$\upsilon = \frac{3}{8}(\delta^2/\kappa - 2\mu a^4). \tag{6.21}$$

From Eqs. (6.19)–(6.21) we see that for

$$a < a_0 \equiv \delta^{1/2}/(2\mu\kappa)^{1/4} = 1.2\delta^{1/2}, \tag{6.22}$$

steady solutions are found only when $\hat{R}_2 < 0$. The latter condition means that the Rayleigh number is below the value for marginal stability; that is, linear theory predicts stability. Hence, when (6.22) is verified, steady solutions of small amplitude exist and we have a subcritical bifurcation to steady convection. These convective solutions are unstable according to (6.19) and we expect to find a second branch of steady solutions for the same values of a, but having considerably larger amplitudes.

The amplitudes indicated by the steady solutions of (6.19) are interpreted as the critical disturbance amplitudes needed to trigger finite-amplitude instability. These critical amplitudes decrease monotonically with decreasing a. In this sense, the degree of subcriticality may be said to increase as a decreases, provided that geometrical constraints do not intervene. This feature of the problem makes it difficult to compute the minimum of $\hat{R}$ for which finite-amplitude instability is possible. For sufficiently small a, nonlinear solutions of (6.15) lead us out of the domain of validity of our expansions. Nevertheless it may be worth reporting briefly on such solutions.

We adopt conditions (5.18) and impose (5.9), which is expressed now as

$$\bar{\theta} = 0. \tag{6.23}$$

Given the boundary conditions, we may immediately perform three integrations of the steady-state version of (6.15). This reduces the problem to quadratures, from which we see that the solutions are not expressible in terms of elementary functions. The simplest representations of the solutions are numerical and to obtain them it is convenient to introduce some changes of variables.

We return to ξ as the independent variable but this time our choice is informed by the calculations just sketched and a definite choice for ϵ is indicated:

$$\epsilon = a_0^2, \tag{6.24a}$$

$$\xi = a_0 x. \tag{6.24b}$$

Now suppose that θ varies from a minimum value of $-\theta_0$ to a maximum value of θ_1, where both θ_0 and θ_1 are positive. We arrange for θ to be minimum at $x = 0$ and maximum at $x = \pi/a$. Then we let

$$\phi(\xi) = b(\theta + \theta_0), b^2 = 2\mu/\kappa. \tag{6.25}$$

As x ranges from 0 to π/a, ξ varies from 0 to π/α, where

$$\alpha = a/a_0; \tag{6.26}$$

ϕ goes from 0 at $\xi = 0$ to a maximum value of

$$\phi\left(\frac{\pi}{\alpha}\right) \equiv \Phi = b(\theta_0 + \theta_1). \tag{6.27}$$

For given α, the bifurcation structure of the solutions is described by the relation between the Rayleigh number and the amplitude of the solution. The latter is characterized by Φ and we may introduce a similarly convenient expression of the former. Let

$$\hat{R} = R_c(1 + \kappa a^2 + \kappa a_0^2 P). \tag{6.28}$$

Then $-P$ measures the degree of subcriticality and we seek the function $P(\alpha,\Phi)$. We know already that $P(\alpha,0) = 0$ and in this case $\hat{R}$ is given by (6.13).

After making the indicated transformations in (6.15), assuming a steady state, and performing a single integration, we obtain

$$\overset{\circ\circ\circ}{\phi} - \frac{1}{2}\overset{\circ}{\phi}{}^3 - 3\phi\overset{\circ}{\phi} + (\alpha^2 + 3b\theta_0 + P)\overset{\circ}{\phi} = 0, \tag{6.29}$$

where the degree symbol denotes differentiation with respect to ξ. By conventional methods we obtain

$$\frac{1}{2}\overset{\circ}{\phi}{}^2 + V(\phi) = E, \tag{6.30}$$

where

$$V = E\cosh\phi + C\sinh\phi + 3\phi, \tag{6.31}$$

$$C = [E(1 - \cosh\Phi) - 3\Phi]/\sinh\Phi \tag{6.32}$$

and

$$E = \alpha^2 + 3b\theta_0 + P. \tag{6.33}$$

The integration constants have been selected to ensure that $\overset{\circ}{\phi}$ vanishes on the end points, so that $V = E$ there. Therefore the conditions

$$\phi(0) = 0, \qquad \phi\left(\frac{\pi}{\alpha}\right) = \Phi \tag{6.34}$$

may also be used as boundary conditions on (6.30).

We may solve (6.30) for given Φ and obtain the eigenvalues of E, but to obtain P from (6.33) we still need θ_0. The latter is obtained by applying condition (6.23). We may do this simply by introducing a new variable, χ, satisfying the equation

$$\overset{\circ}{\chi} = \phi, \tag{6.35}$$

subject to the boundary condition

$$\chi(0) = 0. \tag{6.36}$$

Now we solve the system defined by (6.30)–(6.36) and, on noting that

$$b\theta_0 = \chi\left(\frac{\pi}{\alpha}\right), \tag{6.37}$$

we find $P(\alpha,\Phi)$. This task is readily accomplished with N. H. Baker's GNR1.

For $\alpha > 1$ ($a > a_0$), P increases monotonically with Φ. This is supercritical bifurcation and we do not present results on it. But for $\alpha < 1$, as Φ increases from zero, P decreases at first, starting from the value zero, and reaches a minimum value P_m when $\Phi = \Phi_m$; thereafter P increases monotonically with Φ. For a given α we have finite-amplitude instability when $P_m < P < 0$. The disturbance amplitude needed to initiate sustained motion is somewhere between 0 and Φ_m, while the amplitude of the steady motion that is ultimately established is greater than Φ_m. A useful summary of the numerical results is therefore afforded by a display of P_m and Φ_m for various $\alpha < 1$. This is provided in Fig. 4.

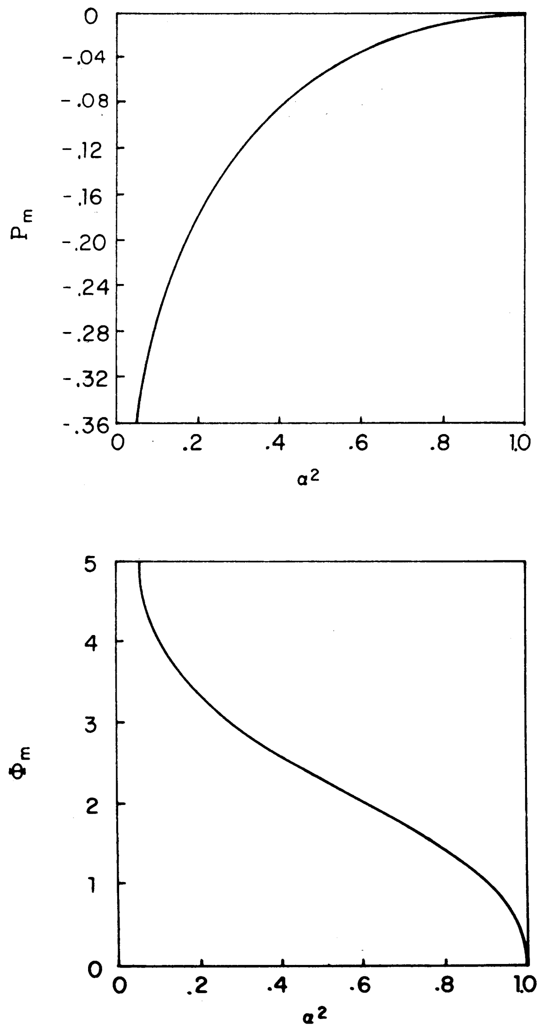

FIG. 4. The subcriticality parameter P_m and the corresponding amplitude Φ_m, both as functions of the square of the scaled horizontal wave number, as explained in the text.

A significant feature of Fig. 4 is that as α decreases to zero, $-P_m$ tends to infinity. However, our results should lose validity when $a_0^2\,|P|$ is no longer much less than unity and we cannot immediately draw conclusions about the behavior at small a. It is therefore not possible to derive from our present results the lowest value of the Rayleigh number for which large-scale convection may be sustained, though the results point to a very low value indeed. And so does the physical argument they suggest.

The subcritical bifurcation in the present model arises because a slight horizontal gradient produced over a sufficiently large distance must lead to an appreciable change in physical conditions. This in turn produces vertical variation which, in the non-Boussinesq treatment, is able to modify the basic state. It seems plausible that, on sufficiently large scales, motions may be sustained at Rayleigh numbers approaching zero. It seems quite important to try to check this conjecture since there are cases, such as the earth's mantle, where it is not clear whether the values of the Rayleigh number are high enough to give rise to convection on the more usual criteria. We should add too that this possibility is not restricted to compressible convection and bioconvection; similar results emerge if the material properties of the fluid depend on temperature (Depassier 1980).

VII. REMARKS

The aim of this work has been to study the nonlinear dynamics of convection with very large horizontal scales. We have had particularly in mind turbulent convective layers and have assumed that the use of constant eddy coefficients was qualitatively sound. This dictated the choice of boundary conditions and led us to consider Prandtl numbers of the order of unity. If these premises are valid, the present results should provide some qualitative guide to the interpretation of large-scale stellar convection.

In a recent study bearing on these questions, Poyet (1980) showed that geometrically thin layers (such as the outer radiative zone of the sun or the earth's crust) can have profound effects on the preferred scales of motion only for extreme values of their thermal diffusivities. Hence our choice of boundary conditions involves subtleties that we have not pursued; a more detailed discussion will be given elsewhere. For the present we shall nevertheless assume that the arguments used are qualitatively reasonable and that our results provide a reliable first impression of large-scale convection such as may be encountered in the sun or other cosmic bodies which do not rotate rapidly.

Perhaps the most important conclusion suggested by our calculations is that convection on the largest horizontal scales permitted by the geometry should be a significant feature of convective envelopes. Though we have considered only two-dimensional motions, this may be a relevant choice for many purposes. A small amount of rotation will inhibit large-scale convection at the poles and, in the spherical case, the velocity will be mainly zonal with a narrow seam of upwelling and one of downwelling.

It may be reasonably asked why such convection has not been explicitly recognized before. Our response would be that convective motions such as we have been studying would be difficult to distinguish from what one normally describes as circulation or differential rotation. Moreover, we would mention a suggestion by Scharlemann (1980) that points to an indirect detection of such convection. In BY Draconis stars and, to a much lesser extent, the sun, spots appear to congregate in one hemisphere. Scharlemann's proposal is that convection on the largest scale allowed by the geometry will gather spots into one hemisphere. In his calculations, spots are assumed to affect the fluid thermally but the consequences seem slight, given the degree of nonlinear instability seen in the non-Boussinesq case. Nevertheless they may be all that is needed to fix the convective pattern in place for a reasonable period of time.

In some respects, the supposed spot drift resembles continental drift; indeed Childress, Chapman, and Proctor (1980) have discussed the possible role of fixed-flux convection in the latter. The interest attaching to such questions motivates the extension of the present study to rotating spherical shells. It is reasonably clear how to proceed in that direction. The difficulty that we confront is the inability of the asymptotic methods to provide solutions for the large-amplitude motions that arise in the case of large horizontal scales, particularly in the highly stratified case. This is a basic question that deserves further investigation.

In conclusion we thank Norman Baker for providing GNR1 and for his valuable comments on the manuscript, S. Childress for discussion, and R. Benguria for kindly verifying some algebraic manipulations. We are grateful also to the NSF for financial assistance under Grant PHY 77-27086 and to the Goddard Institute for Space Studies for the use of its computing facilities.

APPENDIX A: DETAILS OF THE EXPANSION IN SEC. IV

To give explicit expressions for the polynomials in the first-order solution (4.11), we introduce the auxiliary notation

$$y = (z + z_0)/(1 + z_0) \tag{A1}$$

and

$$\Xi_m = 1 - [z_0/(1 + z_0)]^m. \tag{A2}$$

Then

$$P_1(z) = y^m - \frac{(m+2)\Xi_{m+1}}{(m+1)\Xi_{m+2}} y^{m+1}, \tag{A3}$$

$$\mathcal{U}_1(z) = \frac{(1+z_0)^2}{m+1}\left[\frac{y^{m+2}}{m+2} - \frac{\Xi_{m+1}y^{m+3}}{\Xi_{m+2}(m+3)} + \left(\frac{\Xi_{m+1}}{\Xi_{m+2}} - 1\right)y + (m+1)B_m\right], \tag{A4}$$

$$\mathcal{W}_1(z) = \frac{(1+z_0)^3}{m+1}\left[\frac{(y^{2m+4}-1)\Xi_{m+1}}{2(m+2)(m+3)\Xi_{m+2}} - \frac{y^{2m+3}-1}{(m+2)(2m+3)} - \left(\frac{\Xi_{m+1}}{\Xi_{m+2}} - 1\right)\frac{y^{m+2}-1}{m+2} - B_m(y^{m+1}-1)\right]y^{-m}, \tag{A5}$$

where

$$(m+2)B_m = \frac{\Xi_{m+2}}{\Xi_{m+1}} - 1 + \frac{(2m+3)\Xi_{m+1}\Xi_{2m+4} - 2(m+3)\Xi_{m+2}\Xi_{2m+3}}{2(m+3)(2m+3)\Xi_{m+1}\Xi_{m+2}}. \tag{A6}$$

The critical value of the Rayleigh number is given by

$$1/R^{(0)} = (1+z_0)^5\left[\frac{1}{2}\left(\frac{\Xi_{2m+5}}{2m+5} - \Xi_1\right)\frac{\Xi_{m+1}}{\Xi_{m+2}} - \frac{m+3}{2m+3}\left(\frac{\Xi_{2m+4}}{2m+4} - \Xi_1\right) - (m+3)\left(\frac{\Xi_{m+3}}{m+3} - \Xi_1\right)\left(\frac{\Xi_{m+1}}{\Xi_{m+2}} - 1\right) - (m+2)(m+3)B_m\left(\frac{\Xi_{m+2}}{m+2} - \Xi_1\right)\right] \Big/ (m+1)(m+2)(m+3). \tag{A7}$$

In order ϵ^2 the solution becomes rather complicated and we shall not give it all explicitly. The form of the solution in terms of the second arbitrary function, $g(\xi,\tau)$, which must be introduced, is

$$T_2 = g(\xi,\tau) + f_{\xi\xi}\Theta(z), \tag{A8}$$

$$p_2 = \Lambda_0\left[\left(M_1(z) - \frac{(m+2)\Xi_1}{\Xi_{m+2}}N_1(1)y^{m+1}\right)f_{\xi\xi} - \left(M_2(z) - \frac{(m+2)\Xi_1}{\Xi_{m+2}}N_2(1)y^{m+1}\right)f^2 + (g+rf)P_1(z)\right], \tag{A9}$$

$$U_{2\xi} = \frac{m+1}{1+z_0}\left(\frac{(m+2)\Xi_{m+1}}{(m+1)\Xi_{m+2}} - \frac{\Xi_m}{\Xi_{m+1}}\right)f_\tau + \left[(g_{\xi\xi} + rf_{\xi\xi})\mathcal{U}_1 + \left(L_1(z) - (m+1)\frac{\Xi_1}{\Xi_{m+1}}Q_1(1)\right)f_{\xi\xi\xi\xi} - \left(L_2(z) - (m+1)\frac{\Xi_1}{\Xi_{m+1}}[Q_2(1) - Q_4(1)]\right)(f^2)_{\xi\xi}\right]\Lambda_0/\sigma, \tag{A10}$$

and, with

$$Y_m = (y^m + \Xi_m - 1)/\Xi_m, \tag{A11}$$

$$\rho_s W_2 = -\rho_1 W_1 + \{[P_1(1) - P_1(0)]Y_{m+1} + P_1(0) - P_1(z)\}f_\tau + ((g_{\xi\xi} + rf_{\xi\xi})\rho_s\mathcal{W}_1 + [Y_{m+1}Q_1(1) - Q_1(z)]f_{\xi\xi\xi\xi} + \{Q_2(z) - Q_4(z) - [Q_2(1) - Q_4(1)]Y_{m+1}\}(f^2)_{\xi\xi})\Lambda_0/\sigma, \tag{A12}$$

where

$$\Theta = \int_0^z dz' \int_0^{z'} d\zeta[(1+z_0)R^{(0)}\rho_s(\zeta)\mathcal{W}_1(\zeta) - 1], \tag{A13}$$

$$M_1 = y^{m+1}\int_0^z \eta^{-(m+1)}\{(1/3)[4\mathcal{W}_{1\zeta\zeta}(\zeta) + \mathcal{U}_{1\zeta}(\zeta)] - \eta^{m-1}\Theta(\zeta)/(1+z_0)\}d\zeta, \tag{A14}$$

$$M_2 = \frac{y^{m+1}}{1+z_0}\int_0^z \eta^{-(m+2)}P_{1\zeta}(\zeta)d\zeta, \tag{A15}$$

$$N_1 = \int_0^z \{M_1(\zeta) - (1/3)[\mathcal{W}_{1\zeta}(\zeta) + 4\mathcal{U}_1(\zeta)]\}d\zeta, \tag{A16}$$

$$N_2 = \int_0^z M_2(\zeta)d\zeta, \tag{A17}$$

$$Q_4 = \frac{1}{2}\int_0^z P_{1\zeta}(\zeta)\mathcal{U}_1(\zeta)d\zeta, \tag{A18}$$

and, with $i = 1, 2$,

$$L_i = \int_0^z [N_i(\zeta) - Z_{m+2}N_i(1)]d\zeta, \tag{A19}$$

$$Q_i = \int_0^z \eta^m L_i(\zeta) d\zeta, \tag{A20}$$

and where y and Y_m become η and Z_m when z is replaced by ζ.

APPENDIX B: COEFFICIENTS FOR (5.1)

Making use of the notation of Appendix A, we write

$$\Omega = \frac{\Xi_{m+1}}{(m+1)\Xi_1} + \left[\frac{m}{m+1}\Xi_1 - \left(\frac{m+1}{m+2}\right)\left(\frac{\Xi_m \Xi_{m+2}}{\Xi_{m+1}}\right)\right]\frac{\Delta}{\Xi_1}. \tag{B1}$$

We further define the following parameters:

$$C = \frac{(1+z_0)^2}{m+1},$$
$$D = \frac{\Xi_{m+1}}{\Xi_{m+2}},$$
$$E = (1+z_0)C;$$
$$c_1 = C/(m+2),$$
$$c_2 = -CD/(m+3),$$
$$c_3 = C(D-1),$$
$$c_4 = (m+1)CB_m,$$
$$d_1 = \tfrac{1}{2}ED/[(m+2)(m+3)],$$
$$d_2 = -E/[(m+2)(2m+3)],$$
$$d_3 = -E(D-1)/(m+2),$$
$$d_4 = -EB_m.$$

In addition, we need these complicated objects:

$$H(n) = \left[\frac{1}{3} + \frac{1}{2\Xi_1^3}\left(\frac{\Xi_{n+3}}{n+3} - 2\frac{\Xi_{n+2}}{n+2} + (1-\Xi_1^2)\frac{\Xi_{n+1}}{n+1}\right)\right]\Big/ n,$$

$$\Pi(n,m) = \frac{1}{2(n+1)\Xi_1^2}\left[\frac{\Xi_{n+2}}{n+2} + \left(\Xi_1 - \frac{\Xi_{m+2}}{m+2}\right)\frac{\Xi_{n+1}}{\Xi_{m+1}} - \Xi_1\right],$$

$$\Upsilon(n,m) = \frac{1}{(n+1)(n+m+2)\Xi_1^3}\left[\frac{\Xi_{n+m+3}}{n+m+3} + \left(\Xi_1 - \frac{\Xi_{m+2}}{m+2}\right)\frac{\Xi_{n+m+2}}{\Xi_{m+1}} - \Xi_1\right] \quad (n \neq -1),$$

$$\Upsilon(-1,m) = \frac{1}{\Xi_1^3}\left[\frac{\Xi_1(\Xi_{m+1}-1)}{(m+2)\Xi_{m+1}}\ln\left(\frac{z_0}{1+z_0}\right) - \frac{\Xi_{m+2}}{(m+1)(m+2)^2}\right].$$

Then we compound matters and define

$$\Sigma(n,m) = \frac{1}{(n+1)\Xi_1}\left(\Upsilon(n+1,m) - \Upsilon(0,m) - \frac{\Xi_{n+1}}{\Xi_{m+2}}[\Upsilon(m+2,m) - \Upsilon(0,m)]\right),$$

$$\Phi(n,m) = \frac{R^{(0)}(1+z_0)}{\Xi_1^2}\Bigg([(1-\Xi_{n+1})/(n+1) - (1-\Xi_2)/2]\Sigma(m,m) - \frac{1}{n^2(n+1)}\Sigma(n+m+1,m) + \tfrac{1}{2}\Sigma(m+2,m)\Bigg),$$

$$\Psi(m) = \frac{1}{(m+2)^2\Xi_1^4}\Bigg[\frac{(m+2)(3m+8)}{(m+3)^2(2m+5)^2}\Xi_{2m+5} - \frac{(3m+7)(m+1)\Xi_{2m+4}\Xi_{m+2}}{4(m+2)^2(m+3)^2\Xi_{m+1}} + (1-\Xi_{m+2})\left(\frac{2(m+2)^2\Xi_{m+3}}{(m+3)(2m+5)\Xi_{m+2}} - \frac{(2m+5)(m+1)\Xi_{m+2}}{2(m+2)(m+3)\Xi_{m+1}}\right)\ln\left(\frac{z_0}{1+z_0}\right)\Bigg].$$

Then

$$\nu = -R^{(0)}(1+z_0)\Bigg\{\Xi_1\left((m+2)\frac{\Xi_{m+1}}{\Xi_{m+2}}\Sigma(m,m) - \frac{m}{2}\Sigma(m-1,m)\right)$$
$$-\frac{1}{(m+1)\Xi_1}\Bigg[\frac{m}{m+2}\Pi(2m+1,m) - \frac{(2m+3)\Xi_{m+1}}{(m+3)\Xi_{m+2}}\Pi(2m+2,m)$$
$$+ m(m+1)B_m\Pi(m-1,m) + \frac{m+2}{m+3}\left(\frac{\Xi_{m+1}}{\Xi_{m+2}}\right)^2\Pi(2m+3,m)$$
$$-\frac{\Xi_{m+1}}{\Xi_{m+2}}\left(\frac{\Xi_{m+1}}{\Xi_{m+2}}-1\right)(m+2)\Pi(m+1,m)\Bigg]\Bigg\}. \quad \text{(B2)}$$

In addition,

$$\kappa = -\frac{1}{3} - \frac{R^{(0)}(1+z_0)}{(m+1)\Xi_1^3}\Bigg[\frac{\Xi_{m+1}H(2m+4)}{(m+3)\Xi_{m+2}} - \frac{H(2m+3)}{(m+2)}$$
$$-\left(\frac{\Xi_{m+1}}{\Xi_{m+2}}-1\right)H(m+2) - (m+1)B_mH(m+1)\Bigg]$$
$$-R^{(0)}(1+z_0)\Bigg(\frac{1}{3}(m-2)c_1\Sigma(m+2,m) + \frac{1}{6}(m-5)c_2\Sigma(m+3,m)$$
$$-\frac{1}{3}\left(4+\frac{1}{m}\right)c_3\Sigma(1,m) - \frac{4}{3}c_4\Sigma(0,m)$$
$$+\frac{1}{3}\Xi_1\Big\{d_1[(m+4)(2m+5)\Sigma(m+3,m) + m\Sigma(-m-1,m)]$$
$$+ d_2[(m+3)(4m+7)\Sigma(m+2,m) + m\Sigma(-m-1,m)]$$
$$+ d_3\left[-2\left(4+\frac{1}{m}\right)\Sigma(1,m) + m\Sigma(-m-1,m)\right]$$
$$+ d_4[-\Sigma(0,m) + m\Sigma(-m-1,m)]\Big\}$$
$$+ d_1\Phi(2m+5,m) + d_2\Phi(2m+4,m) + d_3\Phi(m+3,m)$$
$$+ d_4\Phi(m+2,2) + \frac{1}{2\Xi_1^2}[\Sigma(m+2) - (1-\Xi_2)\Sigma(m,m)]$$
$$-\frac{\Psi(m)}{\Xi_1^2}\left[R^{(0)}(1+z_0)\left(\frac{2m+4}{2m+5}d_1 + \frac{2m+3}{2m+4}d_2 + \frac{m+2}{m+3}d_3 + \frac{m+1}{m+2}d_4\right)+1\right]\Bigg). \quad \text{(B3)}$$

To give an impression of the magnitudes of the key parameters, we give some typical values for $z_0 = 1$:

m	$R^{(0)}$	Ω	κ	ν
2	349	0.546	0.203	0.738
3	675	0.431	0.210	0.586
4	1280	0.353	0.214	0.443

Further we note that for $z_0 = 0$ the various expressions reduce to

$$R^{(0)} = (m+1)(m+2)^2(2m+3)(2m+5),$$
$$\kappa = (2\,575\,692 + 9\,592\,767m + 15\,561\,220m^2 + 14\,185\,080m^3$$
$$+ 7\,817\,146m^4 + 2\,581\,249m^5 + 451\,394m^6 + 17\,612m^7 - 6472m^8 - 768m^9)/$$
$$[48(m+2)^2(m+3)^2(m+4)(2m+5)(2m+7)(3m+7)(4m+9)],$$
$$\Omega = [1 - \Delta/(m+2)]/(m+1),$$
$$\nu = 0.$$

REFERENCES

Busse, F. H. (1978). Rep. Prog. Phys. **41,** 1979.

Busse, F. H., and Riahi, N. (1980). J. Fluid Mech. **96,** 243.

Chapman, C. J., and Proctor, M. R. E. (1980). J. Fluid Mech. **101,** 759.

Childress, S., and Spiegel, E. A. (1978). Submitted to J. Fluid Mech.

Childress, S., Chapman, C. J., and Proctor, M. R. E. (1980). Preprint.

Childress, S., Levandowsky, M., and Spiegel, E. A. (1975). J. Fluid Mech. **63,** 591.

Depassier, M. C. (1980). Ph.D. dissertation, Astronomy Department, Columbia University.

Graham, E. (1975). J. Fluid Mech. **70,** 689.

Graham, E. (1977). In *Problems of Stellar Convection,* Vol. 71 of *Lecture Notes in Physics,* edited by E. A. Spiegel and J.-P. Zahn (Springer Verlag, Berlin), p. 151.

Hurle, D. T. J., Jakeman, E., and Pike, E. R. (1967). Proc. R. Soc. London A **296,** 469.

Kogelman, S., and Keller, J. B. (1971). SIAM J. Appl. Math. **20,** 619.

Massaguer, J., and Zahn, J.-P. (1980). Astron. Astrophys. **87,** 315.

Poyet, J.-P. (1980). Ph.D. dissertation, Astronomy Department, Columbia University.

Scharlemann, E. T. (1980). Preprint.

Spiegel, E. A. (1965). Astrophys. J. **141,** 1068.

Van der Borght, R. (1974). Aust. J. Phys. **27,** 481.

Zahn, J.-P. (1979). In *Stellar Turbulence,* Vol. 114 of *Lecture Notes in Physics,* edited by D. F. Gray and J. L. Linsky (Springer Verlag, Berlin), p. 1.