Variational approach to a class of nonlinear oscillators with several limit cycles

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We study limit cycles of nonlinear oscillators described by the equation $\ddot{x}+\nu F(\dot{x})+x=0$ with F an odd function. Depending on the nonlinearity, this equation may exhibit one or more limit cycles. We show that limit cycles correspond to relative extrema of a certain functional. Analytical results in the limits $\nu\to 0$ and $\nu\to\infty$ are in agreement with previously known criteria. For intermediate ν, numerical determination of the limit cycles can be obtained.

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I. INTRODUCTION

The study of Lienard's differential equation

$$\ddot{x}+\nu f(x)\dot{x}+x=0 \tag{1}$$

or the equivalent form, given earlier by Rayleigh,

$$\ddot{z}+\nu F(\dot{z})+z=0, \tag{2}$$

goes back to the work of Rayleigh [1] motivated by his work in acoustics and to Cartan and H. Cartan [2], van der Pol [3], and Liénard [4] motivated by their work in electrical circuits. This type of equation arises directly in numerous applications and others can be reduced to them. The problem of studying the number and location of the periodic solutions for polynomial F is a particular case of the general two-dimensional problem $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$, for polynomial P and Q, which constitutes Hilbert's 16th problem. The problem we address here is the determination of the number and position of limit cycles of the equations above for even f, or equivalently, in Rayleigh's form for odd F. The two equations have the same number of limit cycles, Liénard's form (1) is obtained by taking the derivative of Eq. (2) and calling $\dot{z}=x$. These equations have a unique equilibrium point $z=0$ around which their periodic solutions will be nested. One of the most studied equations in this class is van der Pol's equation, which has a unique limit cycle. Liénard [4] established conditions on F which guarantee the existence of a single limit cycle. Many results on the existence and uniqueness have been established; less is known about the number and location of the limit cycles that do not satisfy Liénard's conditions [5–7]. Lins, de Melo, and Pugh [8] conjectured that if F is a polynomial of degree $2n+1$ or $2n+2$, then there can be at most n limit cycles. This conjecture was proved [9] for small ν, that is, for small departures from the Hamiltonian case. For $\nu\to 0$, the number and position of the limit cycles is given by the real roots of a polynomial obtained from Melnikov's function [5,7]. In the present case, it reduces to finding the roots of

$$\oint_{\Gamma_0} F(\dot{z}(t))dt=0, \tag{3}$$

where the integral is performed over a periodic solution Γ_0 of the Hamiltonian system $\ddot{z}+z=0$, a circle in phase space. The conjecture has not been proved for arbitrary values of ν except for special cases. It holds true if F is a polynomial of degree 3 or 4 [10] and if F is odd of degree 5 [11]. These results state the maximum number possible of limit cycles but not the exact number. Precise results on the exact number exist for particular cases; see, for example, [12]. In recent work, the case $\nu\to\infty$ has been studied; a criterion to determine the number of limit cycles in this asymptotic regime was given and tested in several examples [14,15]. In a different approach, an algorithm to determine the number and position of the limit cycles for all values of ν and odd F has been formulated. It is nonperturbative in nature, based on finding the roots of a certain sequence of polynomials [16–18]. Fewer results [13] are known for the generalized equation $\ddot{x}+\nu f(x)\dot{x}+g(x)=0$.

In the present work, we study limit cycles of Eq. (2), with $F(\dot{z})$ an odd function. We show that limit cycles correspond to extrema of a certain functional. For small ν, we recover known analytic results namely Eq. (3); for $\nu\to\infty$, we recover the results of [14,15]. For intermediate ν, we must resort to numerical calculations. The approach is a generalization of a method developed for other nonlinear problems [19–21]. This first approach enabled us to show that in cases of cubic F with a unique limit cycle, its location can be obtained from a minimum principle [22]. Here we show that all limit cycles correspond to extrema of a certain functional, which would allow us, at least numerically, to count the number of limit cycles of a given equation. In Sec. II, we derive the variational principle, in Sec. III, we obtain analytical criteria for the small and large ν regime. In Sec. IV, we present some examples where we see that approximate numerical determination of the limit cycles can be obtained far from the asymptotic regimes. Concluding remarks are made in Sec. V.

II. VARIATIONAL PRINCIPLE

For odd F, due to the symmetry of Eq. (2), the limit cycle extends between a minimum $z_{\min}=-A$ and a maximum $z_{\max}=A$. Moreover, in phase space, if the point $(\dot{z},z)$ belongs to the limit cycle, then the point $(-\dot{z},-z)$ also belongs to it. Therefore, we may consider the positive upper half $\dot{z}>0$ of the phase plane, where half a period will elapse when going

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from the points ($\dot{z}=0,z_{\min}$) to ($\dot{z}=0,z_{\max}$). Then the equation for the limit cycle in phase space $(\dot{z}(z),z)$ can be written as the nonlinear eigenvalue problem, $pp_z+\nu F(p)+z=0$, with $p(\pm A)=0$ and $p>0$. Here we have set $p(z)=\dot{z}(z)$ and the subscript denotes a derivative. The eigenvalue is the amplitude A which appears in the boundary conditions. It is convenient to define a new variable $u=z/A$ in terms of which the equation is

$$\frac{1}{S}p\frac{dp}{du}+F(p)+Ru=0 \quad \text{with} \quad p(\pm 1)=0, \quad p>0. \tag{4}$$

Two parameters appear naturally, $R=A/\nu$ and $S=\nu A$. We may now construct the variational principle. Consider the extrema of the functional $R[p,\phi]$ of two arbitrary functions, $p(u)$ with $p(\pm 1)=0$, and $\phi(u)$, given by

$$R[p(u),\phi(u)]=\frac{\int_{-1}^{1}\left(-\frac{1}{S}p\frac{dp}{du}-F(p)\right)\phi(u)\,du}{\int_{-1}^{1}u\,\phi(u)du} \tag{5}$$

$$=\frac{\int_{-1}^{1}\left[\frac{1}{2S}p^2\frac{d\phi}{du}-F(p)\phi(u)\right]du}{\int_{-1}^{1}u\,\phi(u)\,du}, \tag{6}$$

where the second expression is obtained after integration by parts. Variation with respect to $p(u)$ at fixed $\phi(u)$ yields the equation

$$\frac{1}{S}p\frac{d\phi}{du}-F_p\phi=0. \tag{7}$$

Here F_p denotes a derivative of F with respect to p. Variation with respect to ϕ at fixed p yields

$$\frac{1}{S}p\frac{dp}{du}+F(p)+Ru=0, \tag{8}$$

that is, the equation for the limit cycles. Extrema of $R[p,\phi]$ satisfy both Eq. (7) and (8). Now notice that Eq (7) can be solved for ϕ in terms of p. Its solution is

$$\phi(u)=\exp\left(S\int_{-1}^{u}\frac{F_p(p(t))}{p(t)}dt\right). \tag{9}$$

Finally, replacing Eq. (7) in Eq. (6), we obtain the main result. Solutions of the equation for the limit cycles are extrema of

$$R[p]=\text{ext}\frac{\int_{-1}^{1}\phi(u)\left[\frac{1}{2}p(u)F_p(p(u))-F(p(u))\right]du}{\int_{-1}^{1}\phi(u)u\,du}, \tag{10}$$

where the extremum is taken over all positive functions $p(u)$ that vanish at the end points and $\phi(u)$ is the function of $p(u)$ given by Eq. (9). If we succeed in finding all the extrema, we have found all limit cycles.

III. SMALL AND LARGE ν LIMITS

A. Small ν

Here we shall see that for small ν we recover Melnikov's criterion. From the definition of the parameters R and S, we see that small or large ν, for arbitrary A, corresponds to small or large S, respectively. For small ν or equivalently for small S,

$$\phi\approx 1+S\int_{-1}^{u}\frac{F_p(p(t))}{p(t)}dt \tag{11}$$

and

$$R\approx\text{ext}\frac{\int_{-1}^{1}\left(\frac{1}{2}pF_p-F\right)du}{S\int_{-1}^{1}\frac{1}{2}\frac{F_p}{p}(1-u^2)du}. \tag{12}$$

Let us calculate the first variation δR of R with respect to p. We obtain

$$S(R[p+\delta p]-R[p])=\frac{1}{2D}\int_{-1}^{1}\delta p\,(pF_{pp}-F_p)\times\left[1-RS\frac{(1-u^2)}{p^2}\right], \tag{13}$$

where we called D the integral in the denominator of Eq. (12). The term $pF_{pp}-F_p$ does not vanish identically. Then $\delta R=0$ for arbitrary δp if

$$p^2(u)=RS(1-u^2). \tag{14}$$

We know that this is the correct answer from direct integration of the equation. In the small, ν limit, the cycle is approximately a solution of the equation $(1/S)p(dp/du)+Ru=0$, with $p(\pm 1)=0$, whose solution is what we just obtained, the ellipse $p(u)=RS\sqrt{1-u^2}$. This indicates that we should use a trial function of the form $p(u)=K\sqrt{1-u^2}$ and search for the value of K for which R has an extremum. We first notice that we will find some false extrema which we must discard. To see this, observe that with this trial function, $\delta p=\delta K\sqrt{1-u^2}$, and Eq. (13) becomes

$$S\,\delta R=\frac{1}{2D}\frac{\delta K}{K}\left(1-\frac{RS}{K}\right)\int_{-1}^{1}p(pF_{pp}-F_p)du. \tag{15}$$

The correct solution for $\delta R=0$ is $K=RS$. However, δR also vanishes when the integral in the expression above is zero. For example, if F is a polynomial, this integral can be performed and yields a polynomial in K which vanishes for some K. This is not the desired solution. It is spurious and is only obtained because we have not swept over all possible trial functions. With this in mind, we go back to Eq. (12). With $p(u)=A\sqrt{1-u^2}$ as the trial function, Eq. (12) is of the form

$$R\approx \mathrm{ext}\frac{A^2}{S}\frac{h(A)}{g(A)}, \tag{16}$$

where we set

$$h(A)=\int_{-1}^{1}(\tfrac{1}{2}pF_p-F)du, \tag{17}$$

$$g(A)=\int_{-1}^{1}\frac{1}{2}pF_p du. \tag{18}$$

It is easy to verify that $Ah'-Ag'=-2g$, and we obtain

$$\frac{dR}{dA}=\frac{2gh+A(h'g-hg')}{Sg^2}=\frac{Ah'(A)(h-g)}{Sg^2}. \tag{19}$$

Extrema of R occur when $Ah'(A)=0$ and when $h(A)=g(A)$. The first condition is either $A=0$, the trivial solution which is always present, or $h'(A)=0$. This solution is precisely the false solution which we discard. We retain then the solution $h(A)=g(A)$, which is simply

$$\int_{-1}^{1}F(p(u))du=0 \quad \text{with} \quad p(u)=A\sqrt{1-u^2}. \tag{20}$$

This is exactly condition (3) since we have considered only odd F.

B. Large ν

In a recent work, López *et al.* [15] study limit cycles of Liénard's differential equation (1) in the strongly nonlinear regime. Their approach is based in constructing approximate solutions to the differential equation. Here we give a heuristic derivation of their result. Since S appears only in the exponential in the form $\exp(Sz(u))$, we know, from Watson's lemma [23], that when $S\to\infty$ the leading contribution to the integral comes from the points where the term in the exponential has an extremum. Here the term in the exponential is

$$z(u)=\int_{-1}^{u}\frac{F_p(p(t))}{p(t)}dt. \tag{21}$$

Extrema occur where $z'(u)=0$, that is, where $F_p=0$. For large S, then, R will be given by

$$R=\mathrm{ext}\left(\frac{-F(\hat{p})}{\hat{u}}\right), \tag{22}$$

where $\hat{p}$ is the solution of $F_p(\hat{p})=0$ and $\hat{u}$ is the value of u for which $p=\hat{p}$ on the orbit. The extremum is now taken over $\hat{u}$. Since R is positive, possible extrema of R will occur at $\hat{u}=1$ if $F(\hat{p})<0$ or at $\hat{u}=-1$ if $F(\hat{p})>0$. The values of R at the extremum points, if any, for large S, are given then by

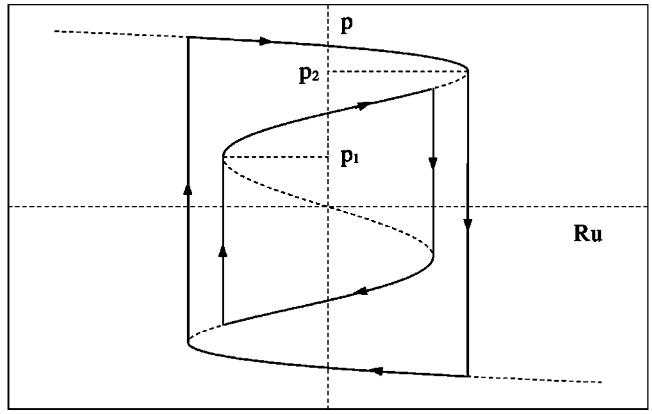

FIG. 1. Graph of allowed limit cycles in the $\nu\to\infty$ regime.

$$R=|F(\hat{p}_1)|, \quad |F(\hat{p}_2)|, \ldots . \tag{23}$$

In order to see which of these correspond to true extrema, hence to limit cycles, we return to the differential equation. For large S, limit cycles are approximately given by

$$F(p)+Ru=0, \tag{24}$$

the solution of which has to be matched to a thin boundary layer. Now that we know the possible values of R, we can read directly which are the limit cycles. This is best seen in a plot in the phase space of Eq. (24). Suppose there are several points $\hat{p}$ at which $F_p=0$. Since F is odd, take the positive values and label them in growing order, $p_1<p_2<p_3\cdots$. Correspondingly we have different values of R. Since limit cycles must be nested, and, for the systems considered, the derivative dp/du can vanish only at one point, the only limit cycles will be those for which $|F(p_j)|>|F(p_k)|$ for all $k<j$. This is best seen in Fig. 1, where we show a case with two allowed limit cycles. In the asymptotic regime when $u\to\pm1$, the horizontal coordinate tends to $R=F(\hat{p})$. The trajectory of each limit cycle is indicated by the arrows. From the definition of R we know then that in the limit $\nu\to\infty$, for each allowed cycle, the amplitude grows as $A_i=|F(p_i)|\nu$. Moreover, we can read the maximum value of p in each cycle. The maximum p are solutions of $F(p)+|F(\hat{p})|=0$. The values of p determined in this way correspond to the amplitude of the associated Liénard equation (1). Thus we have recovered the solution of [14,15], which gives support to the conjecture of Lins, de Melo, and Pugh.

IV. EXAMPLES

Having seen that we recover the small ν and large ν limits, we now give numerical results for arbitrary values of ν in simple examples. Here too, false extrema may appear due to the impossibility of sweeping over all trial functions. This becomes evident, as in the $\nu\to0$ limit, by considering the first variation of R with respect to p. For arbitrary S, $R(p+\delta p)-R(p)$ vanishes when

$$\int_{-1}^{1}du\ \delta p\left[\frac{F_{pp}}{p}-\frac{F_p}{p^2}\right]\left\{\frac{1}{2}p(u)^2\phi(u)+S\int_{-1}^{u}dt\ \phi(t)\times\left[Rt-\left(\frac{1}{2}p(t)F_p(p(t))-F(p(t))\right)\right]\right\}=0. \tag{25}$$

As in the small ν limit, the term in square brackets does not vanish identically, so for arbitrary δp the first variation vanishes when the term in curly brackets vanishes. Taking the derivative of this term, one can see that it corresponds to the equation for the limit cycles. In practice though, when sweeping over a restricted variety of trial functions the integral may vanish at other points, which calls for some care.

Physical systems which exhibit limit cycles arise in different mechanical, electrical, and biological systems. The special class which we consider here is of particular relevance for electrical oscillators. The usual assumption, Ohm's law, that the voltage across a two-terminal device is proportional to the current through it, is not a rule but a simplification valid in restricted conditions. Resistance, inductance, and capacitance and other circuit elements are, in general, nonlinear functions of the applied voltage or current through them. For other circuit elements, like the tunnel or negative resistance diode, nonlinearity is an essential feature. The simplest circuit that leads to an equation of the form we consider is an inductance L, resistance R, capacitor C, and a tunnel diode all in parallel. The equation for such a circuit is $V''+[1/(RC)-(1/C)di/dV]V'+V/(LC)=0$, where V is the potential difference across each element. The current voltage characteristic of the diode, $i(V)$, can be expressed as a power series in V. A truncation of the series in the cubic term yields the van der Pol equation; the effect of the inclusion of higher-order terms or even nonpolynomial expressions for $i(V)$ is what we consider here. As examples we shall take a system with a current voltage relation of the form $i(V)=a_1V-a_3V^3+a_5V^5$. Experimental current voltage characteristics for resonant tunneling diodes are a subject of current interest; samples with different characteristics can be constructed [24–26]

To illustrate the use of the variational principle, we shall use for all ν a simple trial function with only one parameter K. For each value of S, we insert the trial function in Eq. (10) and sweep in K to find all the extrema. We begin with small S where we identify the true minima and follow their evolution as S is increased. We obtain a table of extrema $(R_1,R_2,R_3,\cdots)$ for each S, from which we compute the values $\nu_i=\sqrt{S/R_i}$ and the corresponding amplitude $A_i=\sqrt{SR_i}$. As we show below, with very simple trial functions one may obtain close estimates for the position of the limit cycles.

As a first example we take

$$F(p)=\frac{4}{5}p-\frac{4}{3}p^3+\frac{8}{25}p^5. \tag{26}$$

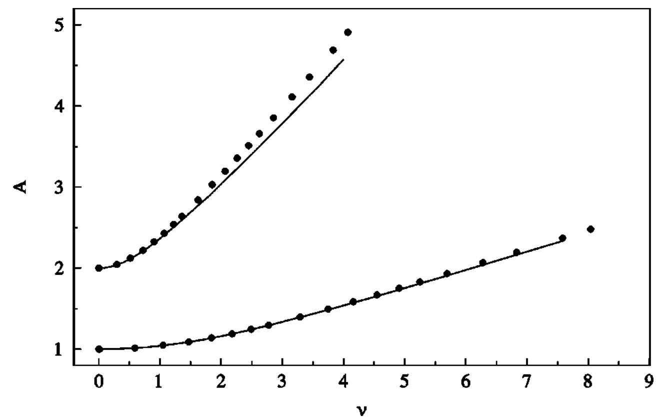

FIG. 2. Amplitude of the limit cycles for $F(p)=(4/5)p-(4/3)p^3+(8/25)p^5$. The continuous line is the result of numerical integration of the differential equation. The dots were obtained variationally with a simple trial function.

For $\nu\to 0$, Eq. (3) predicts the existence of two limit cycles, of amplitudes $A=1$ and $A=2$. As ν increases, the amplitudes change. In Fig. 2, we show the amplitudes as functions of ν. The continuous lines are the amplitudes obtained from the direct integration of the differential equation (2). Unstable limit cycles are obtained by integrating the equation with $\nu<0$ since this only changes the stability properties of the cycles. The origin is a stable fixed point, the inner limit cycle unstable, and the outer stable. To estimate the position of the limit cycles variationally, we used as a trial function the simple $\nu=0$ solution, $p(u)=K\sqrt{1-u^2}$. The dots indicate the values of the amplitudes thus obtained. The agreement is close to fairly high values of ν, in spite of having used a simple one-parameter trial function.

A more interesting example is provided by

$$F(p)=p-\sqrt{\frac{41}{9}}p^3+p^5. \tag{27}$$

Here Eq. (3) again predicts the existence of two limit cycles for $\nu\to 0$ of amplitudes $A_1=\sqrt{(\sqrt{41}-1)/5}\approx 1.039$ and $A_2=\sqrt{(\sqrt{41}+1)/5}\approx 1.216$. For this equation it is known that for large ν there is no limit cycle. A bound on the value of ν for which no limit cycles exist is known [27]. The criterion for $\nu\to\infty$ also indicates that no limit cycles exist in this regime. Numerical integration of the differential equation shows that, as ν increases, these two limit cycles merge and disappear. In Fig. 3, we show the amplitude as a function of ν. The continuous line is obtained by direct numerical integration and the dots and triangles were obtained variationally. The dotted points were obtained using again the linear trial function $p(u)=K\sqrt{1-u^2}$. At larger ν, it is convenient to use a better trial function. We have taken a trial function which is suggested by the first correction for small S. From perturbation theory we obtain

$$p(u)=A\sqrt{1-u^2}-\frac{S}{\sqrt{1-u^2}}\int_{-1}^{u}F(A\sqrt{1-x^2})dx. \tag{28}$$

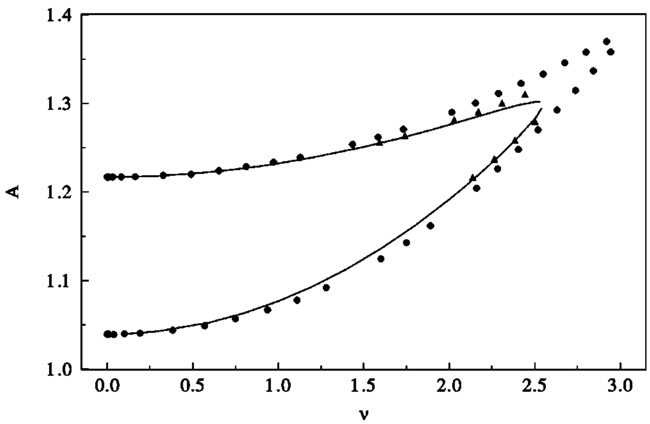

FIG. 3. Amplitude of the limit cycles for $F(p)=p-\sqrt{41/9}p^3+p^5$. The continuous line is the result of numerical integration of the differential equation. The dots and triangles were obtained variationally with simple trial functions.

Evaluated on A_2 this yields

$$p_2(u)=A_2\sqrt{1-u^2}+SA_2\frac{u(1-u^2)}{75}[(\sqrt{41}+21)u^2+4-\sqrt{41}],\tag{29}$$

and on A_1,

$$p_1(u)=A_1\sqrt{1-u^2}-SA_1\frac{u(1-u^2)}{75}[(\sqrt{41}-21)u^2-4-\sqrt{41}].\tag{30}$$

Since this will be a trial function, we set $S=1$ above, and use the trial function $p(u)=Kp_2(u)$, which is approximately $p(u)=K(1.216\sqrt{1-u^2}-0.039u+0.483u^3-0.444u^5)$. The triangles near the upper branch in Fig. 3 were obtained variationally with this trial function. Still within simple one-parameter trial functions one may guess a polynomial correction to the linear solution. For example, the triangles on the lower branch in Fig. 3 were obtained with the trial function $p(u)=K\sqrt{1-u^2}(1.118+0.339u+0.078u^2)$. This trial function does not have the correct symmetry but gives a very close estimate of the amplitude.

V. CONCLUSION

We have shown that all limit cycles of Eq. (2) correspond to extrema of a certain functional. The exact position of each limit cycle is obtained when the trial function coincides with the solution, otherwise an approximate estimation can be made. From the variational expression, analytical results can be obtained in the two asymptotic limits, $\nu\to 0$ and $\nu\to\infty$. In these asymptotic regimes, we obtain both the amplitudes of the Rayleigh and Liénard form of the equations. In the small ν limit, we reobtain the known criterion, namely Melnikov's integral. In the large ν regime, our results coincide with that obtained in recent work. In the intermediate regime, the number and position of the limit cycles can be obtained numerically. Even with simple trial functions, relatively close estimates are obtained.

Here we considered equations of the form (2). The results can be extended directly to a more general equation of the form $\ddot{z}+\nu F(\dot{z})+g(z)=0$, with $g(z)$ an antisymmetric function with $g(0)=0$. In this case, one would be able to find only the limit cycles that encircle the origin, not those encircling other equilibrium points, which do not have the symmetry we require of the solutions.

The most interesting question that remains to be studied is the possibility of giving an analytical criterion to determine the number of limit cycles for arbitrary ν and F. Other topics to be addressed are the extension of this method to nonsymmetric systems and its formulation for an equation written in Liénard's form.

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