Self-Adjointness of Two-Dimensional Dirac Operators on Domains

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Abstract. We consider Dirac operators defined on planar domains. For a large class of boundary conditions, we give a direct proof of their self-adjointness in the Sobolev space H^1.

1. Introduction

We consider a massless two-dimensional Dirac operator on a bounded domain $\Omega \subset \mathbb{R}^2$ with C^2-boundary $\partial\Omega$. Choosing appropriate units, the Dirac operator acts as the differential expression

$$T := -i\boldsymbol{\sigma}\cdot\nabla = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(-i\partial_1) + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}(-i\partial_2).$$

We denote by D_η the operator acting as T on functions in the domain

$$\mathcal{D}(D_\eta) := \{u \in H^1(\Omega, \mathbb{C}^2) \,|\, P_{-,\eta}\gamma u = 0\}.$$

Here γ is the trace operator on the boundary of Ω and the orthogonal projections $P_{\pm,\eta}$ are defined as

$$P_{\pm,\eta} = \frac{1}{2}(1 \pm A_\eta), \quad A_\eta = \cos(\eta)\boldsymbol{\sigma}\cdot\boldsymbol{t} + \sin(\eta)\sigma_3,$$

where $\boldsymbol{t}$ is the unit vector tangent to the boundary and η is a real function on the boundary.

In the physics literature operators of this type were first considered in 1987 by Berry and Mondragon to study two-dimensional neutrino billards [4]. In recent years, they have gained renewed interest due to their applications in the description of graphene quantum dots and nano-ribbons (see e.g. [8, 2, 3] and references therein). The most commonly used boundary conditions are those when $\eta \in \{0, \pi\}$ and $\eta \in \{\pi/2, 3\pi/2\}$, known as infinite mass and zigzag boundary conditions, respectively.

Using integration by parts and the hermiticity of the Pauli matrices, it is straightforward to check that D_η is a symmetric operator. We have, for all $u, v \in H^1(\Omega, \mathbb{C}^2)$,

$$\begin{aligned} \langle u, Tv \rangle &= \int_\Omega -i \,(u, \boldsymbol{\sigma} \cdot \nabla v)_{\mathbb{C}^2} \\ &= \int_\Omega -i\nabla \cdot \,(u, \boldsymbol{\sigma} v)_{\mathbb{C}^2} + i \int_\Omega (\boldsymbol{\sigma} \cdot \nabla u, v)_{\mathbb{C}^2} \\ &= \langle Tu, v \rangle - i \int_{\partial\Omega} (u, \boldsymbol{n} \cdot \boldsymbol{\sigma}\, v)_{\mathbb{C}^2}\,, \end{aligned} \tag{1}$$

where $\boldsymbol{n}$ is the outward normal vector to $\partial\Omega$. If $u, v \in \mathcal{D}(D_\eta)$, the boundary term cancels since the anticommutation relations of the Pauli matrices imply

$$\{A_\eta, \boldsymbol{n} \cdot \boldsymbol{\sigma}\} = 0. \tag{2}$$

To determine when D_η is actually self-adjoint, in the case of C^∞-boundaries, one may adapt the corresponding theorems of [5] to our case (see for instance [11]). However, the operators treated in [5] are more general and the proofs require sophisticated techniques from the analysis of pseudo-differential operators. Our proof, given in Section 2, is simpler and also works in cases with limited regularity of η and $\partial\Omega$.

Theorem 1.1. *Given $\Omega \subset \mathbb{R}^2$, bounded, with C^2-boundary, and $\eta \in C^1(\partial\Omega)$, define D_η as above. If $\cos\eta(s) \neq 0$ for all $s \in \partial\Omega$, then D_η is self-adjoint on $\mathcal{D}(D_\eta)$.*

Remark 1. *Our proof of self-adjointness is really an elliptic regularity result for the Dirac system. We implicitly establish the following inequality:*

Suppose that Ω and η satisfy the conditions of Theorem 1.1. Then there exists a constant $C > 0$ such that

$$\|u\|_{H^1(\Omega)} \leq C\left(\|u\|_{L^2(\Omega)} + \|Tu\|_{L^2(\Omega)}\right), \tag{3}$$

for all $u \in L^2(\Omega, \mathbb{C}^2)$ satisfying the boundary condition $P_{-,\eta}\gamma u = 0$. Notice that we establish below that the boundary trace γu exists in $H^{-1/2}(\partial\Omega)$ if $u, Tu \in L^2(\Omega, \mathbb{C}^2)$.

Remark 2. *We do not know whether the hypothesis $\cos\eta(s) \neq 0$ is optimal, but it can not be relaxed much. If D_η is self-adjoint on a domain contained in $H^1(\Omega, \mathbb{C}^2)$, it follows from the compact embedding of $H^1(\Omega) \subset L^2(\Omega)$ that its resolvent is compact. Thus, the spectrum of D_η consists of eigenvalues of finite multiplicity accumulating only at $\pm\infty$. This is to be contrasted with the case of zigzag boundary conditions, $\cos\eta = 0$, which has 0 in the essential spectrum. In particular, the corresponding operator is not self-adjoint on a domain included in $H^1(\Omega, \mathbb{C}^2)$ (see [12, 9]). More generally, we show in the appendix that, if $\cos\eta(s)$ tends to zero at least quadratically when $s \to s_0 \in \partial\Omega$, $0 \in \sigma_{\mathrm{ess}}(D_\eta)$.*

The rest of the paper presents the proof of Theorem 1.1. Our strategy is to show directly that $\mathcal{D}(D_\eta^*) \subset \mathcal{D}(D_\eta)$. The difficult part is showing the inclusion $\mathcal{D}(D_\eta^*) \subset H^1(\Omega, \mathbb{C}^2)$, for which it is necessary to prove the regularity of the boundary values of functions in $\mathcal{D}(D_\eta^*)$. This step exploits the interplay between the projections giving the boundary conditions and the special structure of the Dirac operator. We first establish the necessary results when $\Omega = \mathbb{D}$, the unit disc. Finally, the Riemann mapping theorem allows to treat the general case as well.

2. Self-adjointness

We first fix some notations. We work with spaces of $\mathbb{C}^2$-valued functions such as $H^1(\Omega, \mathbb{C}^2), C^\infty(\Omega, \mathbb{C}^2), \ldots$. For shortness of notation, we often omit the $\mathbb{C}^2$ and just write $H^1(\Omega), C^\infty(\Omega), \ldots$ when no possible confusion occurs. We will consider a fixed domain Ω with C^2-boundary $\partial\Omega$. We denote by $\boldsymbol{n}(s)$ and $\boldsymbol{t}(s)$ the outward normal and the tangent vector to the boundary at the point $s \in \partial\Omega$, chosen such that $\boldsymbol{n}, \boldsymbol{t}$ is positively oriented. If $\boldsymbol{t}(s) = (t_1(s), t_2(s))$, we define $t(s) = t_1(s) + it_2(s)$, the tangent vector seen as a number in $\mathbb{C}$. Associated to the domain Ω we have the trace operator at the boundary $\gamma : C^1(\overline{\Omega}) \to C^1(\partial\Omega)$, and an extension operator $E : C^1(\partial\Omega) \to C^1(\overline{\Omega})$. We recall that γ extends to a bounded operator from $H^{s+1/2}(\Omega)$ to $H^s(\partial\Omega)$, and E from $H^s(\partial\Omega)$ to $H^{s+1/2}(\Omega)$ for all $s \in (0, 2)$. We denote by $\mathscr{D}'(\Omega)$ the space of distributions, i.e., the dual of $C_0^\infty(\Omega)$.

We will also consider a fixed function η defining the boundary conditions and write simply D for D_η.

In passing, we recall our definition for the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy the (anti)commutation relations

$$\{\sigma_j, \sigma_k\} = 2\delta_{jk}, \quad [\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l, \quad j, k, l \in \{1, 2, 3\},$$

where δ_{jk} is the Kronecker delta and ϵ_{jkl} is the Levi-Civita symbol, which is totally antisymmetric and normalized by $\epsilon_{123} = 1$.

2.1. General considerations

First, we will need some regularity properties of $v \in \mathcal{D}(D^*)$.

Lemma 2.1. *Let $\mathcal{K} := \{u \in L^2(\Omega, \mathbb{C}^2) \,|\, Tu \in L^2(\Omega, \mathbb{C}^2)\}$ equipped with the graph-norm $\|u\|_{\mathcal{K}}^2 = \|u\|^2 + \|Tu\|^2$, where T acts as a differential operator on distributions in Ω. Then $\mathcal{K}$ is a Hilbert space and $C^\infty(\overline{\Omega}, \mathbb{C}^2)$ is dense in $\mathcal{K}$.*

Proof. First we show that $\mathcal{K}$ is a Hilbert space. Take a Cauchy sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{K}$ with $u_n \to u$ and $Tu_n \to v$ in L^2. We have for any test function $\varphi \in C_0^\infty(\Omega)$

$$Tu[\varphi] = u[T\varphi] = \lim_{n\to\infty} \langle u_n, T\varphi\rangle = \lim_{n\to\infty} Tu_n[\varphi] = \lim_{n\to\infty} \langle Tu_n, \varphi\rangle = \langle v, \varphi\rangle\,.$$

Therefore, $Tu = v$ and in particular $u \in \mathcal{K}$.

Recall that by definition $u \in C^\infty(\overline{\Omega})$ iff u is the restriction to Ω of a smooth function (spinor) on $\mathbb{R}^2$. To prove the density of $C^\infty(\overline{\Omega})$ it suffices to show that if

$$\langle v, u \rangle_{\mathcal{K}} = \langle v, u \rangle + \langle Tv, Tu \rangle = 0, \tag{4}$$

for all $u \in C^\infty(\overline{\Omega})$ then v vanishes. Let $w := Tv \in L^2(\Omega)$. It follows from (4) that

$$Tw = -v \quad \text{in} \quad \mathscr{D}'(\Omega). \tag{5}$$

Define $\widetilde{v}$ and $\widetilde{w}$ as the extensions by zero to $L^2(\mathbb{R}^2)$ of v and w, respectively. For any $\varphi \in C_0^\infty(\mathbb{R}^2)$ we calculate using (4)

$$T\widetilde{w}[\varphi] = \langle \widetilde{w}, T\varphi \rangle_{L^2(\mathbb{R}^2)} = \langle w, T\varphi \rangle_{L^2(\Omega)} = \langle -v, \varphi \rangle_{L^2(\Omega)} = \langle -\widetilde{v}, \varphi \rangle_{L^2(\mathbb{R}^2)} .$$

Therefore, $T\widetilde{w} = -\widetilde{v} \in L^2(\mathbb{R}^2)$. By ellipticity we find that $\widetilde{w} \in H^1(\mathbb{R}^2)$. Moreover, using [7, Proposition IX.18] we get that $w \in H_0^1(\Omega)$.

Let $(\varphi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ be a sequence with $\varphi_n \to w$ in $H^1(\Omega)$. For any $u \in \mathcal{K}$

$$\begin{aligned}
\langle v, u \rangle_{\mathcal{K}} &= \langle v, u \rangle_{L^2(\Omega)} + \langle w, Tu \rangle_{L^2(\Omega)} \\
&= \langle v, u \rangle_{L^2(\Omega)} + \lim_{n\to\infty} \langle T\varphi_n, u \rangle_{L^2(\Omega)} \\
&= \langle v, u \rangle_{L^2(\Omega)} + \langle Tw, u \rangle_{L^2(\Omega)} = 0,
\end{aligned}$$

where the last equality follows from (5) and implies that $v = 0$. □

Lemma 2.2. *We have that $\mathcal{D}(D^*) \subset \mathcal{K}$. Moreover, $\mathcal{K} \subset H^1_{\mathrm{loc}}(\Omega, \mathbb{C}^2)$.*

Proof. Fix $v \in \mathcal{D}(D^*)$ and define $\tilde{v} := D^* v \in L^2(\Omega)$. By definition Tv is a distribution, thus for any $u \in C_0^\infty(\Omega)$

$$Tv[u] \equiv \langle v, Tu \rangle = \langle v, Du \rangle = \langle \tilde{v}, u \rangle ,$$

since $C_0^\infty(\Omega) \subset \mathcal{D}(D)$. This shows that the distribution Tv can be identified with the L^2-function D^*v and thus $v \in \mathcal{K}$.

Let now $v \in \mathcal{K}$. By Lemma 2.1 we may choose a sequence of $C^\infty(\Omega)$-functions $(v_n)_{n\in\mathbb{N}}$ that converges to v in $L^2(\Omega)$ and such that Tv_n converges to Tv in $L^2(\Omega)$.

Fix an open set A such that $\bar{A} \subset \Omega$. We will show that ∇v_n converges in $L^2(A)$. Take a cut-off function $\chi_A \in C_0^\infty(\Omega)$ such that $\chi_A = 1$ on A. By equation (1) we have, for all $u \in C_0^\infty(\Omega)$, that $\|Tu\| = \|\nabla u\|$. Thus we can bound

$$\begin{aligned}
\int_A |\nabla(v_n - v_m)|^2_{\mathbb{C}^2} &\leq \int_\Omega |\nabla \chi_A (v_n - v_m)|^2_{\mathbb{C}^2} \\
&= \int_\Omega |T\chi_A(v_n - v_m)|^2_{\mathbb{C}^2} \\
&\leq \|\nabla\chi_A\|^2_\infty \|v_n - v_m\|^2 + \|T(v_n - v_m)\|^2.
\end{aligned}$$

This finishes the proof. □

By Lemma 2.2 the difficult part in proving the inclusion $\mathcal{D}(D^*) \subset \mathcal{D}(D)$ is to show regularity of $v \in \mathcal{D}(D^*)$ up to the boundary. To do so it is sufficient to prove that v has a sufficiently regular trace on the boundary $\partial\Omega$. First we show that traces exist as distributions.

Lemma 2.3. *The trace γ extends to a continuous map*

$$\gamma : \mathcal{K} \to H^{-1/2}(\partial\Omega, \mathbb{C}^2).$$

Moreover, if $v \in \mathcal{D}(D^*)$ then $P_-\gamma v = 0$. An equivalent formulation of this is that $\gamma v_2 = \frac{1-\sin(\eta)}{\cos(\eta)} t\gamma v_1$.

Proof. Let $v \in \mathcal{K}$ and let $(v_n)_{n\in\mathbb{N}}$ be a $C^\infty(\overline{\Omega})$-sequence approximating v in the $\mathcal{K}$-norm. We will show that the traces γv_n of the v_n's converge in $H^{-1/2}(\partial\Omega)$.

Fix $f \in C^\infty(\partial\Omega)$. By [1, Theorem 7.53] it is possible to extend f to a regular function $u \equiv Ef$ on Ω satisfying $\gamma u = f$ with $\|u\|_{H^1(\Omega)} \leq C_E \|f\|_{H^{1/2}(\partial\Omega)}$, with C_E only depending on Ω. By the same calculation as in (1),

$$i \int_{\partial\Omega} (\gamma v_n, \boldsymbol{\sigma}\cdot\boldsymbol{n}\, f) = \langle T v_n, u\rangle - \langle v_n, Tu\rangle \,.$$

This shows

$$\begin{aligned} |\langle \gamma(v_n - v_m), \boldsymbol{\sigma}\cdot\boldsymbol{n}\, f\rangle_{\partial\Omega}| &\leq \|T(v_n - v_m)\|\|u\| + \|v_n - v_m\|\|\nabla u\| \\ &\leq \big(\|T(v_n - v_m)\| + \|v_n - v_m\|\big)\|u\|_{H^1(\Omega)} \\ &\leq C_E\big(\|T(v_n - v_m)\| + \|v_n - v_m\|\big)\|f\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

This in turn proves that the limit $\boldsymbol{\sigma}\cdot\boldsymbol{n}\,\gamma v$ of $\boldsymbol{\sigma}\cdot\boldsymbol{n}\,\gamma v_n$ exists in $H^{-1/2}(\partial\Omega)$. Since $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ is a pointwise invertible matrix (in fact $(\boldsymbol{\sigma}\cdot\boldsymbol{n})^2 = 1$) with C^1-entries, the same conclusion holds for γv.

Assume now that $v \in \mathcal{D}(D^*)$ and that $u \in \mathcal{D}(D)$, then $f := \gamma u = P_+ f$ and

$$i \int_{\partial\Omega} (\gamma v, \boldsymbol{\sigma}\cdot\boldsymbol{n}\, f)_{\mathbb{C}^2} = \int_\Omega (Tv, u)_{\mathbb{C}^2} - (v, Du)_{\mathbb{C}^2} = \langle D^* v, u\rangle - \langle v, Du\rangle = 0.$$

In addition, using (2) we have that $P_+\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{\sigma}\cdot\boldsymbol{n}\, P_-$. Then, $(\gamma v, \boldsymbol{\sigma}\cdot\boldsymbol{n}\, f)_{\mathbb{C}^2} = (\gamma v, \boldsymbol{\sigma}\cdot\boldsymbol{n}\, P_+ f)_{\mathbb{C}^2} = (\gamma v, P_-\boldsymbol{\sigma}\cdot\boldsymbol{n}\, f)_{\mathbb{C}^2} = (P_-\gamma v, \boldsymbol{\sigma}\cdot\boldsymbol{n}\, f)_{\mathbb{C}^2}$. Thus, we have shown that $P_-\gamma v = 0$. This finishes the proof. □

The next lemma shows that improving the regularity of the traces is all that is left to do.

Lemma 2.4. *If $v \in \mathcal{K}$ and $\gamma v \in H^{1/2}(\partial\Omega, \mathbb{C}^2)$, then $v \in H^1(\Omega, \mathbb{C}^2)$.*

Proof. Let $v \in \mathcal{K}$ with $\gamma v \in H^{1/2}(\partial\Omega)$. By replacing v by $v - E(\gamma(v))$, where $E : H^{1/2}(\partial\Omega) \mapsto H^1(\Omega)$ is the (continuous) extension operator, it suffices to consider the case when $\gamma v = 0$.

Write $w := Tv \in L^2(\Omega)$. Next we show that

$$\langle v, T\varphi\rangle = \langle w, \varphi\rangle\,, \quad \text{for all} \quad \varphi \in C^\infty(\overline{\Omega}, \mathbb{C}^2). \tag{6}$$

Let $(v_n)_{n\in\mathbb{N}}$ be a $C^\infty(\overline{\Omega})$-sequence approximating v in the $\mathcal{K}$-norm. Then by Lemma 2.3, $\gamma v_n \to \gamma v = 0$ in $H^{-1/2}(\partial\Omega)$. We calculate for $\varphi \in C^\infty(\overline{\Omega})$ using (1)

$$\begin{aligned}\langle v, T\varphi\rangle &= \lim_{n\to\infty} \langle v_n, T\varphi\rangle = \lim_{n\to\infty}\left(\langle T v_n, \varphi\rangle - i\int_{\partial\Omega} (\gamma v_n, \boldsymbol{n}\cdot\boldsymbol{\sigma}\,\gamma\varphi)_{\mathbb{C}^2}\right)\\ &= \langle w, \varphi\rangle\,,\end{aligned}$$

where the boundary term vanishes since $\gamma\varphi \in H^{1/2}(\partial\Omega)$. This proves (6).

Let $\widetilde{v}$ and $\widetilde{w}$ be the extensions by zero to $L^2(\mathbb{R}^2)$ of v and w, respectively. Then, by (6)

$$T\widetilde{v} = \widetilde{w}, \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^2). \tag{7}$$

From this we conclude that $\widetilde{v} \in H^1(\mathbb{R}^2)$ and thus $v \in H^1(\Omega)$. This finishes the proof. □

In order to take advantage of the special structure of the Dirac operator, it will be convenient to identify $\boldsymbol{x} \in \mathbb{R}^2$ with the complex number $z = x_1 + ix_2$. In this notation, the Dirac operator reads

$$Tu(z) = -2i\begin{pmatrix} 0 & \partial_z \\ \partial_{z^*} & 0\end{pmatrix} u(z) = -2i\begin{pmatrix} \partial_z u_2(z) \\ \partial_{z^*} u_1(z)\end{pmatrix},$$

where we introduced the Cauchy-Riemann derivatives $\partial_z := \frac{1}{2}(\partial_1 - i\partial_2)$ and $\partial_{z^*} := \frac{1}{2}(\partial_1 + i\partial_2)$. In addition, we introduce the Cauchy kernel

$$(Kf)(\zeta) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z-\zeta}\,\mathrm{d}z$$

and its formal conjugate

$$(\overline{K}f)(\zeta) = \frac{-1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z^*-\zeta^*}\,\mathrm{d}z^*.$$

The kernels $K, \overline{K}$ clearly define operators from $C^\infty(\partial\Omega, \mathbb{C})$ to $C^\infty(\Omega, \mathbb{C})$. With these definitions we construct an operator on $C^\infty(\partial\Omega, \mathbb{C}^2)$ by setting

$$S = \begin{pmatrix} K & 0 \\ 0 & \overline{K}\end{pmatrix}.$$

Actually, $-2\gamma S\boldsymbol{\sigma}\cdot\boldsymbol{n}$ coincides with the Calderón projector for the Dirac operator as defined for instance in [6, Chapter 12].

2.2. The Cauchy kernel on the unit circle

On the unit circle $\mathbb{S}$ the operators K and $\overline{K}$ are explicit when acting on the standard basis. For this reason we will first establish all the necessary properties on the disc, $\Omega = \mathbb{D}$, and then translate them to general domains essentially by using the Riemann Mapping Theorem.

Define the orthonormal basis

$$e_n(\theta) = (2\pi)^{-1/2} e^{in\theta} \in L^2(\mathbb{S}),$$

in the standard parametrization of $\mathbb{S}$. An explicit calculation yields,

$$Ke_n(\zeta) = \begin{cases} (2\pi)^{-1/2}\zeta^n, & n \geq 0, \\ 0, & n < 0, \end{cases} \tag{8}$$

and

$$\overline{K}e_n(\zeta) = \begin{cases} 0, & n > 0, \\ (2\pi)^{-1/2}(\zeta^*)^{|n|}, & n \leq 0. \end{cases}$$

Furthermore for L^2-functions on the unit circle, we will denote the Fourier coefficients

$$\widehat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(\theta)e^{-in\theta}\,\mathrm{d}\theta = \langle e_n, f\rangle\,.$$

We set for $s \in \mathbb{R}$

$$\|f\|_{H^s}^2 = \sum_{n\in\mathbb{Z}} (|n|+1)^{2s}|\widehat{f}(n)|^2.$$

The properties of K and $\overline{K}$ that we will need are grouped in the following proposition.

Proposition 2.5. *If $\Omega = \mathbb{D}$ and K, $\overline{K}$ are defined as above, then for all $s \in [-1/2, 1/2]$*

i) K and $\overline{K}$ extend to bounded operators from $H^{-1/2}(\mathbb{S})$ to $L^2(\mathbb{D})$.

ii) For all $f \in H^s(\mathbb{S})$ we have $\partial_{z^*}Kf = 0$ and $\partial_z\overline{K}f = 0$ with derivatives taken in the sense of distributions.

iii) γK and $\gamma\overline{K}$ extend to bounded operators on $H^s(\mathbb{S})$ and they are self-adjoint projections onto $\operatorname{span}\{e_n|n \geq 0\}$ *and* $\operatorname{span}\{e_n|n \leq 0\}$*, respectively.*

iv) $\gamma K + \gamma\overline{K} = 1 + \langle e_0, \cdot\rangle\, e_0$ when acting on $H^s(\mathbb{S})$.

v) For $\beta \in C^1(\mathbb{S})$ and $s = -1/2$ or $s = 0$ the commutators $[\beta, \gamma K]$ and $[\beta, \gamma\overline{K}]$ are bounded from $H^s(\mathbb{S})$ to $H^{s+1/2}(\mathbb{S})$.

Proof. Point iv) is a direct consequence of iii). We will prove the remaining points for K only, since the same ideas apply to $\overline{K}$. Also, it is sufficient to establish these properties for continuous functions, since all statements extend to general elements of H^s by density.

In this setting, point i) follows from (8), since $\left\langle \zeta^n, \zeta^k\right\rangle = \frac{\pi}{n+1}\delta_{n,k}$ and

$$\|Kf\|_{L^2}^2 = \sum_{n,k\geq 0} \widehat{f}(n)^*\widehat{f}(k)\,\langle Ke_n, Ke_k\rangle = \sum_{n\geq 0} \frac{1}{2n+2}|\widehat{f}(n)|^2 \leq \|f\|_{H^{-1/2}}^2.$$

The proof of ii) is straightforward. Using (8) again we have that

$$(\gamma K)e_n = \begin{cases} e_n, & n \geq 0, \\ 0, & n < 0, \end{cases}$$

which establishes point iii).

To see v), we take $s = -1/2$ or $s = 0$, fix $f \in C^1(\partial\Omega)$ and compute the Fourier coefficients of $[\beta, \gamma K]f = \beta\gamma Kf - \gamma K(\beta f)$,

$$\sqrt{2\pi}\big([\beta,\gamma K]f\big)^{\wedge}(n) = \begin{cases} \sum_{k\geq 0}\widehat{\beta}(n-k)\widehat{f}(k) - \sum_{k\in\mathbb{Z}}\widehat{\beta}(n-k)\widehat{f}(k), & n\geq 0,\\ \sum_{k\geq 0}\widehat{\beta}(n-k)\widehat{f}(k), & n<0. \end{cases}$$

By Cauchy-Schwarz,

$$\begin{aligned} &2\pi|\big([\beta,\gamma K]f\big)^{\wedge}(n)|^2 \\ &\quad\leq \begin{cases} \sum\limits_{k<0}|\widehat{\beta}(n-k)|^2(|k|+1)^{-2s}\sum\limits_{k<0}|\widehat{f}(k)|^2(|k|+1)^{2s}, & n\geq 0,\\ \sum\limits_{k\geq 0}|\widehat{\beta}(n-k)|^2(|k|+1)^{-2s}\sum\limits_{k\geq 0}|\widehat{f}(k)|^2(|k|+1)^{2s}, & n<0, \end{cases} \\ &\quad\leq \|f\|_{H^s}^2 \begin{cases} \sum_{k<0}|\widehat{\beta}(n-k)|^2(|k|+1)^{-2s}, & n\geq 0,\\ \sum_{k\geq 0}|\widehat{\beta}(n-k)|^2(|k|+1)^{-2s}, & n<0. \end{cases} \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \|[\beta,\gamma K]f\|_{H^{s+1/2}}^2 &= \sum_{n\in\mathbb{Z}}(|n|+1)^{2s+1}|\big([\beta,\gamma K]f\big)^{\wedge}(n)|^2 \\ &\leq \|f\|_{H^s}^2\Big(\sum_{\substack{n\geq 0\\ k<0}}(|n|+1)^{2s+1}(|k|+1)^{-2s}|\widehat{\beta}(n-k)|^2 \\ &\qquad + \sum_{\substack{n<0\\ k\geq 0}}(|n|+1)^{2s+1}(|k|+1)^{-2s}|\widehat{\beta}(n-k)|^2\Big). \end{aligned}$$

Since either $2s+1=0$ or $s=0$ we get

$$(|n|+1)^{2s+1}(|k|+1)^{-2s}\leq |n|+|k|+1 = |n-k|+1,$$

where the last equality holds since n and k have opposite signs in the sums we are considering. This allows us to conclude that

$$\begin{aligned} \|[\beta,\gamma K]f\|_{H^{s+1/2}}^2 \leq &\|f\|_{H^s}^2\Big(\sum_{\substack{n\geq 0\\ k<0}}(|n-k|+1)|\widehat{\beta}(n-k)|^2 \\ &\qquad + \sum_{\substack{n<0\\ k\geq 0}}(|n-k|+1)|\widehat{\beta}(n-k)|^2\Big) \\ \leq &\|f\|_{H^s}^2\Big(\sum_{m\geq 1}(|m|+1)^2|\widehat{\beta}(m)|^2 + \sum_{m\leq -1}(|m|+1)^2|\widehat{\beta}(m)|^2\Big) \\ \leq &\|f\|_{H^s}^2\|\beta\|_{H^1}^2, \end{aligned}$$

which finishes the proof. □

The following lemma relates the operators K, $\overline{K}$ and S to our problem at hand.

Lemma 2.6. *Let $\Omega=\mathbb{D}$ and assume $v\in\mathcal{K}$. Then $\gamma S(\boldsymbol{\sigma}\cdot\boldsymbol{n}\,\gamma v)\in H^{1/2}(\mathbb{S},\mathbb{C}^2)$.*

Proof. Take a test function $f \in C^\infty(\mathbb{S}, \mathbb{C}^2)$ and a sequence $(v_n) \subset C^1(\overline{\mathbb{D}}, \mathbb{C}^2)$ approaching v in $\mathcal{K}$. By Proposition 2.5 iii), γS is self-adjoint, thus using (1)

$$\begin{aligned}\int_{\mathbb{S}} (\gamma S(\boldsymbol{\sigma}\cdot\boldsymbol{n}\,\gamma v_n), f)_{\mathbb{C}^2} &= \int_{\mathbb{S}} (\gamma v_n, \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\gamma S f)_{\mathbb{C}^2} \\ &= -i\,\langle T v_n, Sf\rangle + i\,\langle v_n, TSf\rangle\,.\end{aligned}$$

The last term above cancels since, by Proposition 2.5 ii), $TSf = 0$. Thus, in view of Proposition 2.5 i)

$$\begin{aligned}\Big|\int_{\mathbb{S}} (\gamma S(\boldsymbol{\sigma}\cdot\boldsymbol{n}\,\gamma v_n), f)_{\mathbb{C}^2}\Big| &\le \|Tv_n\|_{L^2(\mathbb{D})}\|Sf\|_{L^2(\mathbb{D})} \\ &\le C_K\|Tv_n\|_{L^2(\mathbb{D})}\|f\|_{H^{-1/2}(\mathbb{S})}.\end{aligned}$$

Taking the limit as $n \to \infty$ on both sides we see that $\gamma S(\boldsymbol{\sigma}\cdot\boldsymbol{n}\,\gamma v)$ extends to a continuous functional on $H^{-1/2}$, and thus can be identified with a function in $H^{1/2}$. □

The next lemma allows us to conclude the proof of self-adjointness when $\Omega = \mathbb{D}$, see Remark 3.

Lemma 2.7. *Let $\Omega = \mathbb{D}$ and β be a nowhere vanishing $C^1(\mathbb{S})$-function. Assume that $v \equiv \binom{v_1}{v_2} \in \mathcal{K}$ and that $\gamma v_1 = \beta\gamma v_2$ as an equality in $H^{-1/2}(\mathbb{S})$. Then $\gamma v \in H^{1/2}(\mathbb{S}, \mathbb{C}^2)$.*

Remark 3. *In view of Lemma 2.3, $v \in \mathcal{D}(D^*)$ satisfies the hypotheses of Lemma 2.7 with $\beta = \frac{t^*\cos\eta}{1-\sin\eta}$. Thus, according to Lemma 2.4, $v \in H^1(\mathbb{D}, \mathbb{C}^2)$ satisfies the boundary conditions. In particular, $\mathcal{D}(D^*) \subset \mathcal{D}(D)$.*

Proof. Let us write

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \begin{pmatrix} 0 & n^* \\ n & 0 \end{pmatrix}.$$

In order to apply Lemma 2.6, we define the spinor $f = \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\gamma v$. Due to the boundary condition we have that $f_2 = \tilde{\beta} f_1$ where $\tilde{\beta} = (n)^2\beta$ is a C^1-function. In this notation Lemma 2.6 states that

$$\gamma K f_1 \in H^{1/2}, \quad \gamma\overline{K} f_2 \in H^{1/2}. \tag{9}$$

Now we write

$$\gamma K f_2 = \gamma K(\tilde{\beta} f_1) = \tilde{\beta}\gamma K f_1 - [\tilde{\beta}, \gamma K] f_1. \tag{10}$$

Clearly $\tilde{\beta}\gamma K f_1$ is in $H^{1/2}$. By Proposition 2.5 v), the term with the commutator is in L^2, so $\gamma K f_2 \in L^2$ as well. This together with (9) gives that $f_2 \in L^2$, in view of Proposition 2.5 iv). Since $\tilde{\beta}$ does not vanish, f_1 is also in L^2 due to the boundary conditions. With this improved regularity we return to (10) and observe that, due to Proposition 2.5 v), $[\tilde{\beta}, \gamma K] f_1$ is in $H^{1/2}$ so the same holds for $\gamma K f_2$. Again using the complementarity of the projections and the fact $\tilde{\beta}$ does not vanish, we conclude $f_1, f_2 \in H^{1/2}$. □

2.3. Riemann mapping and the proof of Theorem 1.1

We first give the proof in the case where Ω is simply connected. The case of multiply connected domains will be treated at the end of this section. Since $\partial\Omega$ is C^2, there exists a C^1 conformal mapping (up to the boundary) $F : \overline{\Omega} \to \overline{\mathbb{D}}$ with inverse G [10, Theorem 3.5, p. 48]. Consider the map U defined by $(Uf)(z) := f(G(z))$ mapping functions on $\overline{\Omega}$ to functions on $\overline{\mathbb{D}}$. By restriction (and abuse of notation), U also maps functions on $\partial\Omega$ to functions on $\mathbb{S}$.

Lemma 2.8. *When Ω is simply connected and has C^2-boundary, the map U defines a bounded bijection from $L^2(\Omega)$ to $L^2(\mathbb{D})$ with bounded inverse. Furthermore, $U : H^s(\Omega) \to H^s(\mathbb{D})$ is bounded with bounded inverse, for all $s \in [-1, 1]$.*

Similarly, $U : H^s(\partial\Omega) \to H^s(\mathbb{S})$ is bounded with bounded inverse, for all $s \in [-1, 1]$.

Finally, if $v \in \mathcal{D}(D^*)$, then $Uv = (u_1, u_2) \in \mathcal{K}(\mathbb{D})$ and on the boundary $\gamma u_1 = \beta\gamma u_2$ as an identity in $H^{-1/2}(\mathbb{S})$, where $\beta = U(\frac{t^* \cos(\eta)}{1-\sin(\eta)})$ is $C^1(\mathbb{S})$.

Proof. Since F, G have bounded derivatives on $\overline{\Omega}$ (resp $\overline{\mathbb{D}}$) the map $L^2(\Omega) \ni v \mapsto u := Uv = v \circ G \in L^2(\Omega)$ is a bounded bijection with bounded inverse. By direct differentiation one verifies that U is also bounded from $H^1(\Omega)$ to $H^1(\mathbb{D})$ with bounded inverse. By interpolation and duality one finds that also $U : H^s(\Omega) \to H^s(\mathbb{D})$ is bounded with bounded inverse, for all $s \in [-1, 1]$.

The same argument as in the interior applies on the boundary, so we see that $U : H^s(\partial\Omega) \to H^s(\mathbb{S})$ is bounded with bounded inverse, for all $s \in [-1, 1]$.

Suppose now that $v = (v_1, v_2) \in \mathcal{D}(D^*)$. Then, since $\partial_{z^*} G = 0$, we have by the chain rule,

$$\partial_z u_2 = G' \partial_z v_2 \in L^2(\mathbb{D}), \qquad \partial_{z^*} u_1 = (G')^* \partial_{z^*} v_1 \in L^2(\mathbb{D}).$$

Finally, the boundary condition $\gamma u_1 = \beta\gamma u_2$ follows from the boundary condition satisfied by v, see Lemma 2.3. □

Now we can conclude the proof of the self-adjointness of D.

Proof of Theorem 1.1. *Simply connected case.* Fix $v \in \mathcal{D}(D^*)$. By Lemmas 2.3 and 2.4, we only have to prove that v has a well-defined trace in $H^{1/2}(\partial\Omega)$. By Lemma 2.8, this is equivalent to showing that $\gamma u := \gamma Uv \in H^{1/2}(\mathbb{S})$, where U is the map defined above. By the same lemma, $u \in \mathcal{K}$ and its components u_1, u_2 satisfy the boundary condition $\gamma u_1 = \beta\gamma u_2$, with $\beta = U(\frac{t^* \cos(\eta)}{1-\sin(\eta)})$. Since β vanishes nowhere by assumption, we can apply Lemma 2.7 and conclude the proof of the theorem in this case.

Multiply connected case. It clearly suffices to consider connected Ω. Suppose that $\partial\Omega$ is made up of the simple, regular curves $\Gamma_0, \dots, \Gamma_n$, with $n \geq 1$. Let Ω_j be the interior components of $\mathbb{R}^2 \setminus \Gamma_j$ (given by the Jordan Curve Theorem). Let Γ_0 be the exterior boundary. Since Ω is connected, $\Omega \subset \Omega_0$ and $\Omega \subset \mathbb{R}^2 \setminus \overline{\Omega}_j$ for $j \geq 1$.

Let first $F_0 : \Omega_0 \to \mathbb{D}$ be the conformal (Riemann) map and let $U_0 : L^2(\Omega) \to L^2(F_0(\Omega))$ be the push-forward map as in Lemma 2.8. Proceeding as in the proof in the simply connected case, using U_0 instead of U, one concludes the desired $H^{1/2}$-regularity on the boundary component Γ_0.

For $j \in \{1, \dots, n\}$ let $z_j \in \Omega_j$. To obtain the $H^{1/2}$-regularity on the boundary component Γ_j, one first applies the fractional transformation $I_j(z) = (z - z_j)^{-1}$. After this transformation, $I_j(\Gamma_j)$ is the external boundary of $I_j(\Omega)$ and one can proceed as in the previous case. Notice that since $z_j \in \Omega_j$, the map I_j (and its inverse) has bounded derivatives to all orders in Ω and therefore preserves Sobolev spaces in a similar manner to Lemma 2.8. This finishes the proof of Theorem 1.1. □

Appendix A. Construction of a Weyl sequence

In this appendix, we construct a singular Weyl sequence for D_η at 0 in the case where $\cos\eta$ vanishes to second order at a point of $\partial\Omega$. This shows that if D_η has a self-adjoint realisation then 0 is in its essential spectrum. Therefore, by the compact embedding of H^1 in L^2, the domain of such a realisation cannot be included in H^1.

We will assume for definiteness that η tends quadratically to $\frac{\pi}{2}$ at a point of the boundary. Then, the boundary conditions can be written $\gamma u_2 = Bt\gamma u_1$, where $B = (1 - \sin\eta)/\cos\eta$. Our assumption implies then that $|B(s)| \leq C(s - s_0)^2$ and that $|\boldsymbol{t}\cdot \nabla B(s)| \leq C|s - s_0|$ for some $s_0 \in \partial\Omega$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ and assume for definiteness that $s_0 = 0$ and that $t(0) = i$. Then, we can find $R_0 > 0$ such that

$$\Omega \cap \{re^{i\phi} | 0 \leq r \leq R_0, |\phi| \leq \pi/4\} = \varnothing. \tag{11}$$

Taking a smaller R_0 if necessary, we may also assume that B and t can be extended to C^1-functions on $\overline{\Omega} \cap B(0, R_0)$, such that

$$\frac{|B(z)|}{|z|} + |\nabla B(z)| \leq C_B|z|, \quad |t| + |\nabla t| \leq C_t, \text{ for all } z \in \overline{\Omega} \cap B(0, R_0),$$

where C_B and C_t are positive constants. This is always possible since such constants exist for $z \in \partial\Omega$ and Ω has a C^2-boundary. We also fix a cutoff function $\chi \in C^\infty(\mathbb{R}, [0,1])$ such that $\chi(x) = 1$ for $x \leq 1/2$, $\chi(x) = 0$ for $x \geq 1$ and $|\chi'| \leq 3$. For $R \geq 0$, define $\chi_R(z) = \chi(|z|/R)$.

Now for $n \geq 1$, we set

$$u_n(z) = (z - s_n)^{-n} \begin{pmatrix} 1 \\ Bt \end{pmatrix}$$

for some $s_n > 0$. Define $v_n := \chi_{R_n} u_n$. Notice that $v_n \in \mathcal{D}(D_\eta)$ for all $R_n \leq R_0$ and we have that

$$\|v_n\| \geq \|\chi_{R_n}(z - s_n)^{-n}\|.$$

On the other hand,

$$\begin{aligned}\|Tv_n\| &\le \||\nabla\chi_{R_n}|u_n\| + 2\|\chi_{R_n}\begin{pmatrix}\partial_z Bt(z-s_n)^{-n}\\ \partial_{z^*}(z-s_n)^{-n}\end{pmatrix}\| \\ &\le \frac{3}{R_n}\|\mathbf{1}_{[R_n/2,R_n]}(|z|)u_n\| + 2C_BC_tR_n\|\chi_{R_n}(z-s_n)^{-n}\| \\ &\quad + 2nC_t\|\chi_{R_n}B(z-s_n)^{-n-1}\|,\end{aligned}$$

where $\mathbf{1}_I$ is the indicator function on an intervall $I \subset \mathbb{R}$. The last term can be estimated further by observing that, within $\operatorname{supp}\chi_{R_n} \cap \Omega$,

$$\frac{|B|}{|z-s_n|} \le C_B\frac{|z|^2}{|z-s_n|} \le C_BR_n\sqrt{2},$$

where the last inequality holds in view of (11). Thus, we obtain

$$\frac{\|Tv_n\|}{\|v_n\|} \le \frac{3}{R_n}\frac{\|\mathbf{1}_{[R_n/2,R_n]}(|z|)u_n\|}{\|\chi_{R_n}(z-s_n)^{-n}\|} + C_BC_t(2+\sqrt{2}n)R_n.$$

We now fix $R_n \le R_0$ such that the second term in the above equation is smaller than $1/2n$. In the first term, we note that, as $s_n \searrow 0$ for a fixed R_n, the numerator stays bounded while the denominator increases to $+\infty$. Thus, by choosing a sufficiently small s_n, we obtain

$$\frac{\|Tv_n\|}{\|v_n\|} \le \frac{1}{n}.$$

In addition, the sequence $v_n/\|v_n\|$ converges weakly to zero, so it is a singular Weyl sequence, which proves $0 \in \sigma_{\mathrm{ess}}(D_\eta)$.

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