On the Solution of Algebraic Equations

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My first experience with algebraic equations came from a problem that my maternal grandfather told me. He probably remembered it from his time as a student, towards the end of the previous century [T.N. 1]. The problem in question was the following: "A falcon perched in the branch of a tree observes the passage of a flock of doves and says to them, 'Goodbye, hundred doves!' to which greeting one of the doves replies, 'You are mistaken, sir falcon, we are not a hundred. We, plus as many as ourselves again, plus half as many, plus a quarter as many, plus you, sir falcon, are a hundred.' How many doves were in the flock?" When I heard this problem for the first time I knew nothing of algebra, so I solved it by trying various numbers, and after many attempts I obtained the correct answer: 36 doves. In the first courses of algebra one encounters numerous problems like this, and so appreciates how simple it is to solve them by reducing them to algebraic equations. In the case of the problem of the doves, if one calls the number of doves x, one can write the response of the dove to the greeting of the falcon as

$$x + x + \frac{1}{2}x + \frac{1}{4}x + 1 = 100. \tag{1}$$

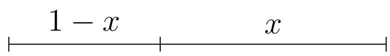
from which we obtain $11x/4 = 99$ and, finally, $x = 36$.

Equation (1) is an algebraic equation of the first degree. Equations like this and their solutions have been known since antiquity.

The type of equation that follows in difficulty is the equation of the second degree

$$x^2 + bx + c = 0. \tag{2}$$

Equations of the second degree were known to the Babylonians, although the algebraic solution such as we know today appears probably for the first time in the Arabic books of mathematics from the ninth century A.D. One of the most well-known second-degree equations is the one which serves to determine the so-called *golden ratio*. It is said that a segment is divided in the golden ratio if the ratio of the lesser and greater pieces is the same as the ratio of the greater piece and the whole segment (see the figure below [T.N. 2]).



If the whole segment has one unit of length (meters, centimeters or whatever it may be) and we call the longer piece x, the segment will be divided in the golden ratio as long as

$$\frac{1-x}{x} = \frac{x}{1},$$

which is to say, as long as x is the solution to the quadratic equation

$$x^2 + x - 1 = 0,$$

whose only positive solution is $x = (\sqrt{5} - 1)/2 \approx 0.61803$. The golden ratio was adopted as one of the norms of aesthetics in architecture and sculpture by the Greeks in the fifth century B.C.

There are two equivalent forms of finding an algebraic solution to a second-degree equation. One consists of adding to both sides the quantity $(b^2)/4 - c$, to obtain a square on the left side, that is

$$(x + \frac{b}{2})^2 = \frac{b^2}{4} - c.$$

Finally, by taking the square root of both sides of the equation, we obtain the well-known solutions:

$$x = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c}). \tag{3}$$

In the second method we make a change of variable, with the aim of reducing equation (2) to an equation of the form $x^2 = \widetilde{c}$, which is easy to solve. Therefore, we let $x = y + \alpha$. The idea is to choose α in such a way that we can eliminate the linear term in x. We obtain

$$y^2 + (2\alpha + b)y + (\alpha^2 + b\alpha + c) = 0.$$

We choose α in such a way that the coefficient in y cancels out, that is $2\alpha + b = 0$, $\alpha = -b/2$, and finally the equation is left as:

$$y^2 = \frac{b^2}{4} - c,$$

from which $y = \pm\sqrt{b^2 - 4c}/2$. As $x = y + \alpha = y - (b/2)$, x remains given by (3). For second-degree equations both methods are practically identical, but the second can generalize to equations of a higher degree.

The general third-degree equation is of the form

$$x^3 + bx^2 + cx + d = 0. \tag{4}$$

The solution of the third-degree equation was published for the first time in the Ars Magna by Gerolamo Cardano in 1545 [3]. Gerolamo Cardano took his solutions from the works of Niccolò Tartaglia, although probably the first to find a solution was Scipione del Ferro (1465–1526) (see boxes).

Gerolamo Cardano: Italian mathematician and physicist. Born in Pavia in 1501 and died in Rome in 1576 [11, p. 295–297]. He wrote books on arithmetic, astronomy and physics. His best-known work is a treatise on algebra, the Ars Magna [3]. He was a professor at the University of Bologna. He is the inventor of the system of suspension that bears his name [T.N. 3].

Niccolò Fontana, called "Tartaglia": Italian mathematician. Born in Brescia around 1499 and died in Venice in 1557. Due to a speech impediment he was called "Tartaglia" (i.e. "stutter"), which pseudonym he used to publish his works. He was probably the first to apply mathematics to artillery. He published a treatise on arithmetic and edited a version of the works of Euclid and Archimedes (1543).

The method of solving the cubic, just as it was published in Ars Magna, is the following (see e.g. [7, p. 480]): first reduce the equation so that the quadratic term does not appear. This is achieved by making the change of variable

$$x = z - \frac{b}{3},$$

so that the equation in z is of the form

$$z^3 + \widetilde{c}z + \widetilde{d} = 0, \tag{5}$$

where the coefficients $\widetilde{c}$ and $\widetilde{d}$ are given in terms of the original coefficients by

$$\widetilde{c} = c - \frac{b^2}{3} \text{ and } \widetilde{d} = d + \frac{2b^3}{27} - \frac{bc}{3}.$$

Now, with the object of solving equation (5) for z, we write $z = u + v$, in which u and v will be determined later. So, equation (5) is written as

$$u^3 + v^3 + (3uv + \widetilde{c})(u + v) + \widetilde{d} = 0. \tag{6}$$

Now we choose u and v so that the coefficient $3uv + \widetilde{c}$ cancels, that is $v = -\widetilde{c}/(3u)$. In this way, equation (6) is written in terms of u and v in the following manner:

$$u^3 + v^3 + \widetilde{d} = 0,$$

or, solely in terms of u,

$$u^6 + \widetilde{d}u^3 - \frac{\widetilde{c}^3}{27} = 0, \tag{7}$$

which is a quadratic equation for u^3. Solving it (observing here that we can choose either of the two solutions; either way, v^3 will be the other solution) we obtain:

$$u = (-\frac{\widetilde{d}}{2} + \sqrt{\frac{\widetilde{d}^2}{4} + \frac{\widetilde{c}^3}{27}})^{(1/3)}$$

and, given that $u^3 + v^3 + \widetilde{d} = 0$,

$$v = (-\frac{\widetilde{d}}{2} - \sqrt{\frac{\widetilde{d}^2}{4} + \frac{\widetilde{c}^3}{27}})^{(1/3)}.$$

Finally, $x = u + v - (b/3)$.

Scipione Del Ferro: Italian mathematician (1465–1526). Around 1515, while a professor at the University of Bologna, he managed to solve the cubic equation of the type $x^3 + mx = n$, from which can certainly be derived the solution of the general cubic. Del Ferro's solution was known to Tartaglia through a student of Del Ferro, Antonio Maria Fior. Finally the solution was known to Cardano, who published it in the Ars Magna in 1545. Around these facts there have been woven various legends (for more details see [4, Lecture 16, p. 172–181]).

The solution in the Ars Magna is relatively simple, but it has a problem: although its deduction is easy to follow, it is difficult to remember the method of solution. Moreover, it does not allow generalization of the method to solve the quartic. Because of that, numerous authors, after the publication of the solution in the Ars Magna, have found their own methods of solution. These gave rise, in the beginning of the nineteenth century, to very interesting connections with Group Theory, connections which finally permitted the Norwegian mathematician

Niels Abel, in 1828, to demonstrate that there is no solution of the general fifth-degree equation [1] (and therefore of any degree greater than or equal to 5) in terms of simple operations (i.e. addition, subtraction, multiplication, division and root extraction). The methods were later perfected by Galois [5]. Galois theory is beyond the scope of this article, but a very good exposition appears in the Collection "The Carus Mathematical Monographs" [6]. In what remains of this article I will present a solution not very widespread in textbooks that allows one at least to have an idea of why it is possible to resolve the cubic and the quartic, but not equations of higher degree.

Let us return to the general cubic equation (4). It is not difficult to notice that if we make the change of variables

$$x = \frac{\alpha y + \beta}{\gamma y + \delta} \tag{8}$$

in the cubic equation, with α, β, γ and δ being any parameters, the equation for y will also be a cubic, of the form

$$y^3 + \widetilde{b} y^2 + \widetilde{c} y + \widetilde{d} = 0. \tag{9}$$

The new coefficients $\widetilde{b}$, $\widetilde{c}$ and $\widetilde{d}$ depend on the old ones, as well as the parameters of transformation α, β, γ and δ. Remember that to solve the second-degree equation, by the second method displayed, we made a transformation of the form $x = y + \alpha$ and used the parameter α to eliminate the linear term in y. Now the idea is similar: we intend to choose the parameters of transformation (8) so that the coefficients $\widetilde{b}$ and $\widetilde{c}$ cancel in (9). In this way we will have an equation of the form $y^3 + \widetilde{d} = 0$ which, of course, is very simple to solve. At first glance we have a lot of freedom to choose the parameters (α, β, γ and δ) and fulfill our objective, as we have four parameters at our disposal and only two coefficients to cancel. But the truth is that of the four parameters only two are useful. In effect, if we observe transformation (8), if we divide the numerator and the denominator from the right side by δ it is written as

$$x = \frac{(\alpha/\delta) y + (\beta/\delta)}{(\gamma/\delta) y + 1}. \tag{10}$$

We see that we have lost a parameter. The second observation is that if we make the change of variables $y = pz$ (that is, if we rescale the variable y) in (9) we will not gain anything in our attempt to make the coefficients disappear: in fact the only thing that happens to the "poor" coefficients before each change of variable is that they are rescaled. In consequence, the most general transformation of type (8) that we can do with the object of eliminating the coefficients of the linear and quadratic terms is

$$x = \frac{(\alpha/\gamma)z + \beta/\delta}{z+1},$$

having set $(\gamma/\delta)y = z$, by virtue of the second observation previous. We can now call $A = (\alpha/\gamma)$, $B = \beta/\delta$, so that our most general transformation is of the form

$$x = \frac{Az + B}{z+1} \tag{11}$$

and, as I hope I have convinced you, we have only two available coefficients (A and B) with which to achieve our objective. Substituting (11) in (4), we obtain the cubic equation for z:

$$(A^3 + bA^2 + cA + d)z^3 + Pz^2 + Qz + (B^3 + bB^2 + cB + d) = 0, \tag{12}$$

with the coefficients P and Q given by

$$P = 3A^2B + b(A^2 + 2AB) + c(B + 2A) + 3d$$

and

$$Q = 3AB^2 + b(B^2 + 2AB) + c(A + 2B) + 3d.$$

We wish to choose A and B such that $P = 0$ and $Q = 0$, which is to say, A and B are solutions to the equations

$$3A^2B + b(A^2 + 2AB) + c(B + 2A) + 3d = 0, \tag{13}$$

and

$$3AB^2 + b(B^2 + 2AB) + c(A + 2B) + 3d = 0. \tag{14}$$

Subtracting (14) from (13):

$$3AB + b(A + B) + c = 0. \tag{15}$$

To arrive at this relation we have simplified by $(A-B)$, supposing that $A \neq B$. Next we multiply (13) by B, (14) by A and subtract, to obtain

$$bAB + c(A + B) + 3d = 0, \tag{16}$$

where again we have simplified by $(A - B)$. Formulas (15) and (16) are linear equations for $A + B$ and AB. Solving them, we find

$$A + B = \frac{9d - bc}{b^2 - 3c} \text{ and } AB = \frac{c^2 - 3db}{b^2 - 3c}. \tag{17}$$

System (17) is easy to solve for A and B. In effect,

$$A = \frac{1}{2}(S + \sqrt{S^2 - 4T}) \text{ and } B = \frac{1}{2}(S - \sqrt{S^2 - 4T}),$$

where S and T denote the right sides that appear in (17), that is

$$S \equiv \frac{9d - bc}{b^2 - 3c} \text{ and } T \equiv \frac{c^2 - 3db}{b^2 - 3c)}.$$

Knowing A and B we can finally find z by dividing (12), remembering that $P = Q = 0$:

$$z^3 = -\frac{B^3 + bB^2 + cB + d}{A^3 + bA^2 + cA + d}.$$

At last, the unknown x is given in terms of z by way of (11). Note that the solution found is valid as long as $b^2 - 3c \neq 0$. Nevertheless, in this last case the cubic is simple to solve directly, by completing the cube in the right side of (4). In effect, equation (4) can be written as

$$(x + \frac{b}{3})^3 = \frac{b^3}{27} - d,$$

from which x is easily cleared.

This solution of the cubic, using transformation (8) (or better, transformation (11)) may seem complicated at first, but it has the virtue that we can apply the same method to solve the quartic and to understand why the fifth-degree equation cannot be solved in general.

Lodovico Ferrari: Italian mathematician. Born in Bologna in 1522 and died in 1560 (or 1565, according to some authors). He was a student of Cardano. He was the first to solve the general algebraic equation of the fourth degree. His solution was published in Cardano's Ars Magna. He was a professor of mathematics in Bologna.

The quartic equation is of the form

$$x^4 + bx^3 + cx^2 + dx + e = 0. \tag{18}$$

As in the case of the cubic, the quartic maintains its form upon making transformation (8) (in reality only (11) interests us). Making transformation (11), the new equation has the form

$$z^4 + \widetilde{b}z^3 + \widetilde{c}z^2 + \widetilde{d}z + \widetilde{e} = 0, \tag{19}$$

in which the coefficients $\widetilde{b}$, $\widetilde{c}$, etc., depend both on the coefficients b, c, d, e and the parameters A and B from transformation (11). We have two parameters at our disposal, so that, in principle, we can only cancel two of the four coefficients of (19). Which ones cancel to resolve the quartic? The only possibility is to cancel $\widetilde{b}$ and $\widetilde{d}$, such that (19) becomes a quadratic for z^2, which, of course, we know how to solve. By stipulating that $\widetilde{b}$ and $\widetilde{d}$ cancel, we at last finds a cubic for A (or for B), which by now we know how to solve. Here we will not go into any detail, but it is a good exercise for the interested reader.

What happens with the fifth-degree equation? Well, repeating the process and using the invariance under transformation (11), we find an equation of the form

$$z^5 + \widetilde{b}z^4 + \widetilde{c}z^3 + \widetilde{d}z^2 + \widetilde{e}z + \widetilde{f} = 0,$$

and, just as before, we have two parameters at our disposal to eliminate (at most two, of course) coefficients. But, in contrast to what occurred with the cubic and quartic equations, in the case of the quintic, even if we cancel any two coefficients of the original five, we don't obtain anything solvable. (One could be tempted to make $\widetilde{f} = 0$, but this is equivalent to solving the original quartic.) Therefore lamentably we cannot solve the quintic, at least not by these methods. Niels Abel demonstrated, in 1828, that the general quintic cannot be solved in terms of simple operations, as we have said above [1,2].

Niels Henrik Abel: Norwegian mathematician. Born in Findöe in 1802 and died in Arendal in 1829. Abel made important contributions to the theory of elliptic functions. We owe to him the demonstration that an algebraic solution to the fifth-degree equation is impossible.

Evariste Galois: French mathematician. Born in Paris in October of 1811 and died in May of 1832. Studied in L'École Normale de Paris. He had a very agitated youth due to his political ideas. He died in a duel at the age of 20. He made important contributions to Group Theory, and we owe to him much of the modern theory of Algebraic Equations.

Algebraic equations have attracted the attention of numerous mathematicians. The idea of the demonstration presented here has been introduced by several authors. In particular, the contemporary mathematician Mark Kac (1914–1984) wrote his first article, at the age of 16, precisely on a new method of solving the cubic equation [10]. Mark Kac's recent autobiography [8] tells an interesting story about his motivation to consider algebraic equations. In particular, the prologue, which reproduces a prior article [9], describes how his first article influenced his future career as a mathematician.

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Translation

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Translator's notes

1. For "the previous century", read "the nineteenth century".
2. This originally said "see the figure on the next page".
3. The suspension in question is the Cardan suspension, or the gimbal.
4. This paragraph was originally prefaced with N.R.; I'm not sure what that stands for.