A Variational Principle for the Asymptotic Speed of
Fronts of the Density Dependent Diffusion-Reaction Equation

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Abstract

We show that the minimal speed for the existence of monotonic fronts of the equation
$u_t = (u^m)_{xx} + f(u)$ with $f(0) = f(1) = 0$, $m > 1$ and $f > 0$ in $(0,1)$, derives from a
variational principle. The variational principle allows to calculate, in principle, the exact
speed for general $f$. The case $m = 1$ when $f'(0) = 0$ is included as an extension of the
results.
Several problems arising in population growth [1, 2], combustion theory [3, 4], chemical kinetics [5], and others [6], lead to an equation of the form
\[
\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = F(\rho),
\]
where the source term \( F(\rho) \) represents net growth and saturation processes. The flux \( \vec{j} \) is given by Fick’s law
\[
\vec{j} = -D(\rho) \vec{\nabla} \rho,
\]
where the diffusion coefficient \( D(\rho) \) may depend on the density or in simple cases be taken as a constant. In one dimension this leads to the equation
\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left( D(\rho) \frac{\partial \rho}{\partial x} \right) + F(\rho).
\] (1a)

In what follows we shall assume that
\[
F(\rho) > 0 \quad \text{in} \quad (0, 1), \quad \text{and} \quad F(0) = F(1) = 0,
\] (1b)
restrictions which are satisfied by several models. When the diffusion coefficient is constant and the additional requirement \( F'(0) > 0 \) is satisfied, the asymptotic speed of propagation of localized small perturbations to the unstable state \( u = 0 \) is bounded below and in some cases coincides [9] with the value \( c_L = 2\sqrt{F'(0)} \) which is obtained from considerations on the linearized equation [8]. However, when either \( F'(0) = 0 \) or \( D(\rho) \) is not a constant, no hint for the speed of propagation of disturbances can be obtained from linear theory alone. A common choice for the diffusion coefficient is a power law, case with which we shall be concerned here. Therefore the equation that we study is
\[
\frac{\partial \rho}{\partial t} = (\rho^m)_{xx} + F(\rho)
\] (2a)
with
\[
F(0) = F(1) = 0, \quad \text{and} \quad F > 0 \quad \text{in} \quad (0, 1).
\] (2b)
Aronson and Weinberger [7, 2] have shown that the asymptotic speed of propagation of disturbances from rest is the minimal speed \( c^*(m) \) for which there exist monotonic travelling
fronts \( \rho(x,t) = q(x-ct) \) joining \( q = 1 \) to \( q = 0 \). The equation satisfied by the travelling fronts is

\[
(q^m)_{zz} + cq_z + F(q) = 0 \tag{3a}
\]

with

\[
q(-\infty) = 1, \quad q > 0, \quad q' < 0 \quad \text{in} \quad (-\infty, \omega), \quad q(\omega) = 0 \tag{3b}
\]

where \( z = x - ct \). The wave of minimal speed is sharp, that is, \( \omega < \infty \) when \( m > 1 \) [2].

An explicit solution is known [1, 2] for the case \( F(q) = q(1-q) \) and \( m = 2 \), the waveform is given by

\[
q(z) = \left[ 1 - \frac{1}{2} e^{z/2} \right]_+
\]

and it travels with speed \( c^*(2) = 1 \) (here \( [x]_+ \equiv \max(x,0) \)). Recently the derivative \( dc/dm \) at \( m = 2 \) has been calculated by two different methods. Its value is -7/24 [9, 10]. Other exact solutions for different choices for \( m \) and \( F \) have been given in [11].

The purpose of this work is to give a variational characterization of the minimal speed \( c^*(m) \) for Eq.(3) when \( m > 1 \), and as a byproduct for the case \( m = 1 \) when \( F'(0) = 0 \), both, cases for which no information is obtained from linear theory. The case \( m = 1 \) with \( F'(0) > 0 \) has been studied elsewhere [13]. Lower bounds have been obtained on the minimal speed \( c^*(m) \) [12]; the present results allow its exact calculation for arbitrary \( f \).

Since the selected speed corresponds to that of a decreasing monotonic front, we may consider the dependence of its derivative \( dq/dz \) on \( q \). Calling \( p(q) = -q^{m-1} dq/dz \), where the minus sign is included so that \( p \) is positive, we find that the monotonic fronts are solutions of

\[
\frac{dp}{dq} + \frac{c^*}{m} p + \frac{1}{m} q^{m-1} F(q) = 0 \tag{4a}
\]

with

\[
p(0) = p(1) = 0, \quad p > 0 \quad \text{in} \quad (0, 1). \tag{4b}
\]

Although the wave of minimal speed is sharp and therefore \( q'(0) < 0 \), by its definition \( p(0) = 0 \) is true. We now show that the minimal speed \( c^*(m) \) follows from a variational principle whose Euler equation is Eq.(4a).
Let $g$ be a positive function such that $h = -g' > 0$. Multiplying Eq.(4a) by $g/p$ and integrating we obtain after integration by parts,

$$
\frac{c}{m} = \frac{\int_0^1 \left[ \frac{1}{m} q^{m-1} F(q \frac{g(q)}{p(q)} + h(q)p(q) \right] dq}{\int_0^1 g(q) dq}.
$$

(5)

By Schwarz’s inequality, since, $q, F, g$ and $h$ are positive we know

$$
\frac{1}{m} q^{m-1} F \frac{g}{p} + hp \geq 2 \sqrt{\frac{1}{m} q^{m-1} F g h}
$$

(6)

and therefore, replacing in Eq.(5) we have

$$
c \geq 2 \frac{\int_0^1 \sqrt{mq^{m-1} F g h} dq}{\int_0^1 g dq}.
$$

(7)

This bound has been already given by us [12]. We now show that it is always possible to find a $g(q)$ such that the equality in Eq.(6) and therefore also in Eq.(7) holds. We do so by explicit construction of such a function $g$. The equality in Eq.(6) holds if

$$
\frac{1}{m} q^{m-1} F \frac{g}{p} = hp
$$

(8)

Let $v(q)$ be the positive solution of

$$
\frac{v'}{v} = \frac{c}{mp}
$$

(9a)

and choose

$$
g = \frac{1}{v'}.
$$

(9b)

We have then

$$
\frac{v''}{v} = \frac{(v')^2}{v^2} - \frac{c}{mp^2} p' = -\frac{c}{mp^2} q^{m-1} F(q)
$$

where we have used Eq.(9a) to eliminate $v'$ and Eq.(4a) to eliminate $p'$. Therefore,

$$
h = -g' = \frac{v''}{(v')^2} = \frac{1}{mp^2} q^{m-1} F g > 0
$$

(9c)

where we have made use of Eqs.(9a) and (9b). With this expression for $h$, we can see that Eq.(8) is satisfied. In addition we must check that $g$ as we have defined it is such that its integral exists. In fact as it exists and moreover one can always normalize $g$ so that $g(0) = 1$ and $g(1) = 0$. From the definition of $g$ we obtain

$$
g(q) = \frac{mp(q)}{c} \exp \left[-\int_{q_0}^q \frac{c}{mp} dq'\right]
$$
where \(0 < q_0 < 1\). Since \(p(1) = 0\) and \(p\) is positive between 0 and 1 it follows that 
\(g(1) = 0\).

At zero no divergence occurs, as we now show. Call \(\hat{c} = c/m\) and \(f(q) = q^{m-1}F(q)/m\). Then Eq.(4a) reads

\[
pp' - \hat{c}p + f = 0
\]

with

\[
f(0) = f(1) = 0 \quad \text{and} \quad f'(0) = 0.
\]

For this case Aronson and Weinberger have shown that \(p(q)\) approaches the fixed point \(q = 0\) as \(p = \hat{c}q = cq/m\). Then, near 0, \(v'/v \approx 1/q\) or \(v \approx q\) and from its definition \(g(0) = 1\). Then the integral of \(g\) exists. We have shown then

\[
c^*(m) = \max \int_0^1 \sqrt{mq^{m-1}Fg} \cdot dq
\]

where the maximum is taken over all functions \(g\) such that

\[
g(0) = 1, \quad g(1) = 0 \quad \text{and} \quad h = -g' > 0.
\]

It is perhaps of some interest to verify explicitly that the Euler equation for the maximizing \(g\) is indeed Eq.(4a). Let us study the maximization of the functional

\[
J_m(g) = 2 \int_0^1 \sqrt{mq^{m-1}Fgh} \cdot dq
\]

where \(h = -g' > 0\) subject to

\[
\int_0^1 g(q) \cdot dq = 1.
\]

The Euler equation for this problem is

\[
\lambda + \sqrt{\frac{mq^{m-1}Fh}{g}} + \frac{d}{dq} \left( \sqrt{\frac{mq^{m-1}Fg}{h}} \right) = 0
\]

where \(\lambda\) is the Lagrange multiplier. Using the expression given in Eq.(9c) for \(h\) we see that this is exactly Eq.(4a) with the Lagrange multiplier \(\lambda = -c\).

As an application we shall consider the case \(F(q) = q(1 - q)\) and \(m = 2\) for which the exact solution is known. Take as the trial function \(g(q) = (1 - q)^2\). Then we obtain

\[
c \geq 4 \int_0^1 q(1-q)^2 dq \int_0^1 (1-q)^2 dq = 1
\]
the exact value, which shows that this is the function \( g \) for which the maximum is attained. In addition, due to the existence of the variational principle we may use the Feynman-Hellman formula to calculate the dependence of \( c(m) \) on parameters of \( F \). We illustrate this by applying it to the calculation of \( dc/dm \) at \( m = 2 \). Taking the derivative of Eq.(10) with respect to \( m \) we obtain

\[
\frac{dc}{dm} = \frac{1}{f_0^g dq} \int_0^1 \frac{ghF}{\sqrt{mFq^{m-1}gh}} [q^{m-1}(1 + m \log q)] dq.
\]

Evaluating at \( m = 2 \), with \( g(q) = (1 - q)^2 \) we obtain

\[
\frac{dc}{dm}(2) = 3 \int_0^1 q(1 - q)^2(1 + 2 \log q) dq = -\frac{7}{24}
\]

the value previously obtained by other methods.

A fast estimation of the speed for other values of \( m \) can be obtained with simple trial functions. In Fig. 1 we show lower bounds for \( F = q(1 - q) \) using as trial functions \( g_1 = (1 - q)^2 \) and \( g_2 = (1 - q) \). With the first trial function we have the exact value at \( m = 2 \). The dotted line is the line of slope \(-7/24\) that coincides with the tangent at \( m = 2 \). For larger \( m \) a better estimate is obtained using \( g_2 \). The dashed line is the curve \( \sqrt{2/m} \) which has been suggested by Newman \[1\] as the best fit to his numerical results. With better choice of trial functions the exact value can be approached arbitrarily close.

Finally we observe that the case \( m = 1 \) when \( F'(0) = 0 \) follows directly here. Repeating the procedure starting now from equation (10), one obtains,

\[
c = \max 2 \frac{\int_0^1 \sqrt{Fgh} \ dq}{\int_0^1 g \ dq}
\]

where the maximum is taken over all functions \( g \) such that

\[
g(0) = 1, \quad g(1) = 0 \quad \text{and} \quad h = -g' > 0.
\]

To show this we have used \( \nu'/\nu = c/p \) and \( g = 1/\nu' \) and the asymptotic behavior described above.

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References


