Classification of the solutions of semilinear elliptic problems in a ball.

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Abstract

In this paper we fully describe the set of the positive and nodal (regular and singular) radial solutions of the superlinear elliptic P.D.E.

$$-\Delta u \;=\; \lambda u + |u|^{p-1} u \;\text{ in }\; B_1, \quad u = 0 \;\text{ on }\; \partial B_1, \quad p > 1\,, \tag{1}$$

without restriction on the range of $\lambda \in \mathbb{R}$. Here, B_1 is the unit ball in $\mathbb{R}^N$.

More precisely, in all subcritical, critical and supercritical cases, we analyze the possible singularities of radial solutions at the origin and the number of bounded and unbounded solutions. The solutions will be of three different types : bounded with a finite number of zeroes in $(0, 1)$, singular at the origin, still with a finite number of zeroes and singular with sign changing oscillations at the origin.

AMS Subject Classification : 35J60, 35B05, 35P30, 34C23, 34C15, 70K10. **Keywords and Phrases :** Nodal solutions, oscillatory solutions,

multiplicity, branches, bifurcations, critical exponent, Pohozaev's identity, semilinear elliptic equations, removable singularities.

[1] Partially supported by Fondecyt (Chile), project # 199–00427, a John Simon Guggengeim Memorial Fellowship and Cátedra Presidencial en Ciencias (Chile).

[2] Partially supported by ECOS-CONYCIT under contrat no. C95E02.

1 Introduction

Problem (1) has been extensively studied in the past twenty years. However, most of the work has been done for classical (bounded) solutions and mainly for positive solutions only. The critical case especially, $p_N = (N+2)/(N-2)$, has received a lot of attention.

It is the aim of this paper to study all the distributional radial solutions of (1). In addition to all the bounded solutions, for any p and λ, there exist solutions which are singular at the origin and there is even an uncountable number of such solutions. Among the singular solutions, we may distinguish those who have a fixed sign near the origin and those which are oscillating near the origin (with sign changing oscillations). In all cases, $1 < p < p_N$, $p = p_N$ and $p > p_N$, we will describe the entire solution set, putting an emphasis on its structure.

For all $p > 1$, the existence of branches of bounded solutions with a given number of zeroes is well known. For the construction of the branches one may for instance refer to [29]. In the subcritical case, $1 < p < p_N$, precise information about the branches of solutions may be found in [5] in dimension $N = 1$, and in [13], [20], [32] and [19] in any dimension. For some p's, the existence of singular positive solutions was also given in [23], Rem. 3.1.

In this paper we prove that when $1 < p < p_N$, all radial solutions of (1) have a finite number of zeroes. Moreover, for $1 < p < N/(N-2)$, all the solutions of (1) are bounded. On the contrary, when $N/(N-2) \leq p < p_N$, appart from the bounded solutions, there is also an uncountable number of unbounded solutions with any given number of zeroes. The behavior of the solutions at the singularity is well known (this question was already discussed in [18], [21], [22]; see also [6], [25], [23]).

In the critical case, $p = p_N$, the non existence of bounded solutions for all $\lambda \leq 0$ follows from Pohozaev's identity [28]. Brezis and Nirenberg [7] proved that nontrivial positive bounded solutions only exist in the interval (μ_1, λ_1), where $\mu_1 = \lambda_1/4$ for $N = 3$ and $\mu_1 = 0$ for all $N \geq 4$. The articles by Atkinson, Brezis and Peletier ([2], [3]) describe the set of bounded solutions for a given number of zeroes in the interval $(0, 1)$ (see also [8]). In particular, they prove that nontrivial bounded solution with k nodes exist only in some interval $(\tilde{\mu}_k, \lambda_k)$, where $0 \leq \tilde{\mu}_k < \lambda_k$. [14] provided the existence of bounded solutions for λ near all the eigenvalues of the Laplacian. In [15] was established the existence of at least one bounded solution in the interval $(0, \lambda_1)$ if $N \geq 6$ and in [12], the existence of a bounded solution for any $\lambda \geq 0$ if $N \geq 6$. Cerami, Solimini and Struwe proved in [14] that for $N \geq 7$, there exists an infinity of H_0^1-radial solutions to (1) for all $\lambda > 0$. In [2], [3] and [4] we find an extensive list of qualitative results in dimensions $N = 3, 4, 5, 6$, and in [1], an elementary proof of a nonexistence result (based on Pohozaev's identity). In Appendix A, we present a computation in that direction in dimension $N = 3$ which is optimal. Other results on the existence and the behavior of non necessarily radially symmetric solutions of (1) can be found for instance in [30], [33], [34] and [17].

Here, we systematically describe the structure of the set of the radial solutions in the critical case, including the solutions which are oscillating near the origin and have an infinite number of zeroes. We reduce the problem to the analysis of a related asymptotic dynamical system and prove that for λ outside the union of a countable number of intervals (including $\lambda \leq 0$), all solutions are sign changing, oscillating and singular near the origin. Moreover, in the interior of those intervals, appart from bounded solutions with a given number of zeroes, there are oscillating solutions and singular solutions with a finite number of zeroes. Finally, we show that for a given λ either all the solutions of (1) are singular and oscillating, or the three classes of solutions (bounded, unbounded with a finite number of zeroes and unbounded oscillating) coexist.

In the supercritical case, $p > p_N$, Merle and Peletier proved in [26] (see also [27]) that there is a unique value of λ, namely $\lambda = \lambda^* < \lambda_1$ (which is the asymptotic value of the branch) for which there is an unbouded solution of (1). Moreover, there are bounded positive solutions for λ in the interval (λ^*, λ_1). Formal expansions of the branch in the limit of the L^∞-norm growing to $+\infty$ have been given by Budd and Norbury [10]. The computations in [10] seem to indicate that bounded positive solutions of (1) exist in an interval $(\bar{\lambda}, \lambda_1)$, with $0 < \bar{\lambda} < \lambda^*$. Other results about the supercritical case can be found in [11, 9].

In this paper, we prove the existence of an uncountable number of oscillating solutions for all λ and furthermore, we describe the solution set of (1). Let us finally note that in the case $N/(N-2) < p < 3N/(N-2)$, the singularities at the origin of solutions of (1) in $B_1 \setminus \{0\}$ can be removed easily. But for $p \geq 3N/(N-2)$, the oscillating solutions are not in $L^p_{loc}(B_1)$ anymore and in order to say that they are solutions of (1), we have to define explicitely what we mean by the distribution $|u|^{p-1}u$. The rigth way to tackle this problem is to consider a natural extension of $|u|^{p-1}u$ which is defined by means of the principal value distribution.

Let us note that in [31], Serrin and Zou fully describe the set of positive radial solutions of $-\Delta u = \lambda u + |u|^{p-1}u$ in $\mathbb{R}\setminus\{0\}$. Some of the arguments used in [31] are close to ours, but the fact that we are looking for solutions on $(0,1)$ or in $H_0^1(B)$ allows us to introduce new tools for the description of the singularity.

The paper is organized as follows. Section 2 is devoted to the proof of several technical auxiliary results. The critical case is studied in Section 3 and in Sections 4 and 5, we discuss the subcritical and supercritical cases. Our results hold for $\lambda \in \mathbb{R}$, without further restriction on the range of λ. Finally, in Appendix A we present an alternative proof of a nonexistence result proved in [7] and Appendix B contains all the figures.

Notations. For any function f, defined in $\mathbb{R}$, f' denotes its derivative. Also, all throughout this paper, we will write undistinctly $u(x)$ and $u(r)$, $r = |x|$, for any radially symmetric function u defined in $\mathbb{R}^N$ or in $B_1 := \{x \in \mathbb{R}^N \,:\, |x| < 1\}$.

2 Auxiliary and technical results

In this section, we prove preliminary results for (1) with $p > 1$. Results which are specific to the supercritical case can be found at the beginning of Section 5. The equation (1) is supposed to be satisfied in the weak sense : in $\mathcal{D}'(B_1)$, or even in a weaker sense (see Section 5, about the supercritical case). Any radial solution of (1) is a solution of

$$-u'' - \frac{(N-1)}{r}\,u' = \lambda u + |u|^{p-1}u \quad \text{in} \quad (0,1)\,, \quad u(1) = 0. \tag{2}$$

On the contrary, any solution of (2) is a solution of (1) if u is not too singular at the origin. More precisely, if for some solution u of (2), the integrals $\int_{B_\varepsilon} \nabla u \cdot \nabla\varphi\, dx$ and $\int_{B_\varepsilon} |u|^{p-1}u\varphi\, dx$ are small for small ε and if $\lim_{\varepsilon\to 0} \varepsilon^{N-1}\,|u'(\varepsilon)| = 0$, then u is a distributional solution of (1) :

Lemma 2.1 *Let u be a solution of (2) such that*

$$\limsup_{r\to 0}\, r^a\,|\log r|^b\,|u(r)| < +\infty, \quad \limsup_{r\to 0}\, r^{a+1}\,|\log r|^c\,|u'(r)| < +\infty\,.$$

u is a distributional solution of (1) if the following conditions are satisfied :

- $a \leq \min(N-2, N/p)$,
- *if* $a = N-2$, *then* $c > 0$,
- *if* $a = N/p$, *then* $bp > 1$.

For more precise and general results on this subject, see for instance [6], [25], [23]. On the other hand, note also that any bounded solution of (1) is smooth in B_1, as it can be seen by an easy bootstrap argument.

Consider the radial problem in the whole space (with $\lambda = 1$).

Lemma 2.2 *Let* $N \geq 3$, $p = (N+2)/(N-2)$, $a > 0$ *and let* v *be the solution of*

$$-v'' - \frac{(N-1)}{r} v' = v + |v|^{p-1} v \quad \text{in} \quad (0, +\infty)\,,\ v(0) = a\,,\ v'(0) = 0\,. \quad (3)$$

Then, v *has an infinite number of zeroes in* $(0, +\infty)$ *.*

Proof. Performing the following change of variables :

$$v(r) = r^{-2/(p-1)} w(s)\,,\ s = -\log r\,, \qquad (4)$$

one easily sees that v is a solution of (3) if and only if w is a solution of

$$-w'' = |w|^{p-1} w - \frac{(N-2)^2}{4} w + e^{-2s} w \qquad (5)$$

in $(-\infty, +\infty)$.

Now, when s is negative, with $|s|$ large enough, w'' has the sign of $-w$: if w did not change sign in an interval $(-\infty, -s_0)$, w should be bounded away from 0, which in view of (5) is clearly impossible. Hence w has a sequence of zeroes going to $-\infty$. □

Lemma 2.3 *Let* $N \geq 3$, $p > 1$ *and* v *be the solution of (3). If for some* $r_0 \geq 0$, $v'(r_0) = 0$, *then*

$$|v(r)| \leq |v(r_0)| \quad \text{for all} \quad r \geq r_0\,.$$

Proof. Let $0 \leq a < b < +\infty$ be two critical points of v. We multiply the equation in (3) by v' and integrate on (a, b) to get

$$-\int_a^b \frac{N-1}{r} |v'(r)|^2 \, dr = F(v(b)) - F(v(a))\,,$$

with $F(t) = |t|^2/2 + |t|^{p+1}/(p+1)$. Clearly, $F(v(b)) < F(v(a))$. □

Remark 2.4 *The above result also holds for solutions of (1), for all* λ*. Indeed, for* $\lambda \geq 0$*, the proof is the same. When* $\lambda < 0$*, the argument used above still holds. Indeed, notice that the only points* $\bar{r} > r_0$ *which have to be taken into account are those for which* $u'(\bar{r}) = 0$*,* $\left(\lambda + |u(\bar{r})|^{p-1}\right) \geq 0$*.*

Next we describe how to relate (3) to (2). The proof of the following lemma follows from a straightforward computation.

Lemma 2.5 *Let* v *be a solution of*

$$-v'' - \frac{(N-1)}{r}\, v' = \mu v + |v|^{p-1} v \quad in \quad (0, r_0)\,,\ v(r_0) = 0\,. \qquad (6)$$

Then, $u(\cdot) = r_0^{2/(p-1)} v(r_0\,\cdot)$ *is a solution of (2) with* $\lambda = \mu r_0^2$*.*

Reciprocally, to any solution u *of (2) corresponds a solution* v *of (6) with* $\mu = 1$*. If moreover* u *is bounded, then there exists a unique* $a \in \mathbb{R}$ *such that* $v(0) = a$*.*

Let us now consider the solution of (3), v, and its sequence of nodes $0 < r_1(a) < \ldots < r_k(a) < \ldots$. Then, by Lemma 2.5, for every $k \geq 1$, $u_{k,a}(r) = (r_k(a))^{2/(p-1)} v(r_k(a) r)$ is a solution of (2) with $\lambda = \lambda_k(a) := r_k(a)^2$ and $u_{k,a}$ has $k-1$ nodes in the interval $(0,1)$. Let us finally denote by $c_k(a)$ the value of $u_{k,a}$ at 0. Obviously, $c_k(a) = a\, r_k(a)^{2/(p-1)}$.

Definition 2.6 *For every* $i \geq 1$ *we denote by* λ_i *the* i*-th eigenvalue of the operator* $-\Delta$ *when acting on*

$$H^1_{0,r}(B_1) := \{u \in H^1_0(B_1)\ ;\ u \textit{ is radially symmetric }\}.$$

Remark 2.7 *In the case of the critical exponent* $p = (N+2)/(N-2)$*, several results due to Atkinson and Peletier [4] and Atkinson, Brezis and Peletier [3] (see also [7]) yield the following qualitative information about the functions introduced above :*

- *As* $a = v(0) \to 0$*,* $r_i(a)^2 \to \lambda_i$ *(with* $\lambda_i = (i\pi)^2$ *if* $N = 3$*).*
- *As* $a = v(0) \to +\infty$*,* $r_i(a)^2 \to \mu_i$*, where* $\mu_i \in [0, \lambda_i)$*. Moreover, more precise information is available for* $3 \leq N \leq 6$ *: when* $N = 3$*,* $\mu_i = (i - (1/2))^2 \pi^2$*. When* $N = 4, 5$*,* $\mu_i = \lambda_{i-1}$ *for* $i \geq 2$ *and* $\mu_1 = 0$*. Finally, when* $N = 6$*,* $\mu_1 = 0$ *and for* $i \geq 2$*,* $\mu_i \in [0, \lambda_{i-1})$*.*

The comparison of the bounded solutions with the eigenfunctions of linear eigenvalue problems provides several useful informations. The crucial points are summarized in the

Lemma 2.8 *Let $N \geq 3$ and consider a bounded solution of*

$$-u'' - \frac{N-1}{r}u' = \lambda u + |u|^{p-1}u \quad in \quad (0,1)$$

such that $u'(0) = 0$, $u(0) = u_0 > 0$ and $u(1) = 0$. Let $r_1 = \inf\{r \in]0,1] : u(r) = 0\}$. Then

$$r_1^2 \geq \frac{2u_0}{u_0^p + \lambda_+ u_0} \, . \tag{7}$$

If u changes sign $(k-1)$ times, then $\lambda < \lambda_k$ and

$$u_0 \geq \Big(\lambda_k - \lambda\Big)^{\frac{1}{p-1}} \, . \tag{8}$$

As a consequence, if $k_0 = \min\{k \in I\!N \, : \, \lambda_k > \lambda\}$, then

$$u_0 \geq \Big(\lambda_{k_0} - \lambda\Big)^{\frac{1}{p-1}} \, .$$

Reciprocally, bounded solutions with $k-1$ zeroes exist for any $\lambda < \lambda_k$ if $p < \frac{N+2}{N-2}$, and for any $\lambda < \lambda_k$, close enough to λ_k, if $p \geq \frac{N+2}{N-2}$.

Proof. Let $\bar{r} \in [0, r_1)$ be such that $u'(\bar{r}) = 0$, $u' < 0$ on $(\bar{r}, r_1)$. For any $r \in [\bar{r}, r_1]$,

$$-u''(r) \leq (u^p + \lambda u)(r) \leq u_1^p + \lambda_+ u_1 \, , \quad u_1 := u(\bar{r}) \, ,$$

thus giving

$$u(r) \geq u_1 - (u_1^p + \lambda_+ u_1)\frac{r^2}{2} \, .$$

(7) follows from $u(r_1) = 0$ and Remark 2.4, which implies $u_1 \leq u_0$. On the other hand, u is the unique solution of

$$-v'' - \frac{N-1}{r}v' = (\lambda_k(\epsilon) + \epsilon |u|^{p-1})v \, ,$$

$$v(0) = u_0 \, , \quad v'(0) = 0 \, , \quad v(1) = 0$$

changing sign $k-1$ times with $\epsilon = 1$ and $\lambda_k(1) = \lambda$. Since $\lambda_k(\epsilon)$ is a decreasing function of $\epsilon \in (0,1)$,

$$\lambda_k(1) < \lambda_k(0) = \lambda_k \; .$$

For the same reason, 1 is the k^{th} eigenvalue of the linear eigenvalue problem with weight $\lambda + |u|^{p-1}$, which, by Remark 2.4, is bigger than the k^{th} eigenvalue μ_k of the linear eigenvalue problem with weight $\lambda + |u_0|^{p-1}$. But $\mu_k(\lambda + |u_0|^{p-1}) = \lambda_k$, thus proving that $\lambda + |u_0|^{p-1} > \lambda_k$, which gives (8).

The existence results for $\lambda < \lambda_k$ follow from classical results (see for instance [29], [5], [13], [20], [32], [19]; [7], [3], [4] in the critical case; [26] in the supercritical case). $\square$

In order to analyze the behavior of all the solutions of (2), we introduce the following problem :

$$-u'' - \frac{(N-1)}{r}\, u' = \lambda u + |u|^{p-1}u \quad \text{in} \quad (0,1)\,,\;\; u(1) = 0\,,\; u'(1) = -\gamma, \tag{9}$$

where $\gamma > 0$. Obviously, up to a change of sign, any solution of (2) is a solution of (9) for an appropriate value of γ. Throughout this paper, we will denote by u_γ the solution of (9).

Consider the change of variables introduced in (4) :

$$u(r) = r^{-2/(p-1)} w(s)\,, \quad s = -\log r\,. \tag{10}$$

It is straightforward to see that u is a solution of (9) if and only if w is a solution of

$$\begin{cases} w'' + \left(\frac{4}{p-1} - N + 2\right) w' + |w|^{p-1}w \\ + \frac{2}{(p-1)^2}\Big((p+1) - (N-1)(p-1)\Big)\, w + \lambda e^{-2s} w = 0 \quad \text{in} \quad (0,+\infty)\,, \\ \\ w(0) = 0\,,\; w'(0) = -u'(1) = \gamma\,, \end{cases} \tag{11}$$

and, when necessary, we will denote by w_γ the solution of (11).

The relationship between the solutions of (11) and those of the autonomous problem

$$\begin{cases} w'' + \left(\frac{4}{p-1} - N + 2\right) w' + |w|^{p-1}w \\ + \frac{2}{(p-1)^2}\Big((p+1) - (N-1)(p-1)\Big)\, w = 0 \quad \text{in} \quad (0,+\infty)\,, \\ \\ w(0) = 0\,,\; w'(0) = -u'(1) = \gamma\,, \end{cases} \tag{12}$$

is given by classical theorems of O.D.E. theory : for s large, the solution of (12) behaves like a solution of the autonomous O.D.E. Let us show this result in a more general setting :

Lemma 2.9 *Let* $b, c \in \mathbb{R}$, $p > 1$ *and* $g \in C^1(0, +\infty)$ *be a function such that* $|b| + |c| > 0$, $\lim_{s \to +\infty} g(s) = 0$ *and* $g' \in L^1(1, +\infty)$. *If* $w \in L^\infty(0, +\infty)$ *is a solution of the equation*

$$w'' + bw' + cw + |w|^{p-1}w + g(s)w = 0 \quad in \quad \mathbb{R}^+\,, \tag{13}$$

then there exists a function $\tilde{w}$ *solution of*

$$w'' + bw' + cw + |w|^{p-1}w = 0 \quad in \quad \mathbb{R}^+\,, \tag{14}$$

such that

$$\lim_{s \to +\infty} |w(s) - \tilde{w}(s)| + \left|w'(s) - \tilde{w}'(s)\right| = 0\,. \tag{15}$$

Moreover, if we define the 'energy' functional $\mathcal{E}[w]$ *by*

$$\mathcal{E}[w] := \frac{|w'|^2}{2} + \frac{c\,|w|^2}{2} + \frac{|w|^{p+1}}{p+1}$$

then, $\mathcal{E}[\tilde{w}](\cdot)$ *is constant in* $\mathbb{R}^+$ *and the function* $\tilde{w}$ *is periodic.*

Remark 2.10 *Note that when* $b \neq 0$, *the only periodic solutions of (14) are constant functions, corresponding to critical points of* $w \mapsto \frac{c}{2}|w|^2 + \frac{1}{p+1}|w|^{p+1}$. *On the other hand, when* $b = 0$ *the equation (14) has non constant periodic solutions.*

Proof of Lemma 2.9 By (13), for all $0 \leq s_1 < s_2 < +\infty$ we have

$$\mathcal{E}[w](s_2) - \mathcal{E}[w](s_1) \tag{16}$$

$$= -\int_{s_1}^{s_2} b\,|w'(s)|^2 ds + \frac{1}{2}\int_{s_1}^{s_2} g'(s)|w(s)|^2 ds - \frac{1}{2}g(s_2)|w(s_2)|^2 + \frac{1}{2}\,g(s_1)|w(s_1)|^2.$$

Hence, under our assumptions, the function $s \mapsto \mathcal{E}[w](s)$ has a limit E as $s \to +\infty$. Therefore, the function w' is also in $L^\infty(0, +\infty)$.

Let us now define a sequence of continuous functions w_n by $w_n(\cdot) = w(n + \cdot)$. From (13), the functions w_n, w_n', and w_n'' are uniformly bounded in $(0, +\infty)$. Then, by using Ascoli's theorem, we find a function $\tilde{w}$, solution of (14), such that for any fixed $d > 1$, we have $\lim_{n\to+\infty} \|w_n - \tilde{w}\|_{W^{1,\infty}(0,d)} = 0$, and for all $s \in (0,d)$,

$$E = \lim_{n\to+\infty} \mathcal{E}[w_n](s) = \mathcal{E}[\tilde{w}](s)\,.$$

By classical O.D.E. results, all the solutions of (14) which have constant 'energy' $\mathcal{E}$ in some non empty interval are periodic functions. $\tilde{w}$ is therefore unique up to translation, which proves (15). □

When in the above lemma, $\tilde{w} \equiv 0$, a more precise description of the asymptotic behavior of w is given by the following lemma.

Lemma 2.11 *Under the assumptions of Lemma 2.9, if $\tilde{w} \equiv 0$ and $b^2 > 4c$, there exist two constants, C_1, C_2 such that for large s_0, s, with $s_0 < s \to +\infty$,*

$$w(s) \sim C_1 w(s_0) e^{\int_{s_0}^{s} \nu_1(\tau)\,d\tau} + C_2 w(s_0) e^{\int_{s_0}^{s} \nu_2(\tau)\,d\tau}\,, \tag{17}$$

where $\nu_1(s) \leq \nu_2(s)$ are the two roots of the equation

$$\nu^2 + b\nu + \left(c + |w(s)|^{p-1}\right) = 0\,.$$

Proof. When $\tilde{w} \equiv 0$, $w(s)$ does not change sign for s large. Indeed, assume that for $a < b$ large enough, $w(a) = w(b) = 0$ and $w > 0$ in (a,b). Then, w satisfies

$$(pw')' + pqw = 0, \ \text{ in } \ (a,b), \ \ w(a) = w(b) = 0, \tag{18}$$

with $p(s) = \exp(bs)$, $q(s) = c + |w(s)|^{p-1} + g(s)$. As s is large, $q(s) \leq c + \delta$, for some δ small. Let v be a positive solution of $(pv')' + p(c+\delta)v = 0$ in $(1,+\infty)$. We have :

$$\begin{aligned} 0 \leq \int_a^b p(s)(c + \delta - q(s))v(s)w(s)\,ds &= \int_a^b \left((pw')'v - (pv')'w\right)ds \\ &= p(b)w'(b)v(b) - p(a)w'(a)v(a) < 0, \end{aligned} \tag{19}$$

a contradiction. Hence, w cannot change sign near infinity. Since equation (13) can be written as $w''(s) + bw(s) + (c + V(s) + R(s))\,w = 0$, with $V', R \in L^1(1,+\infty)$, this enables us to use a general O.D.E. result like Theorem 8.1, page 92, in [16] to prove (17). □

Remark 2.12 *The above result and some straightforward computations show in fact that if $b \geq 0$ and $c < 0$, C_2 must be equal to 0 in (17), since in this case $\nu_2(\infty) > 0$, but w is bounded. If $b > 0$, and $0 < c < b^2/4$, $w(s)$ is either equivalent to $Ce^{\nu_1(\infty)s}$ or to $Ce^{\nu_2(\infty)s}$, with $C \neq 0$. Finally, if $b > 0$, $c = 0$, then $w(s)$ is equivalent either to Ce^{-bs} with $C \neq 0$, or to $C_0 s^{-1/(p-1)}$, with $C_0 = (b/(p-1))^{\frac{1}{p-1}}$.*

In the rest of this section, we will only be interested in the critical exponent $p = (N+2)/(N-2)$. Note that in this case, the factor of w' in (11), $\frac{4}{p-1} - N + 2$, cancels, so that w is a solution of

$$\begin{cases} -w'' = |w|^{\frac{4}{N-2}} w - \frac{(N-2)^2}{4} w + \lambda e^{-2s} w \quad \text{in} \quad (0, +\infty)\,, \\ w(0) = 0\,,\ w'(0) = -u'(1) = \gamma\,, \end{cases} \tag{20}$$

The asymptotic problem

$$-w'' = |w|^{\frac{4}{N-2}} w - \frac{(N-2)^2}{4}\, w \quad \text{in} \quad (0, +\infty) \tag{21}$$

will play an important role in the analysis of the solution set for (20). All the solutions of (21) are periodic and can be either

(i) positive or negative, but not constant, or

(ii) sign changing, or

(iii) $w \equiv 0$, or

(iv) $w \equiv \pm \left(\frac{(N-2)^2}{4}\right)^{\frac{N-2}{4}}$.

The above classification of all the solutions of (21) can be made with the help of the following 'energy' functional,

$$\mathcal{E}[w] = \frac{1}{2}\,|w'|^2 + \frac{N-2}{2N}\,|w|^{\frac{2N}{N-2}} - \frac{(N-2)^2}{8}\,|w|^2\,. \tag{22}$$

Indeed, for any solution of (21), $\mathcal{E}[w](s)$ is constant in s. Moreover, the positive and negative solutions of (21) are those having negative 'energy'. $w \equiv 0$ has zero 'energy' and the solutions with positive 'energy' $\mathcal{E}$ change sign an infinity number of times at infinity.

As we shall prove below, the use of the functional $\mathcal{E}$ enables us to prove that for all $\gamma > 0$, w_γ is a bounded function in $(0, \infty)$:

Lemma 2.13 *Let $N \geq 3$, $\lambda \geq 0$, $p = (N+2)/(N-2)$. For any $\gamma > 0$, for any $s > 0$, $\mathcal{E}[w_\gamma](s) \leq \mathcal{E}[w_\gamma](0) = \gamma^2/2$. Therefore, the functions w_γ and w'_γ are bounded in $(0, +\infty)$, independently of λ.*

Proof. This results follows immediately from the identity : for all $0 \leq s_1 < s_2 < +\infty$, and for any solution w of (20),

$$\mathcal{E}[w](s_2) - \mathcal{E}[w](s_1) = -\lambda \int_{s_1}^{s_2} w^2(s) e^{-2s} ds - \frac{\lambda w^2(s_2)}{2} e^{-2s_2} + \frac{\lambda w^2(s_1)}{2} e^{-2s_1}. \tag{23}$$

□

The above lemma does not hold when $\lambda < 0$, but in this case, the functions w_γ are still bounded :

Lemma 2.14 *Let $N \geq 3$, $\lambda < 0$, $p = (N+2)/(N-2)$. For any $\gamma > 0$, the functions w_γ and w'_γ are bounded on $(0, +\infty)$, independently of λ, and they are sign changing and periodic near infinity.*

Proof. Since for all $s > 0$, $\frac{d}{ds}\left(\mathcal{E}[w](s)\right) = |\lambda| e^{-2s} w(s) w'(s)$, either $w = w_\gamma$ is oscillatory and sign changing at infinity or w does not change sign as $s \to +\infty$ and $\lim_{s \to +\infty} w(s) = 0$, or w does not change sign as $s \to +\infty$ and $\lim_{s \to +\infty} w(s) = \pm\infty$. The latter can be excluded easily by just looking at equation (21), since in that case, $w''(s)$ would approach $\mp\infty$, which is incompatible with $\pm w' > 0$. In the case $\lim_{s \to +\infty} w(s) = 0$, $\pm w(s) > 0$ for s large enough, Lemma 2.11 shows that $\pm w(s) \sim \exp\left(-\frac{1}{2}(N-2)\, s\right)$ as $s \to +\infty$ and u is therefore a bounded function, which is impossible for $\lambda < 0$ by Pohozaev's identity.

Hence, w is oscillatory and sign changing at infinity. If w were not bounded, it would be possible to find c_n such that $\lim_{n \to +\infty} \mathcal{E}[w](c_n) = +\infty$ and $w(c_n) = \max_{(0, c_n]} w$. According to (23), since $p \geq 1$,

$$\mathcal{E}[w](0) \geq \mathcal{E}[w](c_n) - |\lambda| |w(c_n)|^2 \sim \mathcal{E}[w](c_n) \to +\infty,$$

as $n \to +\infty$, a contradiction. □

Lemma 2.15 *Assume that $p = (N+2)/(N-2)$, $N \geq 3$. For every $\gamma > 0$, let w_γ be the solution of (20). Then,*

$$E(\gamma) := \lim_{s \to +\infty} \mathcal{E}[w_\gamma](s) \quad \textit{exists}, \tag{24}$$

and if we denote by $w_{\infty,\gamma}$ the solution of (21) which satisfies $w_{\infty,\gamma}(0) = 0$, $\mathcal{E}[w_{\infty,\gamma}] \equiv E(\gamma)$, then

$$\lim_{s \to +\infty} |w_\gamma(s) - w_{\infty,\gamma}(s+c)| + \left| w'_\gamma(s) - w'_{\infty,\gamma}(s+c) \right| = 0\,, \tag{25}$$

for some $c \in \mathbb{R}$. Moreover, the function $E(\gamma)$ is continuous in (γ, λ).

Proof. The first part of the proof follows immediately from Lemma 2.9. The continuity of $E(\gamma)$ with respect to γ and λ follows from the fact that $\mathcal{E}[w_\gamma](t)$ approaches $E(\gamma)$ exponentially, with an exponential rate depending continuously on γ and λ. □

Lemma 2.16 *Assume that $p = (N+2)/(N-2)$, $N \geq 3$. For every $\gamma > 0$, let w_γ be the solution of (20) with $\lambda > 0$. Then, $E(\gamma) \geq \mathcal{E}[w_\gamma](s)$ for all $s \geq 0$ such that $w'_\gamma(s) = 0$ and $\mathcal{E}[w_\gamma](s) > 0$.*

Proof. Let $s_1 > 0$ be such that $w'_\gamma(s_1) = 0$ and $\mathcal{E}[w_\gamma](s_1) > 0$. Since $\dfrac{d}{ds}\left(\mathcal{E}[w_\gamma](s)\right) = -\lambda w_\gamma(s) w'_\gamma(s) e^{-2s}$ for all s, there must exist $s_2 > s_1$ such that $w'_\gamma(s_2) = 0$. We claim that $|w_\gamma(s_2)| \geq |w_\gamma(s_1)|$ and $\mathcal{E}[w_\gamma](s_2) \geq \mathcal{E}[w_\gamma](s_1)$. Indeed, taking into account the properties of the function $t \mapsto (N-2)|t|^{\frac{2N}{N-2}}/2N - (N-2)^2|t|^2/8$, if we assume that $|w_\gamma(s_2)| < |w_\gamma(s_1)|$, then $|w_\gamma(s)| < |w_\gamma(s_1)|$ for all $s \in (s_1, s_2]$. Then by (23) we have

$$\mathcal{E}[w_\gamma](s_2) - \mathcal{E}[w_\gamma](s_1) > \frac{\lambda e^{-2s_2}}{2}(w_\gamma^2(s_1) - w_\gamma^2(s_2)) > 0\,,$$

a contradiction. Hence, $|w_\gamma(s_1)| \leq |w_\gamma(s_2)|$ and $\mathcal{E}[w_\gamma](s_2) \geq \mathcal{E}[w_\gamma](s_1)$. The lemma follows from Lemma 2.15. □

Lemma 2.17 *For every $\gamma > 0$,*

- $E(\gamma) = 0$ *if and only if the solution u of (2) corresponding to (11)-(10) is bounded. In that case, u has a finite number of zeroes in the interval* $(0,1)$.
- $E(\gamma) > 0$ *if and only if u is unbounded and oscillates near the origin with sign changing oscillations.*
- $E(\gamma) < 0$ *if and only if u is unbounded at the origin, but has only a finite number of zeroes in* $(0,1)$.

Proof. The cases $E(\gamma) \neq 0$ immediatly follows from Lemma 2.15. Let us consider the case $E(\gamma) = 0$: $|w(s)| + |w'(s)|$ tends to 0 as s goes to $+\infty$. The function $r \mapsto f(r) = w_\gamma(-\ln r)$ satisfies $|f(r)| + r|f'(r)| \to 0$ as r goes to 0 and

$$-\left(rf'(r)\right)' = \frac{1}{r}\left(|f(r)|^{\frac{4}{N-2}} f(r) - \frac{(N-2)^2}{4} f(r) + \lambda r^2 f(r)\right)$$

in the interval $(0,1)$. Hence,

$$\left(\frac{(rf')^2}{2}\right)' = \frac{(N-2)^2}{8}\left(|f|^2\right)' - \frac{N-2}{2N}\left(|f|^{\frac{2N}{N-2}}\right)' - \lambda r^2 f f'.$$

Integrating the above equation, one obtains

$$rf'(r) = \left(\frac{N-2}{2} + h(r)\right) f(r)\,,$$

with $h(r) = o(1)$ for r small enough. Hence, $|f(r)|^2$ is increasing near the origin.

Therefore, for r small enough, $f(r) \sim \ell_\gamma(r) r^{(N-2)/2}$, for some continuous function ℓ_γ, such that $\ell_\gamma(0) > 0$. Finally, by (4), u_γ is bounded, has a finite number of zeroes in the interval $(0,1)$ and $u_\gamma(0) = \ell_\gamma(0)$. $\square$

Lemma 2.18

(i) $\lim_{\gamma \to +\infty} E(\gamma) = +\infty$,

(ii) *If* $\lambda \geq 0$, *then* $\limsup_{\gamma \to 0^+} E(\gamma) \leq 0$.

Proof. By Lemma 2.13, $E(\gamma) \leq \frac{\gamma^2}{2}$, which proves (ii).

Assume first that $\lambda \geq 0$. To prove (i), let us define

$$a(\gamma) = \inf \left\{ s > 0 \,;\, w'_\gamma(s) = 0 \right\} \in (0, +\infty),$$

and

$$r(\gamma) = \inf \left\{ s > 0 \,;\, w'_\gamma(s) = \frac{\gamma}{2} \right\}.$$

Assume that there is a sequence $\{\gamma_n\}$ such that $\gamma_n \to +\infty$ as $n \to +\infty$ and $E(\gamma_n)$ uniformly bounded. By Lemma 2.16,

$$\limsup_{n \to +\infty} w_{\gamma_n}(a(\gamma_n)) = M < +\infty .$$

Moreover,

$$\gamma_n/2 = \left| w'_{\gamma_n}(r(\gamma_n)) - w'_{\gamma_n}(0) \right| = r(\gamma_n) w''_{\gamma_n}(\theta_n r(\gamma_n)) \,,$$

for some $\theta_n \in (0,1)$. But $||w''_{\gamma_n}||_\infty \leq C_\lambda(M) := M^{\frac{N+2}{N-2}} + \lambda M$. Therefore,

$$\gamma_n \leq r(\gamma_n) C_\lambda(M),$$

and

$$w_{\gamma_n}(a(\gamma_n)) \geq w_{\gamma_n}(r(\gamma_n)) \geq \frac{\gamma_n r(\gamma_n)}{2} \geq \frac{\gamma_n^2}{4 C_\lambda(M)} \xrightarrow[n \to \infty]{} +\infty \,,$$

a contradiction.

If $\lambda < 0$, $|w|^{\frac{2N}{N-2}}$ dominates $|\lambda|\ |w|^2 > |\lambda| e^{-2s} |w|^2$, which by similar arguments to the ones in the case $\lambda \geq 0$ again provides a uniform in γ_n bound on $w_{\gamma_n}(a(\gamma_n))$. $\square$

Proposition 2.19 *Assume that for some* $\lambda > 0$, *there exists a positive bounded solution* $\overline{u}$ *of (2). Then, for all* $\gamma \in (0, -\overline{u}'(1))$, *the solutions of (2) with* $u'(1) = -\gamma$ *are unbounded at the origin and positive in* $(0,1)$.

Proof. As a consequence of the uniqueness result for positive bounded solutions of (1) in an annulus proved in [35] for $p = (N+2)/(N-2)$, it is clear that for all $\gamma \in (0, \overline{\gamma})$, $u_\gamma > 0$ in the interval $(0,1)$. Indeed, the uniqueness result implies that the largest zero of u_γ in $(0,1)$ is increasing in γ. In particular, if for some $\gamma \in (0, \overline{\gamma})$, $u_\gamma(c_\gamma) = 0$, $c_\gamma \in (0,1)$, there should be a zero of $u_{\overline{\gamma}}$ in $(c_\gamma, 1)$, which contradicts our assumptions. Therefore, by Lemmas 2.15 and 2.18, $E(\gamma) < 0$ and the result follows. □

Finally, the following last lemma gives a result equivalent to (7) in the case of unbounded solutions of (9).

Lemma 2.20 *Let* $\gamma, \lambda > 0$ *be such that* $E(\gamma) < 0$, *where* $E(\cdot)$ *is defined by (24). If we denote by* r_1 *the smallest zero of* u_γ *in* $(0,1)$, *then*

$$r_1^2 \geq \frac{2\,|E(\gamma)|}{\lambda C(\gamma)},$$

for some $C(\gamma) > 0$.

Proof. Let w_γ be defined by (10). Since $\mathcal{E}(w_\gamma(-\ln r_1)) > 0$, according to Lemma 2.13,

$$|E(\gamma)| < |E(\gamma) - \mathcal{E}(w_\gamma(-\ln r_1))| = \lambda \left| \int_{-\ln r_1}^{+\infty} w(s) w'(s) e^{-2s} ds \right| \leq \frac{\lambda}{2} C(\gamma) r_1^2$$

which proves the result. □

3 Critical case

In this section we assume that $N \geq 3$ and consider the critical case $p = (N+2)/(N-2)$. The known results on the branches of nodal bounded solutions can be summarized in the

Proposition 3.1 *For every* $i \geq 1$ *there exists a nonnegative contant* $\mu_i \in [0, \lambda_i)$ *and a continuous curve,* $\mathcal{C}_i \subset \{(\lambda, c) \in [0, +\infty)^2\}$ *corresponding to the set of the bounded solutions* u *of (2) with* $i-1$ *zeroes in the interval* $(0,1)$ *and such that* $u_c(0) = c$. *The point* $(\lambda_i, 0)$ *belongs to* $\mathcal{C}_i$ *and the half-line* $(\lambda = \mu_i, c > 0)$ *is asymptotic to* $\mathcal{C}_i$.

When $i = 1$, $\mathcal{C}_1$ can be parametrized by $c > 0$ and the function $\lambda(c)$ is decreasing. In other words, for every $\lambda \in (\mu_1, \lambda_1)$, there is a unique bounded positive solution of (1).

If $N = 3$, $\mu_i = \left(i - \frac{1}{2}\right)^2 \pi^2$. When $N = 4, 5$, $\mu_1 = 0$ and $\mu_i = \lambda_{i-1}$ for $i \geq 2$, and when $N = 6$, $\mu_1 = 0$, $0 < \mu_i < \lambda_{i-1}$ for all $i \geq 2$.

The information on the value of μ_i given in the last part of the proposition above is based on previous results of Atkinson and Peletier[4] and Atkinson, Brezis and Peletier [3] (see also [7]). For the uniqueness, see [36], [37], [24]. The existence of the curve $\mathcal{C}_i$ is easily recovered using the parametrization given in Lemma 2.5, the comparison with the eigenvalues of the Laplacian in B_1 and Pohozaev's identity. For the behavior of $\mathcal{C}_i$ as $c \to +\infty$, we can refer for instance to [2], [3], [4], [7].

Definition 3.2 *We say that a point (λ, u) in a branch of solutions of (1) in $(0, +\infty) \times L^\infty$ is* regular *if locally the branch can be parametrized by λ. A point $\lambda \in (0, +\infty)$ is* k-regular *if there exists a bounded solution with $k-1$ zeroes u_k such that (λ, u_k) is regular. We define I_k as the projection of $\mathcal{C}_k$ on $\mathbb{R}$ ($I_k = (\tilde{\mu}_k, \lambda_k)$ or $I_k = [\tilde{\mu}_k, \lambda_k)$, with ($\tilde{\mu}_k \leq \mu_k$)) and denote by J_k the set*

$$J_k = \{\lambda \in I_k \ : \ \lambda \text{ is } k- \text{ regular } \} .$$

Remark 3.3 *The number of k-singular points in I_k* i.e. *of the points which are in $I_k \setminus J_k$.*

Theorem 3.4 *Let $N \geq 3$, $p = \frac{N+2}{N-2}$. Then,*

(i) If $\lambda \in \mathbb{R} \setminus \cup_{k=1}^{\infty} I_k$, there is an uncountable number of solutions of (2) which are unbounded, oscillating at the origin, with sign changing oscillations. No other solutions to (2) exist.

(ii) If $\lambda \in I_k$ for some $k \geq 1$, then, there exist

- *a bounded solution u_k with $k-1$ zeroes in $(0,1)$ (this solution is unique at least if $k = 1$),*
- *an uncountable number of solutions (2) which are unbounded, oscillating at the origin, with sign changing oscillations,*
- *an uncountable number of unbounded solutions (2) with $k-1$ zeroes in $(0,1)$ if $\lambda \in J_k$.*

Finally, all the above solutions of (2) are also solutions of (1) in $\mathcal{D}'(B_1)$.

Remark 3.5 *The classification of all solutions of (2) can be made by using the parameter* $\gamma := -u'(1)$ *and problem (9). In case (i), for all* $\gamma > 0$, $E(\gamma) > 0$ *and there exists a bounded, periodic and sign changing function* w *, such that the solution* $u_\gamma(r)$ *of (9) behaves like* $r^{-2/(p-1)}\,w(-\ln r)$ *when* $r \to 0^+$.

In case (ii), there exists γ_k *such that* $E(\gamma_k) = 0$,

- u_{γ_k} *is bounded and has* $k-1$ *zeroes in* $(0,1)$,
- *if* $\lambda \in J_k$, *for an uncountable number of* γ*'s to the left of* γ_k, $E(\gamma) < 0$, u_γ *is singular, has* $(k-1)$ *zeroes in* $(0,1)$ *and near the origin it behaves like* $r^{2/(p-1)}\,a(r)$ *as* $r \to 0^+$ *for some positive or negative bounded periodic function* $a(r)$.
- *If* $\gamma > \sup\{\gamma > 0 : E(\gamma) = 0\}$, u_γ *behaves like* $r^{-2/(p-1)}\,w(-\ln r)$ *when* $r \to 0^+$, w *being periodic and sign changing.*

In view of the numerical computations, one can conjecture that the whole branch of the bounded solutions with $k-1$ *zeroes can be parametrized as a monotone decreasing function of* λ. *In that case,* $J_k = I_k$.

Proof of Theorem 3.4. By Lemmas 2.15, 2.17 and Proposition 3.1, if $\lambda \notin \cup_{k=1}^{\infty} I_k$, $E(\gamma)$ does not change sign. Since by Lemma 2.18, $E(\gamma)$ is positive for large γ, it follows that $E(\gamma) > 0$ for all $\gamma > 0$. Hence, by Lemma 2.15, for s large, w_γ is close to the bounded, periodic and sign changing function $w_{\infty,\gamma}$, which ends the proof of (i).

Let $k \geq 1$. The set of k-regular points by definition is open. Consider a small interval $\mathcal{V}$ of such points and denote by (λ, u_k^λ) a point on the branch such that $\lambda \in \mathcal{V}$. Consider then

$$r_1^k(\lambda) = \min\{r \in (0,1) \;:\; u_k^\lambda(r) = 0\}\;, \quad \nu_k(\lambda) = \frac{du_k^\lambda}{dr}(r_1^k(\lambda))\;,$$

and assume, without loss of generality, that $\nu_k(\lambda) > 0$. The solution $u_k^{\lambda,\nu}$ of

$$\begin{aligned} &-u'' - \frac{(N-1)}{r}\,u' = \lambda u + |u|^{\frac{4}{N-2}}u \quad \text{in} \quad (0, r_1^k(\lambda)) \\ &u(r_1^k(\lambda)) = 0\,, \quad \frac{du}{dr}(r_1^k(\lambda)) = \nu. \end{aligned} \tag{26}$$

does not change sign and is singular on $(0, r_1^k(\lambda))$ for any $\nu < \nu_k(\lambda)$ according to Proposition 2.19, and $u_k^{\lambda,\nu_k(\lambda)} = u_k^\lambda$.

Consider now the extension of $u_k^{\lambda,\nu}$ to $(0,+\infty)$ and define $\rho_k^{\lambda,\nu}$ as its k^{th} zero. For $\nu \le \nu_k(\lambda)$, $\tilde{u}_k^{\lambda,\nu}(r) = (\rho_k^{\lambda,\nu})^{\frac{N-2}{2}} u_k^{\lambda,\nu}(r \cdot \rho_k^{\lambda,\nu})$ is the unique solution (singular with $k-1$ zeroes) of

$$-u'' - \frac{(N-1)}{r} u' = \tilde{\lambda}_k(\nu,\lambda) u + |u|^{\frac{N-2}{2}} u \quad \text{in} \quad (0,1) \tag{27}$$
$$u(1) = 0\,, \quad \frac{du}{dr}(1) = \gamma_k(\nu,\lambda)$$

where $\tilde{\lambda}_k(\nu,\lambda) = \lambda \cdot (\rho_k^{\lambda,\nu})^2$ and $\gamma_k(\nu,\lambda) = (\rho_k^{\lambda,\nu})^{\frac{N}{2}} \cdot \frac{du_k^{\lambda,\nu}}{dr}(\rho_k^{\lambda,\nu})$. The map

$$\begin{aligned} \mathcal{M}^k : \mathcal{W} = \{(\nu,\lambda) \in (0,+\infty) \times \mathcal{V} \ : \ \nu < \nu_k(\lambda)\} \quad &\to (0,+\infty) \times (0,+\infty) \\ (\nu,\lambda) \quad &\mapsto (\gamma_k(\nu,\lambda), \tilde{\lambda}_k(\nu,\lambda)) \end{aligned}$$

is continuous. Moreover, there cannot exist two pairs (λ_1,ν_1), (λ_2,ν_2) such that $\mathcal{M}_k(\lambda_1,\nu_1) = \mathcal{M}_k(\lambda_2,\nu_2)$ if $\nu_1 = \nu_k(\lambda_1)$ but $\nu_2 < \nu_k(\lambda_2)$. Indeed, if $\mathcal{M}_k(\lambda_1,\nu_1) = \mathcal{M}_k(\lambda_2,\nu_2)$, $u_k^{\lambda_1,\nu_1}$ will be equal to $u_k^{\lambda_2,\nu_2}$ up to scaling, because $\tilde{u}_k^{\lambda_2,\nu_2} \equiv \tilde{u}_k^{\lambda_2,\nu_2}$. But, under the above assumptions $u_k^{\lambda_1,\nu_1}$ is bounded in $(0,1)$, while $u_k^{\lambda_2,\nu_2}$ is unbounded in that same interval. An inspection of the boundaries of $\mathcal{M}^k(\mathcal{W}) \cap (0,+\infty) \times \mathcal{V}$ shows that for any interval (a,b) such that $[a,b] \subset \mathcal{V}$, for $\lambda \in (a,b)$ and for γ in the interval of bounds

$$\overline{\gamma}_k(\lambda) := \gamma_k(\nu_k(\lambda),\lambda) = \frac{du_k^\lambda}{dr}(1)$$

and

$$\underline{\gamma}_k(\lambda) = \inf\{\gamma < \overline{\gamma}_k(\lambda) \ : \ (p,\lambda) \in \mathcal{M}^k(\mathcal{W}) \quad \forall\, p \in (\gamma, \overline{\gamma}_k(\lambda)\}\ ,$$

the corresponding solution of (2) is singular with $k-1$ zeroes in $(0,1)$. But by the above arguments, $\underline{\gamma}_k(\lambda) < \overline{\gamma}_k(\lambda)$ for all $\lambda \in (a,b)$, thus proving (ii).

Finally, the fact that all the solutions of (2) are also solutions of (1) follows from Lemma 2.1. $\square$

4 Subcritical case

As in the critical case, we will use the change of variables (10) to study the solution set of (2) when $1 < p < (N+2)/(N-2)$, $N \ge 2$. The function u is a solution of (9) if and only if w is a solution of

$$\begin{cases} w'' + L_1 w' + |w|^{p-1} w + L_2 w + \lambda e^{-2s} w = 0 \quad \text{in} \quad (0,+\infty)\,, \\ w(0) = 0\,,\ w'(0) = -u'(1) = \gamma\,, \end{cases} \tag{28}$$

We notice that in the subcritical case, $1 < p < (N+2)/(N-2)$, the coefficient of w' in (28) is always positive, that is $L_1 = \left(\frac{4}{p-1} - N + 2\right) > 0$. On the other hand, the sign of $L_2 = \frac{2}{(p-1)^2}((p+1)-(N-1)(p-1))$ depends on p : $L_2 > 0$ if and only if $1 < p < N/(N-2)$. Moreover, $L_2 = 0$ when $p = N/(N-2)$. The fact that L_2 changes sign in the subcritical exponent interval explains why the solution set of (2) will be quite different when $p \leq N/(N-2)$ and when $p > N/(N-2)$.

But before going into the details, let us again define an 'energy' functional adapted to equation (28). If for any function $w \in C^1(0, +\infty)$, we define

$$\mathcal{E}_{sb}[w] = \frac{1}{2}\,|w'|^2 + \frac{1}{p+1}\,|w|^{p+1} + \frac{L_2}{2}|w|^2\,,$$

we notice that for every solution of (28), we have

$$\frac{d}{ds}\left(\mathcal{E}_{sb}[w](s)\right) = -L_1|w'(s)|^2 - \lambda e^{-2s} w(s) w'(s)\,.$$

Throughout this section, we shall refer to [29], [5], [13], [32] and [19] for results concerning the existence, to Lemma 2.8 for the comparison with linear eigenvalue problems and estimates on the bounded solutions, and to [24], [36] and [37] for the uniqueness of the positive bounded solution when it exists.

Theorem 4.1 *Let $1 < p < N/(N-2)$ and assume that for some $i_0 \geq 1$, $\lambda < \lambda_{i_0}$. Then, all the solutions of (2) in $\mathcal{D}'(0,1)$ are bounded and radial solutions of (1) in $\mathcal{D}'(B_1)$. Moreover, for every $i \geq i_0$, there exists at least a solution of (1), u_i, which is bounded and has $i-1$ zeroes in $(0,1)$. The positive bounded solution, which exists if $\lambda < \lambda_1$, is unique.*

Proof. By Lemma 2.9 and Remark 2.10, when $1 < p < N/(N-2)$, $w_\gamma(s)$ converges to 0 as s goes to $+\infty$, for all $\gamma > 0$. Then, by Lemma 2.11 and Remark 2.12, when s is large enough, w will be asymptotically small and close to some function $Ce^{\nu s}$, where $C > 0$ and ν is a solution of the characteristic equation

$$\nu^2 + L_1\nu + L_2 = 0\,. \tag{29}$$

A straightforward computation shows that (29) has two solutions $\nu_1 = -2/(p-1)$ and $\nu_2 = N - 2 - 2/(p-1)$. Moreover, $\nu_2 < 0$ only if $1 < p < N/(N-2)$.

By the change of variables (4), if w behaves near 0 as $Ce^{\nu_1 s}$, the corresponding u satisfies $u(0) = C > 0$ and $u \in L^\infty(B_1)$. Also, if, near 0, w

behaves as $Ce^{\nu_2 s}$, the corresponding u satisfies $u(r) \sim Cr^{2-N}$ near 0. Such a function is a solution of (2) but is not a solution of (1), because the singularity at the origin is not removable. Indeed, it introduces a Dirac mass in the second term of the equation. Hence, no singular solution of (1) exists when $1 < p < N/(N-2)$.

Theorem 4.2 *Let $N \geq 3$ and $N/(N-2) < p < (N+2)/(N-2)$. Then, if $\lambda < \lambda_{i_0}$ for some $i_0 \geq 1$, there exists :*

- *a bounded and smooth solution of (2), u_i, with $(i-1)$ zeroes in $(0,1)$, for all $i \geq i_0$ (if it exists, i. e. if $\lambda < \lambda_1$, the bounded positive solution u_1 is unique),*
- *an uncountable set of unbounded solutions with a finite number of zeroes in $(0,1)$. All these solutions behave at the origin like $a(r)r^{-2/(p-1)}$ for some bounded function $a(r)$ which is bounded away from 0 at the origin.*

Under the assumptions of this theorem, the number of zeroes is finite for all solutions of (2) and all the solutions of (2) are also solutions of (1).

Proof. By Lemma 2.9 and Remark 2.10, for all $\gamma > 0$, w_γ converges to a zero of $V(w) = |w|^{p-1}w + L_2 w$ as s goes to $+\infty$. In this case, the zeroes of V are 0 and $\pm(-L_2)^{1/(p-1)}$. Moreover, if w goes to 0 at infinity, it has to go necessarily like $C\,e^{-2s/(p-1)}$ (see Lemma 2.11), for some $C > 0$, which implies that the corresponding solution of (2), u, is bounded and $u(0) = C$. On the other hand, if w tends to $\pm(-L_2)^{1/(p-1)}$ at infinity, the corresponding u will be singular at the origin, with a singularity like $r^{-2/(p-1)}$ giving rise to a singular solution of (1) by Lemma 2.1 (removability of the singularity), which has a finite number of zeroes in $(0,1)$. Since for given i, λ, there is an *a priori* bound on $||u||_{C^1([0,1])}$ for any $(\lambda, u) \in \mathcal{C}_i$, whenever γ is large enough, u_γ is not bounded, i.e., w_γ converges to $\pm(-L_2)^{1/(p-1)}$ at infinity. The rest of the proof is done by following the same arguments as those used to prove Theorems 3.4 and 4.1. $\square$

Theorem 4.3 *Let $N \geq 3$ and $p = N/(N-2)$. Then, if $\lambda < \lambda_{i_0}$ for some $i_0 \geq 1$, there exists :*

- *a bounded and smooth solution of (2), u_i, with $(i-1)$ zeroes in the interval $(0,1)$, for all $i \geq i_0$, and the bounded positive solution is unique if it exists,*

- *an uncountable set of unbounded solutions with a finite number of zeroes in* $(0,1)$*. All these solutions behave at the origin like* $C_0|\ln r|^{-1/(p-1)}r^{-2/(p-1)}$ *with* $C_0 = ((N-2)/\sqrt{2})^{N-2}$.

Under the assumptions of this theorem, the number of zeroes is finite for all solutions of (2) and all the above solutions of (2) are also solutions of (1).

We skip the proof of the above theorem since it is quite similar to the above ones.

5 Supercritical case

When $p > (N+2)/(N-2)$, we shall consider a change of variables different from (4) : for solutions u of (2), we define w as

$$u(r) = r^{-\alpha}w(s)\,, \quad s = r^{-\beta}\,, \tag{30}$$

with

$$\alpha = \frac{2(N-1)}{p+3}\,, \quad \beta = N-2-\frac{4(N-1)}{p+3}\,.$$

(Note that in the limit $p \to (N+2)/(N-2)$, $\alpha \to (N-2)/2$ and $\beta \to 0^+$). To a solution u_γ of (9) corresponds now a solution of

$$\begin{cases} \beta^2 w'' + |w|^{p-1}w - \frac{\mu^2 w}{s^2} + \frac{\lambda w}{s^{2/\beta+2}} = 0 & \text{in} \quad (1,+\infty)\,, \\ w(1) = 0\,,\ w'(1) = -u'_\gamma(1)/\beta = \gamma/\beta\,, \end{cases} \tag{31}$$

with $\mu^2 = \alpha(N-2-\alpha)$. Next, we define the 'energy' functional

$$\mathcal{F}[w] = \frac{\beta^2}{2}\,|w'|^2 + \frac{1}{p+1}\,|w|^{p+1},$$

and we notice that for any solution of (31), w, we have

$$\frac{d}{ds}\left(\mathcal{F}[w](s)\right) = \left(-\frac{\lambda w(s)}{s^{2/\beta+2}} + \frac{\mu^2 w(s)}{s^2}\right) w'(s)\,. \tag{32}$$

Let us now prove two qualitative results on the solutions of (31), in the spirit of Lemmas 2.13 and 2.16.

Lemma 5.1 *Let* $p > (N+2)/(N-2)$, $N \geq 3$*. Then, if* w *is a solution of (31) for some* $\gamma > 0$ *and* $\mathcal{F}$ *is the 'energy' functional defined above, for any* a, b *large with* $a < b < +\infty$, *such that* $w(a) = w(b) = 0$, *we have*

$$\mathcal{F}[w](b) > \mathcal{F}[w](a)\,.$$

Proof. From (32), an integration by parts provides

$$\mathcal{F}[w](s_2) - \mathcal{F}[w](s_1) = -\frac{\lambda}{\beta}\int_{s_1}^{s_2} \frac{|w(s)|^2\, ds}{s^{2/\beta+3}}$$

$$+\mu^2 \int_{s_1}^{s_2} \frac{|w(s)|^2\, ds}{s^3} - \lambda \frac{|w(s)|^2}{2s^{2/\beta+2}}\Bigg|_{s_1}^{s_2} + \mu^2 \frac{|w(s)|^2}{2s^2}\Bigg|_{s_1}^{s_2}, \tag{33}$$

and since β is positive, the lemma follows for a, b large enough. □

Lemma 5.2 *Let $p > (N+2)/(N-2)$, $N \geq 3$ and $\lambda \geq 0$. Then, for all $\gamma > 0$, there exists a constant $C_\gamma > 0$ such that if w is a solution of (31) , then*

$$|w(s)| \leq C_\gamma\,, \quad \textit{for all} \;\; s \geq 0\,.$$

Moreover,

$$\lim_{\gamma\to+\infty} \lim_{s\to+\infty} \mathcal{F}[w](s) = +\infty. \tag{34}$$

Proof. Consider (33) with s_1 fixed and $s_2 = s_n$ such that $\lim_{n\to+\infty} |w(s_n)| = +\infty$ and $w(s_n) = \max_{(s_1,s_n]} w(s)$:

$$\mathcal{F}[w](s_n) \leq C + C|w(s_n)|^2 = o\left(|w(s_n)|^{p+1}\right),$$

for some constant $C > 0$, a contradiction. The proof of (34) is similar to the proof of Lemma 2.18 and is left to the reader. □

For $p \geq \tilde{p}_N := 3N/(N-2)$, solutions of (2) which are singular, sign changing, oscillating near 0, do not belong to $L^p_{loc}(B_1)$ and are not weak solutions in the usual sense. Because these solutions are sign changing, it is however possible to consider them as solutions "in the sense of the principal value", giving sense to $|u|^{p-1}u$ as the distribution corresponding to its principal value VP $(|u|^{p-1}u)$:

$$< \text{VP } (|u|^{p-1}u), \varphi > := \lim_{\varepsilon\to 0} \int_{B_1\setminus\overline{B_\varepsilon}} |u|^{p-1} u\, \varphi\, dx\,, \quad \text{for all } \; \varphi \in \mathcal{D}(B_1)\,. \tag{35}$$

We can now prove the following

Theorem 5.3 *Let $N \geq 3$ and $p > (N+2)/(N-2)$.*

For every $\lambda \in \mathbb{R}$, there is an uncountable number of solutions of (2) which are unbounded, sign changing and oscillating near the origin. More *precisely, for an uncountable number of* $\gamma > 0$*, the solution to (9),* $u_\gamma(r)$*, is equivalent near* 0 *to* $\tilde{w}(r^{-\beta})r^{-2(N-1)/(p+3)}$, $\beta = N-2-4(N-1)/(p+3)$*, where* $\tilde{w}$ *is a non trivial periodic, bounded and sign changing solution of* $\beta^2\tilde{w}'' + |\tilde{w}|^{p-1}\tilde{w} = 0$ *in the interval* $(0,+\infty)$*.*

For every $k \geq 1$ *let us denote by* I_k *the interval of* λ*'s such that (2) has a bounded solution with* $k-1$ *zeroes in* $(0,1)$*. By Pohozaev's identity,* $I_k \subset (0,+\infty)$*. Then,*

- *For* $\lambda \in \mathbb{R} \setminus \cup_{k=1}^{+\infty} I_k$*, all the solutions of (2) are oscillating near the origin, as described above.*

- *For every* $k \geq 1$ *there is a unique* $\lambda_k^* \in \overline{I}_k$ *such that (2) has a unique unbounded solution* u_k^* *with* $k-1$ *zeroes in the interval* $(0,1)$*. Moreover,* $r^{2/(p-1)}u_k^*(r) \in L^\infty(0,1)$*.*

Finally, all the above solutions of (2) are also solutions of (1) if $p < \tilde{p}_N$ *or if* $p > \tilde{p}_N$ *and* u *has a finite number of zeroes in the interval* $(0,1)$*. If* $p \geq \tilde{p}_N$ *and* u *is oscillatory and sign changing near the origin, then* $u \notin L^p_{loc}(B_1)$ *and* u *is a distributional solution of (1) "in the sense of the principal value".*

The numerical computations performed for this problem by C.J. Budd and J. Norbury (see [10]) suggest that $\lambda_1^* - \delta \in I_1$ for $\delta > 0$ small enough, and that for $\lambda = \lambda_1^*$ there is an infinite sequence of bounded positive solutions of (1) and a unique singular positive solution.

Proof. First we recall a result of F. Merle and L.A. Peletier [26] showing the existence of a unique $\lambda_1^* \in (0,\lambda_1)$ for which there exists a singular positive solution of (1) (note that a simple rescaling argument provides the uniqueness of singular solutions with a fixed number of zeroes). Moreover, the branch of solutions bifurcating from λ_1, obtained by classical bifurcation theory (see [29]), becomes unbounded exactly at $\lambda = \lambda_1^*$. The behavior at the singularity can also be found in [26]. Lemma 2.8 shows the nonexistence of bounded nontrivial solutions of (2) for all $\lambda \geq \lambda_1$, while for $\lambda \leq 0$, the nonexistence result follows from Pohozaev's identity.

Thus, in view of Lemmas 5.1, 5.2 and their proofs, we only have to prove that there is an uncountable number of sign changing oscillatory solutions for all $\lambda > 0$ and that all the singularities at the origin are removable. Moreover, from Lemmas 2.9 and 5.1, for every $\gamma > 0$, the sign changing oscillatory solutions of (31) are asymptotically close, as s goes to $+\infty$, to a periodic, bounded, nonconstant function. Hence, by (30), the corresponding solutions of (2) behave near the origin as $u(r) \sim r^{-\alpha}\tilde{w}(r^{-\beta})$, where $\tilde{w}$ is a periodic, bounded, nonconstant and sign changing solution of $\beta^2\,\tilde{w}'' + |\tilde{w}|^{p-1}\,\tilde{w} = 0$.

For every $\lambda \in \mathbb{R}$, there is a 'continuum' of sign changing oscillating such solutions according to Lemma 5.2.

The removability of the singularities of solutions with a finite number of zeroes in the interval $(0,1)$ follows from Lemma 2.1. The same argument can be used to remove the singularity at the origin for the oscillating, sign changing solutions of (2) if $p < 3N/(N-2)$.

There is a different difficulty about the sign changing oscillating solutions of (2) when $p \geq 3N/(N-2)$. In this case, u is not in $L^p_{loc}(B_1)$, and hence, one has to use (35) to define u as a weak solution of (1). Elementary asymptotics indeed show that the integral of $|u|^{p-1}u$ on $B_1 \setminus \bar{B}_\varepsilon$ gives rise to an oscillating, sign changing function of ε which converges as $\varepsilon \to 0^+$ (this can be proved by means of a convergent alternate numerical series). $\square$

6 Appendix A : a sharp estimate of Pohozaev type

From the results of Section 2, it follows that when $N=3$, $p=5$, equation (1) has no bounded or unbounded positive solution for any $\lambda \leq \pi^2/4$. The nonexistence of bounded solutions was proved by H. Brezis and L. Nirenberg in [7]. The nonexistence of unbounded positive solutions follows from Lemmas 2.15 2.17 and 2.18.

This annex is devoted to a direct proof of this result for $0 < \lambda \leq \pi^2/4$ using the solution w of (20). To do this, we prove that for all $\gamma > 0$, $E(\gamma)$ has to be positive, and then apply Lemma 2.17. For $\lambda \leq 0$, the result follows from Pohozaev's identity.

Theorem 6.1 *Let $N=3$, $p=5$ and $0 < \lambda \leq \pi^2/4$. Then for all $\gamma > 0$, $E(\gamma) > 0$. Therefore, all solutions of (2) are oscillatory at the origin, with unbounded sign changing oscillations and weak solutions of (1) (in the distributional sense).*

Proof. Let us consider a function $g \in C^2([0,+\infty)) \cap L^\infty(0,\infty)$ such that $g(0) \geq 0$, $g(t)$ is increasing in t and $g'(t)\,,g''(t)$ tend to 0 as t goes to $+\infty$.

Consider the equation for w given by (20) when $N=3$:

$$-w'' = |w|^4 w - \frac{w}{4} + \lambda\, e^{-2s} w, \tag{36}$$

multiply it by gw' and integrate between 0 and T :

$$-\frac{g(0)|w'(0)|^2}{2} - \int_0^T \frac{g'(t)|w'(t)|^2}{2} dt = \int_0^T g'(t) \left\{ \frac{|w(t)|^6}{6} - \frac{|w(t)|^2}{8} \right\} dt$$

$$-g(T)\mathcal{E}[w](T) - \frac{\lambda}{2}\, g(T)e^{-2T}|w(T)|^2 + \lambda \int_0^T |w(t)|^2 e^{-2t} \left(\frac{g'(t)}{2} - g(t) \right) dt. \tag{37}$$

Now, multiplying equation (36) by $\frac{1}{2}wg'$ and integrating, we get

$$\frac{w'(T)w(T)g'(T)}{2} - \frac{|w(T)|^2 g''(T)}{4} - \int_0^T \frac{|w'(t)|^2 g'(t)}{2} dt + \int_0^T \frac{|w(t)|^2 g'''(t)}{4} dt$$

$$= \int_0^T \frac{|w(t)|^2 g'(t)}{8} dt - \int_0^T \frac{|w(t)|^6 g'(t)}{2} dt - \lambda \int_0^T \frac{|w(t)|^2 g'(t) e^{-2t}}{2} dt\,. \tag{38}$$

Adding up (37) and (38) and taking into account the assumptions made on g, we obtain :

$$\lim_{T\to+\infty} \left(\int_0^T |w(t)|^2 \left(\frac{g'''(t)}{4} - \frac{g'(t)}{4} + \lambda e^{-2t}(g'(t) - g(t)) \right) dt \right.$$

$$\left. + \frac{2}{3} \int_0^T |w(t)|^6 g'(t)\, dt \right) = \lim_{T\to+\infty} g(T)\mathcal{E}[w](T) - \frac{1}{2} g(0) w'(0)^2\,, \tag{39}$$

since from Lemma 2.15, for any $\gamma > 0$, the functions w_γ and w'_γ are bounded in the interval $(0, \infty)$.

Finally, we choose the function $g(t) = \frac{\sin(2\sqrt{\lambda}e^{-t})}{2\sqrt{\lambda}e^{-t}}$ which satisfies all the required assumptions if and only if $\lambda \le \pi^2/4$ ($g(0) \ge 0$). This function is solution to the differential equation

$$\frac{g'''}{4} - \frac{g'}{4} + \lambda(g' - g)e^{-2t} = 0\,.$$

With this choice of the function g, (39) reads :

$$0 < \lim_{T\to+\infty} \int_0^T \frac{2}{3} g'(t)|w(t)|^6 dt = E(\gamma) - \frac{1}{2} g(0)|w'(0)|^2\,. \tag{40}$$

So, if $g(0) \ge 0$, $E(\gamma)$ has to be positive. $\square$

7 Appendix B: Some plots

CRITICAL CASE

Let us first consider a solution $w(t)$ of $w'' - \frac{w}{4} + \lambda e^{-2t} + |w|^4 w = 0$ corresponding to a radial solution of $\Delta u + |u|^4 u + \lambda u = 0$ in the critical case $5 = \frac{N+2}{N-2}$ for $N = 3$, $E[w] = \frac{|w'|^2}{2} + \frac{|w|^6}{6} - \frac{|w|^2}{8}$ and $z(t) = w'(t)$. We assume that $\frac{\pi^2}{4} < \lambda = 8 < \pi^2$. Define the asymptotic 'energy' $I(g) = \lim_{t \to +\infty} E[w](t)$ where w is the solution defined by to $w(0) = 0$, $w'(0) = z(0) = g$. Depending whether $I(g)$ is positive, zero or negative, we have the three following cases (where the left plot corresponds to the parametric curve $t \mapsto (w(t), E[w](t))$ – the potential $w \mapsto \frac{|w|^6}{6} - \frac{|w|^2}{8}$ is also represented, and the right plot is the representation in the phase space of $t \mapsto (w(t), z(t) = w'(t))$.

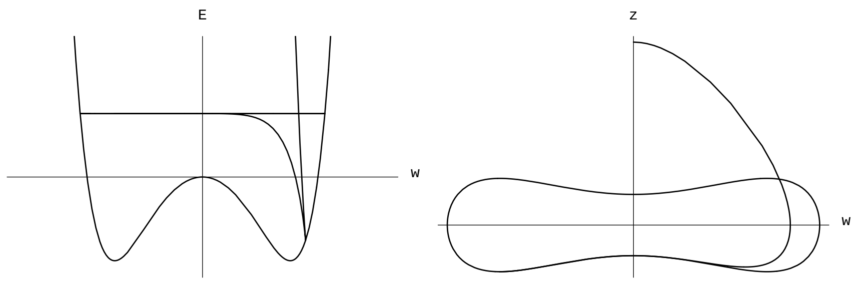

Figure 1. $I(g_1) > 0$: u is singular and oscillating near 0.

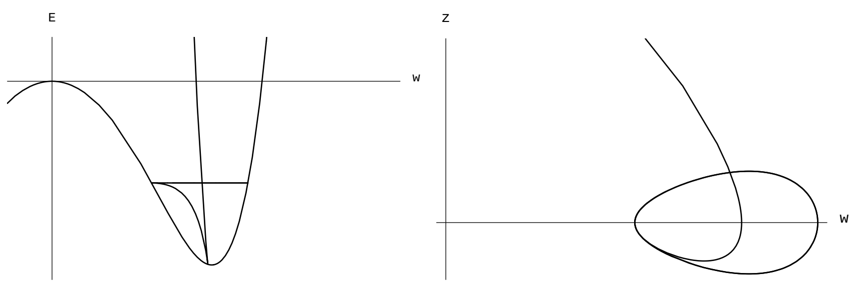

Figure 2. $I(g_2) < 0$: u is singular but non oscillating (positive).

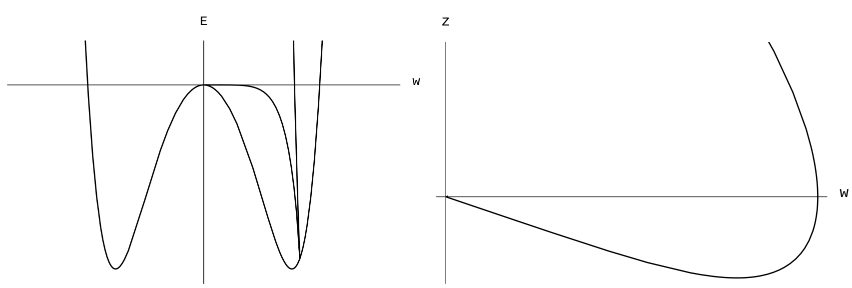

Figure 3. $I(g_3) = 0$: u is the unique bounded (smooth) solution.

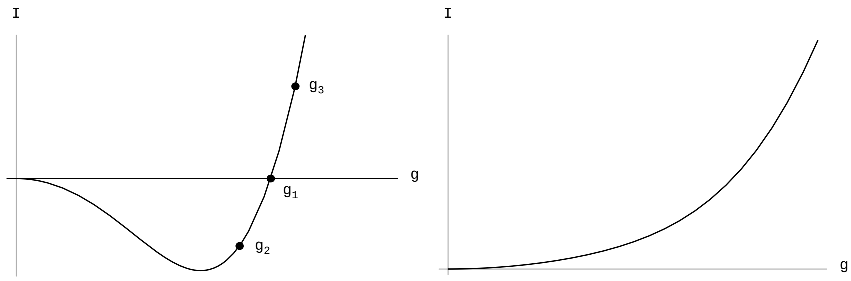

Figure 4. The curve $g \mapsto I(g)$ is the crucial tool for the classification of the solutions in the critical case. It strongly depends on the value of λ. There is a unique classical (bounded, positive, smooth) solution for the left plot corresponding to $I(g = g_3) = 0$, for a given $\lambda = 8 \in (\frac{\pi^2}{4}, \pi^2)$, and no such solution (all solutions are singular and oscillating) for the right plot, corresponding to the case $\pi^2 < \lambda = 11 < \frac{9\pi^2}{4}$.

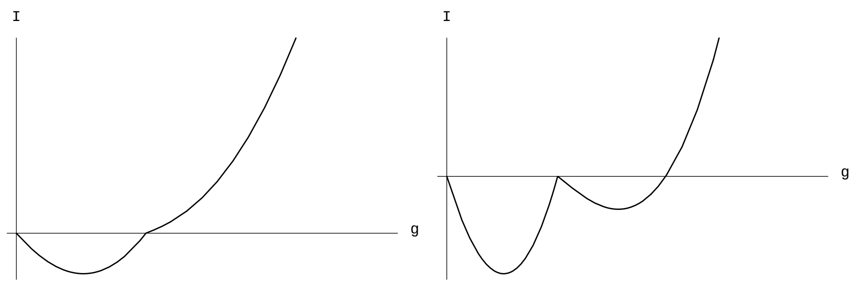

Figure 5. In the critical case $p = \frac{N+2}{N-2} = 2$ for $N = 6$, several branches of bounded solutions may intersect the vertical line $\lambda \times \mathbb{R}$. The curve $g \mapsto I(g) = \lim_{t\to+\infty} E[w](t)$ where w is the solution of $w'' - 4w + \lambda e^{-2t} w + |w|w = 0$ corresponding to $w(0) = 0$, $w'(0) = g$ and $E[w] = \frac{|w'|^2}{2} + \frac{|w|^3}{3} - 2|w|^2$ then has several zeroes corresponding to bounded solutions. The left and the right plots correspond to $\lambda = 20$ and $\lambda = 23.5$ respectively.

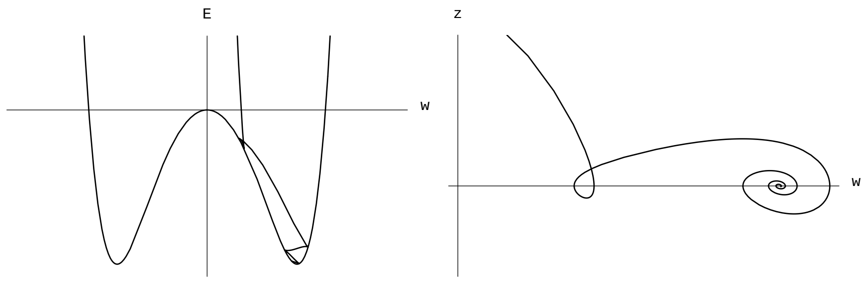

Figure 6. Consider for $N = 3$ the solutions of $\Delta u + |u|^3 u + \lambda u = 0$ ($3 = \frac{N}{N-2} < 4 < \frac{N+2}{N-2} = 5$) with $\lambda = 8$. $E[w](t) = \frac{|w'|^2}{2} + \frac{|w|^5}{5} - \frac{|w|^2}{8}$ converges as $t \to +\infty$ to a critical point of $w \mapsto \frac{|w|^5}{5} - \frac{|w|^2}{8}$, which is different from 0 if $w(t)$ is a solution of $w'' - \frac{w}{4} + \lambda e^{-2t} + |w|^3 w = 0$ corresponding to a radial singular solution u. On the left is shown a representation of $t \mapsto (w(t), E[w](t))$ decaying to a minimum of $w \mapsto \frac{|w|^5}{5} - \frac{|w|^2}{8}$, while on the right the same solution is represented in the phase space.

Acknowledgements. The authors want to thank warmfully Michel Vanbreugel for his help in the numerical investigation of the properties of the solutions.

One of us (R. B.) would like to thank Ceremade (Université Paris IX-Dauphine) for their kind hospitality while part of this work was done.

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