RAYLEIGH–FABER–KRAHN INEQUALITY – Inequality concerning the lowest eigenvalue of the Laplacian, with Dirichlet boundary condition, on a bounded domain in $\mathbf{R}^n$ $(n \geq 2)$.

Let $0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \ldots$ be the **Dirichlet eigenvalues** of the Laplacian in $\Omega \subset \mathbf{R}^n$, i.e.,

$$-\Delta u = \lambda u \qquad \text{in } \Omega, \tag{1}$$

$$u = 0 \qquad \text{on the boundary of } \Omega. \tag{2}$$

Here Δ is the Laplace operator, Ω an open bounded subset of $\mathbf{R}^n$ $(n \geq 2)$. If $n = 2$, the **Dirichlet eigenvalues** are proportional to the square of the eigenfrequencies of an elastic, homogeneous, vibrating membrane with fixed boundary.

The Rayleigh–Faber–Krahn inequality for the membrane (i.e., $n = 2$) states that

$$\lambda_1 \geq \frac{\pi j_{0,1}^2}{A}, \tag{3}$$

where $j_{0,1} = 2.4048\ldots$ is the first zero of the Bessel function of order zero, and A is the area of the membrane. Equality is attained in (3) if and only if the membrane is circular. In other words, among all membranes of given area, the circle has the lowest fundamental frequency. This inequality was conjectured by Lord Rayleigh [14], based on exact calculations for simple domains, and a variational argument for near circular domains. In 1918, Courant [5] proved the weaker result that among all membranes of the same perimeter L, the circular one yields the least lowest eigenvalue, i.e.,

$$\lambda_1 \geq \frac{4\pi^2 j_{0,1}^2}{L^2}, \tag{4}$$

with equality if and only if the membrane is circular. Rayleigh's conjecture was proven independently by Faber [6] and Krahn [7]. The corresponding isoperimetric inequality in dimension n,

$$\lambda_1(\Omega) \geq \left(\frac{1}{|\Omega|}\right)^{2/n} C_n^{2/n} j_{n/2-1,1}, \tag{5}$$

was proven by Krahn [8]. In (5), $j_{m,1}$ is the first positive zero of the Bessel function J_m, $|\Omega|$ is the volume of the domain and $C_n = \pi^{n/2}/\Gamma(n/2+1)$ is the volume of the n–dimensional unit ball. Equality is attained in (5) if and only if Ω is a ball.

The proof of the Rayleigh–Faber–Krahn inequality rests upon two facts: a variational characterization for the lowest Dirichlet eigenvalue and the properties of symmetric decreasing rearrangements of functions. The variational characterization of the lowest eigenvalue is given by

$$\lambda_1(\Omega) = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega (\nabla u)^2 \, dx}{\int_\Omega u^2 \, dx}. \tag{6}$$

Concerning decreasing rearrangements, let Ω be a measurable subset of $\mathbf{R}^n$, then the symmetrized domain Ω^* is a ball with the same measure as Ω. If u is a real valued mesaurable function defined on a bounded domain $\Omega \subset \mathbf{R}^n$, its spherical decreasing rearrangement $u^\star$ is a function defined on the ball Ω^* centered at the origin and having the same measure as Ω, such that $u^\star$ depends only on distance from the origin, is decreasing away from the origin and is equimeasurable with u. We refer to [13, 18, 4] for properties of rearrangements of functions. Since the function u and its spherical decreasing rearrangement are equimeasurable, their L^2 norms are the same. What Faber and Krahn actually proved is that the L^2 norm of the gradient of a function is decreased under rearrangements (see [18] for details, and also [9] for a different approach to this fact). The fact that the L^2 norm of the gradient of a function decreases under rearrangements, combined with the variational characterization (6) immediately gives the Rayleigh–Faber–Krahn inequality.

There are several isoperimetric inequalities for the lowest eigenvalue of boundary value problems, similar to the Rayleigh–Faber–Krahn inequality. We survey a few of them in the sequel. The lowest non trivial **Neumann eigenvalue** also satisfies an isoperimetric inequality. Let $0 = \mu_1(\Omega) < \mu_2(\Omega) \leq \mu_3(\Omega) \leq \ldots$ be the **Neumann eigenvalues** of the Laplacian in $\Omega \subset \mathbf{R}^n$, i.e.,

$$-\Delta u = \mu u \qquad \text{in } \Omega, \tag{7}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on the boundary of } \Omega. \tag{8}$$

If $n = 2$, Szegö [17] proved

$$\mu_2(\Omega) \leq \frac{\pi p_1^2}{A}, \tag{9}$$

where $p_1 = 1.8412\ldots$, with equality if and only if Ω is a circle. The corresponding result for dimension n,

$$\mu_2(\Omega) \leq \left(\frac{1}{|\Omega|}\right)^{2/n} C_n^{2/n} p_{n/2,1}^2, \tag{10}$$

was proven by Weinberger [19], with equality if and only if Ω is a ball. Here C_n is the volume of the unit ball in dimension n. In (9) and (10), $p_{m,1}$ denotes the first positive zero of the derivative of the Bessel function J_m. For $n = 2$ Weinberger [19] also proved

$$\frac{1}{\mu_2(\Omega)} + \frac{1}{\mu_3(\Omega)} \geq \frac{2A}{\pi p_1^2}, \tag{11}$$

with equality if and only if Ω is a circle.

There is also an analog of the Rayleigh–Faber–Krahn inequality for domains in spaces of constant curvature [15]. The optimal Rayleigh–Faber–Krahn inequalities for domains in $\mathbf{S}^n$ was proven by Sperner [16].

In his book, *The Theory of Sound*, Lord Rayleigh also conjectured an isoperimetric inequality for the lowest eigenvalue, Λ_1, of the clamped plate. The eigenvalue problem for the clamped plate is given by

$$\Delta^2 u_1 = \Lambda_1 u_1 \qquad \text{in } \Omega$$

with

$$u_1 = \left|\frac{\partial u}{\partial n}\right| = 0 \qquad \text{in the boundary of } \Omega.$$

Here Ω is a bounded open subset of $\mathbf{R}^2$. Rayleigh's conjecture for the clamped plate reads

$$\Lambda_1(\Omega) \geq \Lambda_1(\Omega^*), \tag{12}$$

where Ω^* is a ball of the same area as Ω. Rayleigh's conjecture was proven by N. Nadirashvili [12]. Equality is attained in (12) if and only if Ω is a circle. For dimension 3, the corresponding isoperimetric inequality was proven by Ashbaugh and Benguria [2]. To prove the analogous result for dimensions 4 and higher is still an open problem (see [3] however).

Back in the membrane problem, if we go beyond the lowest eigenvalue, there are several isoperimetric inequalities as well as a number of open problems. The simplest combination $\lambda_2/\lambda_1(\Omega)$ satisfies the following inequality [1]

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{j_{n/2,1}^2}{j_{n/2-1,1}^2}, \tag{13}$$

in n dimensions, where equality is obtained if and only if Ω is a ball. Stability results for both, the Rayleygh–Faber–Krahn inequality (3), (4), and inequality (13) have been obtained by Melas [11] (in simple words stability means that if Ω is convex and the appropriate left side on either (3), (4) or (13) is not too different from its corresponding isoperimetric value, then Ω is approximately a ball).

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