**DIRICHLET EIGENVALUE**– Consider a bounded domain $\Omega \subset \mathbf{R}^n$ with a piecewise smooth boundary $\partial\Omega$. $\lambda$ is a Dirichlet eigenvalue of $\Omega$ is there exists a function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ (Dirichlet eigenfunction) satisfying the following **Dirichlet boundary value problem**

$$-\Delta u = \lambda u \qquad \text{in } \Omega \tag{1}$$

and

$$u = 0 \qquad \text{in } \partial\Omega, \tag{2}$$

where $\Delta$ is the Laplace operator (i.e., $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$). Dirichlet eigenvalues (with $n = 2$) were introduced in the study of the vibrations of the clamped membrane in the XIX century. In fact they are proportional to the square of the eigenfrequencies of the membrane with fixed boundary. See [9] for a review and historical remarks. Provided $\Omega$ is bounded and the boundary $\partial\Omega$ is sufficiently regular, the Dirichlet Laplacian has a discrete spectrum of infinitely many positive eigenvalues with no finite accumulation point [13]

$$0 < \lambda_1(\Omega) \le \lambda_2(\Omega) \le \lambda_3(\Omega) \le \ldots \tag{3}$$

($\lambda_k \to \infty$ as $k \to \infty$).

The Dirichlet eigenvalues are characterized by the max–min principle [4]

$$\lambda_k = \sup\inf \frac{\int_\Omega (\nabla u)^2\, dx}{\int_\Omega u^2\, dx}, \tag{4}$$

where the inf is taken over all $u \in H_0^1(\Omega)$ orthogonal to $\varphi_1, \varphi_2, \ldots, \varphi_{k-1} \in H_0^1(\Omega)$, and the sup is taken over all choices of $\{\varphi_i\}_{i=1}^{k-1}$. For simply connected domains it follows from the max–min principle (4) that $\lambda_1(\Omega)$ is nondegenerate and the corresponding eigenfunction $u_1$ is positive in the interior of $\Omega$. For higher values of $k$ the nodal lines of the $k$–th eigenfunction divide $\Omega$ into no more than $k-1$ subregions (*nodal domains*) (Courant's nodal line theorem [4]). Along this subject, notice the proof of Melas [11] of the nodal line conjecture for plane domains (if $\Omega$ is a bounded, smooth, convex domain, the nodal line of $u_2$ always meets $\partial\Omega$).

For large values of $k$, if $\Omega \subset \mathbf{R}^n$, Weyl [17, 18] proved

$$\lambda_k \approx \frac{4\pi^2 k^{2/n}}{(C_n|\Omega|)^{2/n}}, \tag{5}$$

where $|\Omega|$ and $C_n = \pi^{n/2}/\Gamma(n/2+1)$ are, respectively, the volumes of $\Omega$ and of the unit ball in $\mathbf{R}^n$.

For any plane–covering domain (i.e., a domain that can be used to tile the plane without gaps, nor overlaps, allowing rotations, translations, and reflections of itself), Pólya [14] proved that

$$\lambda_k \ge \frac{4\pi k}{A} \qquad \text{for } k = 1, 2, \ldots, \tag{6}$$

and conjectured the same bound for any bounded domain in $\mathbf{R}^2$ (here $A$ is the area of the domain). Pólya's conjecture in $n$ dimensions is equivalent to saying that the Weyl asymptotics of $\lambda_k$ (5) is a lower bound for $\lambda_k$, i.e.,

$$\lambda_k \ge \frac{4\pi^2 k^{2/n}}{(C_n|\Omega|)^{2/n}} \qquad \text{for } k = 1, 2, \ldots. \tag{7}$$

A result analogous to (6) for the **Neumann Eigenvalues** of tiling domains, with the sign of the equalities reversed, also holds. The best result to date towards the proof of the Pólya conjecture is the bound [10]

$$\sum_{i=1}^{k} \lambda_i \ge \frac{n}{n+2} \frac{4\pi^2 k^{1+2/n}}{(C_n|\Omega|)^{2/n}} \qquad k = 1, 2, 3, \ldots, \tag{8}$$

proven using the asymptotic behavior of the heat kernel of $\Omega$ and the connection between the heat kernel and the Dirichlet eigenvalues of a domain (see e.g., [6] for a review and related results).

Dirichlet eigenvalues are completely characterized by the geometry of the domain $\Omega$. The inverse problem, i.e., up to what extent the geometry of $\Omega$ can be recovered from the knowledge of $\{\lambda_n\}_{n=1}^{\infty}$ was posed by Kac in [8]. If $n = 2$, for example, and $\partial\Omega$ is smooth (in particular $\partial\Omega$ does not have corners) the distribution function behaves as

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \approx \frac{A}{4\pi t} + \frac{L}{8\sqrt{\pi t}} + \frac{1}{6}(1 - r) + O(t), \tag{9}$$

as $t \to 0$, where $A$ is the area, $L$ the perimeter, and $r$ the number of holes of $\Omega$, so at least these features of the domain can be recovered from knowledge of all the eigenvalues (the first term in (9) is just a consequence of Weyl's asymptotics). However, complete recovery of the geometry is impossible, as was later shown by Gordon, Web and Wolpert, who constructed two isospectral domains in $\mathbf{R}^2$ with different geometries [7].

The inverse of the square root of a Dirichlet eigenvalue is a length that may be compared with other characteristic lengths of the domain $\Omega$. A typical such comparison is the **Rayleigh–Faber–Krahn inequality**. Another inequality along these lines is the following: if $\Omega$ is a simply connected domain in $\mathbf{R}^2$ and $r_\Omega$ the radius of the largest disc contained in $\Omega$, there is a universal constant $a$, such that

$$\lambda_1(\Omega) \geq \frac{a}{r_\Omega^2}. \qquad (10)$$

(the best, not yet optimal, constant to date in (10) is $a = 0.6197$; see [2] for details and historical facts). For other isoperimetric inequalities see e.g., [1, 12, 15]. In the same vein, one can also compare Dirichlet and Neumann eigenvalues (see the entry **Neumann Eigenvalue**).

Because of the connection between Potential Theory and Brownian Motion, it is possible to use probabilistic methods to find properties of Dirichlet Eigenvalues. One such property was found by Brascamp and Lieb [3] for $\lambda_1$: if $\Omega_1$ and $\Omega_2$ are domains in $\mathbf{R}^n$, and we set $\Omega_t = t\Omega_1 + (1-t)\Omega_2$, then $\lambda_1(\Omega_t) \leq t\lambda_1(\Omega_1) + (1-t)\lambda_2(\Omega_2)$ for all $t \in (0,1)$. Another example of the use of probabilistic methods is the proof of (10) by Bañuelos and Carroll [2].

To conclude, note that it is possible to define Dirichlet Eigenvalues for much more general domains in $\mathbf{R}^n$ (see e.g., [16], p. 263), and also for the Laplace–Beltrami operator defined on domains in Riemannian manifolds (see e.g., [5]).

## References


[1] ASHBAUGH, M.S., AND BENGURIA, R.D.: Isoperimetric inequalities for eigenvalue ratios, Symposia Mathematica **35**, Cambridge University Press, Cambridge, 1994, pp. 1–36.

[2] BAÑUELOS R., AND CARROLL, T.: Brownian motion and the fundamental frequency of a drum. Duke Math. J. **75**, 575–602.

[3] BRASCAMP, H., AND LIEB, E.H.: On extensions of the Brunn–Minkowski and Prékopa–Leindler theorem, including inequalities for log–concave functions, and with an application to the diffusion equation. J. Funct. Anal. **22**, 366–389 (1976).

[4] COURANT R., AND HILBERT, D.: Methoden der mathematischen Physik, vol. I, Springer, Berlin, 1931. (English edition: Methods of Mathematical Physics, vol. I., Interscience, NY, 1953).

[5] CHAVEL, I.: Eigenvalues in Riemannian Geometry, Pure and Applied Mathematics **115**, Academic Press, Orlando, 1984.

[6] DAVIES, E.B.: Heat kernels and spectral theory, Cambridge Tracts in Mathematics **92**, Cambridge University Press, Cambridge, 1989.

[7] GORDON, C., WEBB, D., AND WOLPERT, S.: Isospectral plane domains and surfaces via Riemannian orbifolds, Invent. Math. **110** (1992), 1–22.

[8] KAC, M.: Can one hear the shape of a drum?, Amer. Math. Monthly (4) **73** (1966), 1–23.

[9] KUTTLER, J.R., AND SIGILLITO, V.G.: Eigenvalues of the Laplacian in two dimensions, SIAM Review **26** (1984), 163–193.

[10] LI, P., AND YAU, S.T., On the Schrödinger equation and the eigenvalue problem, Commun. Math. Phys. **88** (1983), 309–318.

[11] MELAS, A.D.: On the nodal line of the second eigenfunction of the Laplacian in $\mathbf{R}^2$, J. Differential Geometry **35** (1992), 255–263.

[12] OSSERMAN, R.: Isoperimetric Inequalities and Eigenvalues of the Laplacian. Proceedings of the International Congress of Mathematicians, Acad. Sci. Fennica, Helsinki, 1978, pp. 435–441.

[13] POCKELS, F.: Über die partielle Differentialgleichung $\Delta u + k^2 u = 0$ und deren Auftreten in die mathematischen Physik, Z. Math. Physik **37** (1892), 100–105.

[14] POLYA, G.: On the eigenvalues of vibrating membranes, Proc. London Math. Soc. (3) **11** (1961), 419–433.

[15] POLYA, G., AND SZEGÖ, G.: Isoperimetric Inequalities in Mathematical Physics. Annals of Math. Studies **27**, Princeton University Press, Princeton, 1951.

[16] REED, M., AND SIMON, B.: Methods of Modern Mathematical Physics. IV: Analysis of Operators. Academic Press, New York, 1978.

[17] WEYL, H.: Ramifications, old and new, of the eigenvalue problem. Bull. Amer. Math. Soc. **56** (1950), 115–139.

[18] WEYL, H.: Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen, Math. Ann. **71** (1911), 441–479

*Rafael D. Benguria*
*Facultad de Física*
*P. U. Católica de Chile*