DIRICHLET EIGENVALUE– Consider a bounded domain $\Omega \subset \mathbf{R}^n$ with a piecewise smooth boundary $\partial\Omega$. λ is a Dirichlet eigenvalue of Ω is there exists a function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ (Dirichlet eigenfunction) satisfying the following **Dirichlet boundary value problem**

$$-\Delta u = \lambda u \qquad \text{in } \Omega \tag{1}$$

and

$$u = 0 \qquad \text{in } \partial\Omega, \tag{2}$$

where Δ is the Laplace operator (i.e., $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$). Dirichlet eigenvalues (with $n = 2$) were introduced in the study of the vibrations of the clamped membrane in the XIX century. In fact they are proportional to the square of the eigenfrequencies of the membrane with fixed boundary. See [9] for a review and historical remarks. Provided Ω is bounded and the boundary $\partial\Omega$ is sufficiently regular, the Dirichlet Laplacian has a discrete spectrum of infinitely many positive eigenvalues with no finite accumulation point [13]

$$0 < \lambda_1(\Omega) \le \lambda_2(\Omega) \le \lambda_3(\Omega) \le \dots \tag{3}$$

($\lambda_k \to \infty$ as $k \to \infty$).

The Dirichlet eigenvalues are characterized by the max–min principle [4]

$$\lambda_k = \sup\inf \frac{\int_\Omega (\nabla u)^2\, dx}{\int_\Omega u^2\, dx}, \tag{4}$$

where the inf is taken over all $u \in H_0^1(\Omega)$ orthogonal to $\varphi_1, \varphi_2, \dots, \varphi_{k-1} \in H_0^1(\Omega)$, and the sup is taken over all choices of $\{\varphi_i\}_{i=1}^{k-1}$. For simply connected domains it follows from the max–min principle (4) that $\lambda_1(\Omega)$ is nondegenerate and the corresponding eigenfunction u_1 is positive in the interior of Ω. For higher values of k the nodal lines of the k–th eigenfunction divide Ω into no more than $k-1$ subregions (*nodal domains*) (Courant's nodal line theorem [4]). Along this subject, notice the proof of Melas [11] of the nodal line conjecture for plane domains (if Ω is a bounded, smooth, convex domain, the nodal line of u_2 always meets $\partial\Omega$).

For large values of k, if $\Omega \subset \mathbf{R}^n$, Weyl [17, 18] proved

$$\lambda_k \approx \frac{4\pi^2 k^{2/n}}{(C_n|\Omega|)^{2/n}}, \tag{5}$$

where $|\Omega|$ and $C_n = \pi^{n/2}/\Gamma(n/2+1)$ are, respectively, the volumes of Ω and of the unit ball in $\mathbf{R}^n$.

For any plane–covering domain (i.e., a domain that can be used to tile the plane without gaps, nor overlaps, allowing rotations, translations, and reflections of itself), Pólya [14] proved that

$$\lambda_k \ge \frac{4\pi k}{A} \qquad \text{for } k = 1, 2, \dots, \tag{6}$$

and conjectured the same bound for any bounded domain in $\mathbf{R}^2$ (here A is the area of the domain). Pólya's conjecture in n dimensions is equivalent to saying that the Weyl asymptotics of λ_k (5) is a lower bound for λ_k, i.e.,

$$\lambda_k \ge \frac{4\pi^2 k^{2/n}}{(C_n|\Omega|)^{2/n}} \qquad \text{for } k = 1, 2, \dots. \tag{7}$$

A result analogous to (6) for the **Neumann Eigenvalues** of tiling domains, with the sign of the equalities reversed, also holds. The best result to date towards the proof of the Pólya conjecture is the bound [10]

$$\sum_{i=1}^k \lambda_i \ge \frac{n}{n+2} \frac{4\pi^2 k^{1+2/n}}{(C_n|\Omega|)^{2/n}} \qquad k = 1, 2, 3, \dots, \tag{8}$$

proven using the asymptotic behavior of the heat kernel of Ω and the connection between the heat kernel and the Dirichlet eigenvalues of a domain (see e.g., [6] for a review and related results).

Dirichlet eigenvalues are completely characterized by the geometry of the domain Ω. The inverse problem, i.e., up to what extent the geometry of Ω can be recovered from the knowledge of $\{\lambda_n\}_{n=1}^\infty$ was posed by Kac in [8]. If $n = 2$, for example, and $\partial\Omega$ is smooth (in particular $\partial\Omega$ does not have corners) the distribution function behaves as

$$\sum_{k=1}^\infty e^{-\lambda_k t} \approx \frac{A}{4\pi t} + \frac{L}{8\sqrt{\pi t}} + \frac{1}{6}(1-r) + O(t), \tag{9}$$

as $t \to 0$, where A is the area, L the perimeter, and r the number of holes of Ω, so at least these features of the domain can be recovered from knowledge of all the eigenvalues (the first term in (9) is just a consequence of Weyl's asymptotics). However, complete recovery of the geometry is impossible, as was later shown by Gordon, Web and Wolpert, who constructed two isospectral domains in $\mathbf{R}^2$ with different geometries [7].

The inverse of the square root of a Dirichlet eigenvalue is a length that may be compared with other characteristic lengths of the domain Ω. A typical such comparison is the **Rayleigh–Faber–Krahn inequality**. Another inequality along these lines is the following: if Ω is a simply connected domain in $\mathbf{R}^2$ and r_Ω the radius of the largest disc contained in Ω, there is a universal constant a, such that

$$\lambda_1(\Omega) \geq \frac{a}{r_\Omega^2}. \tag{10}$$

(the best, not yet optimal, constant to date in (10) is $a = 0.6197$; see [2] for details and historical facts). For other isoperimetric inequalities see e.g., [1, 12, 15]. In the same vein, one can also compare Dirichlet and Neumann eigenvalues (see the entry **Neumann Eigenvalue**).

Because of the connection between Potential Theory and Brownian Motion, it is possible to use probabilistic methods to find properties of Dirichlet Eigenvalues. One such property was found by Brascamp and Lieb [3] for λ_1: if Ω_1 and Ω_2 are domains in $\mathbf{R}^n$, and we set $\Omega_t = t\Omega_1 + (1-t)\Omega_2$, then $\lambda_1(\Omega_t) \leq t\lambda_1(\Omega_1) + (1-t)\lambda_2(\Omega_2)$ for all $t \in (0,1)$. Another example of the use of probabilistic methods is the proof of (10) by Bañuelos and Carroll [2].

To conclude, note that it is possible to define Dirichlet Eigenvalues for much more general domains in $\mathbf{R}^n$ (see e.g., [16], p. 263), and also for the Laplace–Beltrami operator defined on domains in Riemannian manifolds (see e.g., [5]).

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