

**NEUMANN EIGENVALUE**– Consider a bounded domain  $\Omega \subset \mathbf{R}^n$  with a piecewise smooth boundary  $\partial\Omega$ .  $\mu$  is a Neumann eigenvalue of  $\Omega$  if there exists a function  $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$  (Neumann eigenfunction) satisfying the following **Neumann boundary value problem**

$$-\Delta u = \mu u \quad \text{in } \Omega \quad (1)$$

and

$$\frac{\partial u}{\partial n} = 0 \quad \text{in } \partial\Omega, \quad (2)$$

where  $\Delta$  is the Laplace operator (i.e.,  $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ ). For more general definitions see [8]. Neumann eigenvalues (with  $n = 2$ ) appear naturally when considering the vibrations of a free membrane. In fact, for  $n = 2$ , the nonzero Neumann eigenvalues are proportional to the square of the eigenfrequencies of the membrane with free boundary. Provided  $\Omega$  is bounded and the boundary  $\partial\Omega$  is sufficiently regular, the Neumann Laplacian has a discrete spectrum of infinitely many nonnegative eigenvalues with no finite accumulation point

$$0 = \mu_1(\Omega) \leq \mu_2(\Omega) \leq \mu_3(\Omega) \leq \dots \quad (3)$$

( $\mu_k \rightarrow \infty$  as  $k \rightarrow \infty$ ). The Neumann eigenvalues are characterized by the max–min principle [3]

$$\mu_k = \sup \inf \frac{\int_{\Omega} (\nabla u)^2 dx}{\int_{\Omega} u^2 dx}, \quad (4)$$

where the inf is taken over all  $u \in H^1(\Omega)$  orthogonal to  $\varphi_1, \varphi_2, \dots, \varphi_{k-1} \in H^1(\Omega)$ , and the sup is taken over all the choices of  $\{\varphi_i\}_{i=1}^{k-1}$ . For simply connected domains the first eigenfunction  $u_1$ , corresponding to the eigenvalue  $\mu_1 = 0$  is constant throughout the domain. All the other eigenvalues are positive. While **Dirichlet Eigenvalues** satisfy stringent constraints (e. g.,  $\lambda_2/\lambda_1$  cannot exceed 2.539... for any bounded domain in  $\mathbf{R}^2$ , [1]), no such constraints exist for Neumann Eigenvalues, other than the fact that they are nonnegative. In fact, given any finite sequence  $\{\mu_1 = 0 < \mu_2 < \mu_3 < \dots < \mu_N\}$  there is an open, bounded, smooth, simply connected domain of  $\mathbf{R}^2$  having this sequence as the first  $N$  Neumann Eigenvalues of the Laplacian on that domain [2]. Though

it is obvious from the variational characterization of both **Dirichlet Eigenvalues** and Neumann Eigenvalues (Eqn. (4)) that  $\mu_k \leq \lambda_k$ , Friedlander [4] proved the stronger result,

$$\mu_{k+1} \leq \lambda_k, \quad k = 1, 2, \dots \quad (5)$$

How far is the first nontrivial Neumann eigenvalue from zero for a convex domain in  $\mathbf{R}^2$  is given through the optimal inequality [6]

$$\mu_1 \geq \frac{\pi^2}{d^2}, \quad (6)$$

where  $d$  is the diameter of the domain. There are many more isoperimetric inequalities for Neumann Eigenvalues (see the article on the **Rayleigh-Faber-Krahn Inequality**).

For large values of  $k$ , Weyl proved [9].

$$\mu_{k+1} \approx \frac{4\pi^2 k^{2/n}}{(C_n |\Omega|)^{2/n}}, \quad (7)$$

where  $|\Omega|$  and  $C_n = \pi^{n/2}/\Gamma(n/2+1)$  are, respectively, the volumes of  $\Omega$  and of the unit ball in  $\mathbf{R}^n$ .

For any plane-covering domain (i.e., a domain that can be used to tile the plane without gaps, nor overlaps, allowing rotations, translations, and reflections of itself), Pólya [7] proved that

$$\mu_{k+1} \leq \frac{4\pi k}{A} \quad \text{for } k = 0, 1, 2, \dots, \quad (8)$$

and conjectured the same bound for any bounded domain in  $\mathbf{R}^2$ . This is equivalent to saying that the Weyl asymptotics of  $\mu_k$  is an upper bound for  $\mu_k$ . The analogous conjecture in dimension  $n$  is

$$\mu_{k+1} \leq \frac{4\pi^2 k^{2/n}}{(C_n |\Omega|)^{2/n}} \quad \text{for } k = 0, 1, 2, \dots \quad (9)$$

The most significant result towards the proof of Pólya's conjecture for Neumann Eigenvalues is the result by Kröger [5]

$$\sum_{i=1}^k \mu_i \leq \frac{n}{n+2} \frac{4\pi^2 k^{2/n}}{(C_n |\Omega|)^{2/n}} \quad \text{for } k = 1, 2, \dots \quad (10)$$

A proof of Pólya's conjecture for both Dirichlet and Neumann eigenvalues would imply Friedlander's result (5).

## References

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