NEUMANN EIGENVALUE– Consider a bounded domain $\Omega \subset \mathbf{R}^n$ with a piecewise smooth boundary $\partial\Omega$. μ is a Neumann eigenvalue of Ω is there exists a function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ (Neumann eigenfunction) satisfying the following **Neumann boundary value problem**

$$-\Delta u = \mu u \qquad \text{in } \Omega \tag{1}$$

and

$$\frac{\partial u}{\partial n} = 0 \qquad \text{in } \partial\Omega, \tag{2}$$

where Δ is the Laplace operator (i.e., $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$). For more general definitions see [8]. Neumann eigenvalues (with $n = 2$) appear naturally when considering the vibrations of a free membrane In fact, for $n = 2$, the nonzero Neumann eigenvalues are proportional to the square of the eigenfrequencies of the membrane with free boundary. Provided Ω is bounded and the boundary $\partial\Omega$ is sufficiently regular, the Neumann Laplacian has a discrete spectrum of infinitely many nonnegative eigenvalues with no finite accumulation point

$$0 = \mu_1(\Omega) \leq \mu_2(\Omega) \leq \mu_3(\Omega) \leq \ldots \tag{3}$$

($\mu_k \to \infty$ as $k \to \infty$). The Neumann eigenvalues are characterized by the max–min principle [3]

$$\mu_k = \sup \inf \frac{\int_\Omega (\nabla u)^2 \, dx}{\int_\Omega u^2 \, dx}, \tag{4}$$

where the inf is taken over all $u \in H^1(\Omega)$ orthogonal to $\varphi_1, \varphi_2, \ldots, \varphi_{k-1} \in H^1(\Omega)$, and the sup is taken over all the choices of $\{\varphi_i\}_{i=1}^{k-1}$. For simply connected domains the first eigenfunction u_1, corresponding to the eigenvalue $\mu_1 = 0$ is constant throughout the domain. All the other eigenvalues are positive. While **Dirichlet Eigenvalues** satisfy stringent constraints (e. g., λ_2/λ_1 cannot exceed $2.539\ldots$ for any bounded domain in $\mathbf{R}^2$, [1]), no such constraints exist for Neumann Eigenvalues, other than the fact that they are nonnegative. In fact, given any finite sequence $\{\mu_1 = 0 < \mu_2 < \mu_3 < \ldots < \mu_N\}$ there is an open, bounded, smooth, simply connected domain of $\mathbf{R}^2$ having this sequence as the first N Neumann Eigenvalues of the Laplacian on that domain [2]. Though it is obvious from the variational characterization of both **Dirichlet Eigenvalues** and Neumann Eigenvalues (Eqn. (4)) that $\mu_k \leq \lambda_k$, Friedlander [4] proved the stronger result,

$$\mu_{k+1} \leq \lambda_k, \qquad k = 1, 2, \ldots \tag{5}$$

How far is the first nontrivial Neumann eigenvalue from zero for a convex domain in $\mathbf{R}^2$ is given through the optimal inequality [6]

$$\mu_1 \geq \frac{\pi^2}{d^2}, \tag{6}$$

where d is the diameter of the domain. There are many more isoperimetric inequalities for Neumann Eigenvalues (see the article on the **Rayleigh-Faber-Krahn Inequality**).

For large values of k, Weyl poved [9].

$$\mu_{k+1} \approx \frac{4\pi^2 k^{2/n}}{(C_n|\Omega|)^{2/n}}, \tag{7}$$

where $|\Omega|$ and $C_n = \pi^{n/2}/\Gamma(n/2+1)$ are, respectively, the volumes of Ω and of the unit ball in $\mathbf{R}^n$.

For any plane–covering domain (i.e., a domain that can be used to tile the plane without gaps, nor overlaps, allowing rotations, translations, and reflections of itself), Pólya [7] proved that

$$\mu_{k+1} \leq \frac{4\pi k}{A} \qquad \text{for } k = 0, 1, 2, \ldots, \tag{8}$$

and conjectured the same bound for any bounded domain in $\mathbf{R}^2$. This is equivalent to saying that the Weyl asymptotics of μ_k is an upper bound for μ_k. The analogous conjecture in dimension n is

$$\mu_{k+1} \leq \frac{4\pi^2 k^{2/n}}{(C_n|\Omega|)^{2/n}} \qquad \text{for } k = 0, 1, 2, \ldots. \tag{9}$$

The most significant result towards the proof of Pólya's conjecture for Neumann Eigenvalues is the result by Kröger [5]

$$\sum_{i=1}^{k} \mu_i \leq \frac{n}{n+2} \frac{4\pi^2 k^{2/n}}{(C_n|\Omega|)^{2/n}} \qquad \text{for } k = 1, 2, \ldots. \tag{10}$$

A proof of Pólya's conjecture for both Dirichlet and Neumann eigenvalues would imply Friedlander's result (5).

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