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Propagation of fronts of a reaction-convection-diffusion equation

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Abstract. We study the speed of propagating fronts of the convection reaction diffusion equation $u_t + \mu u u_x = u_{xx} + f(u)$ for reaction terms $f(u)$ such that in the non convective case fronts joining two equilibrium states exist. A variational principle for the wave speed is constructed from which upper and lower bounds are obtained. We find that, in general, there is a transition value μ_c below which advection has no effect on the speed of the travelling front. Results for the more general case $u_t + \mu\phi(u)_x = u_{xx} + f(u)$ are also given.

1. Introduction

The reaction diffusion equation $u_t = u_{xx} + f(u)$ has been employed as a simple model of phenomena in different areas, population growth, chemical

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reactions, flame propagation and others. For nonlinear reaction terms with two equilibrium points, localized initial conditions evolve into monotonic fronts joining the stable and unstable points. In the classical Fisher case [6], $f(u) = u(1-u)$, a front propagating with speed $c_{\mathrm{kpp}} = 2$ joins the two equilibrium points [11]. The time evolution for general reaction terms was solved by Aronson and Weinberger [1] who showed that localized initial conditions evolve into a front which propagates with speed c_* such that $2\sqrt{f'(0)} \le c_* \le 2\sqrt{sup(f(u)/u)}$. The asymptotic speed of propagation is the minimal speed for which a monotonic front joining the stable to unstable equilibrium point exists. It has been shown that this minimal speed can be derived either from a local variational principle of the minimax type [8], or from an integral variational principle [2,3].

In many processes, in addition to diffusion, motion can also be due to advection or convection. Patlak [17] developed an extended version of the random walk model, accounting for correlation between successive steps and external forces, which led to the advection-diffusion equation

$$u_t + (\mu u)_x = u_{xx}. \tag{1}$$

A similar derivation is made for turbulent diffusion, where the flux generated by an instantaneous random velocity is μu. The superposition of this flux to the molecular diffusion flux leads again to (1). Nonlinear advection terms arise naturally in the motion of chemotactic cells. In a simple one dimensional model, denoting by ρ the density of bacteria, chemotactic to a single chemical element of concentration s we have that

$$\rho_t = [D\rho_x - \rho\chi s_x]_x + f(\rho),$$

where diffusion, chemotaxis and growth have been considered. There is some evidence [18] that, in certain cases, the rate of chemical consumption is due mainly to the ability of the bacteria to consume it. In that case

$$s_t = -k\rho,$$

where diffusion of the chemical has been neglected (arguments to justify this approximation, together with the choice of constant D and χ are given in [18]). If we now look for travelling wave solutions $s = s(x-ct)$, $\rho = \rho(x-ct)$, then $s_t = -cs_x$, therefore $s_x = k\rho/c$, and the problem reduces to a single differential equation for ρ, namely,

$$\rho_t = D\rho_{xx} - \frac{\chi k}{c}(\rho^2)_x + f(\rho). \tag{2}$$

The more elaborate models of Keller and Segel for chemotaxis [10], which include diffusion of the chemical and other effects, have been considered to explain chemotactic collapse ([5,9] and references therein) and other phenomena. Advective terms have also been used to model dispersion due to population pressure [16,7]. Traveling waves for equation (2) without the diffusion term $D\rho_{xx}$ have been considered to model certain population processes [12]. Reaction diffusion equations with non-local density dependent advection have been used to model swarming behavior ([13] and references therein). In addition to these biological processes, equations analogous to (2) appear when modeling the Gunn effect in semiconductors and in other physical phenomena [4]. Equation (2) for a Fisher type reaction term $f(u) = u(1-u)$ has been studied [14,15], the effect of the convective term on the speed of travelling fronts has been established.

In this work we concentrate on equation (2), which suitably scaled we write as

$$u_t + \mu u u_x = u_{xx} + f(u) \tag{3}$$

where the reaction term satisfies

$$f(0) = f(1) = 0, \qquad \text{and} \qquad \mathrm{f} > 0 \quad \text{in} \quad (0,1).$$

We show that the minimal speed c_* for the existence of a monotonic decaying front $u(x-ct)$ joining the stable equilibrium $u=1$ to the unstable equilibrium $u=0$ obeys the variational principle

$$c_* = \sup_g \left[\frac{2\int_0^1 \sqrt{fg(-g')}\,d\,u}{\int_0^1 g(u)d\,u} + \mu \frac{\int_0^1 ug(u)d\,u}{\int_0^1 g(u)d\,u} \right] \tag{4}$$

where $g(u)$ is a positive monotonic decreasing function. From here it will follow that

$$2\sqrt{f'(0)} \le c_* \le 2 \sup_u (1 + f'(u) + \mu u) \tag{5}$$

From the variational expression (4) one may obtain the value of the speed with any desired accuracy, and the inequalities (5) enable us to characterize the functions for which the speed is the linear marginal stability value $2\sqrt{f'(0)}$. For them, convection has no effect on the speed of propagation. A similar result is given for the more general case $u_t + \mu\phi(u)_x = u_{xx} + f(u)$.

Further generalizations to density dependent diffusion follows in a simple way.

2. Speed of the fronts

2.1. Variational principle

Travelling monotonic fronts $u(x-ct)$ of (3) satisfy the ordinary differential equation

$$u_{zz}+(c-\mu)u_z+f(u)=0 \qquad \lim_{z\to-\infty}u=1, \quad \lim_{z\to\infty}u=0, \quad u_z<0, \quad (6)$$

where $z=x-ct$. It is convenient to work in phase space, defining $p(u)=-u_z$, the problem reduces to finding the solutions of

$$p(u)\,\frac{dp}{du}-(c-\mu\,u)\,p(u)+f(u)=0, \qquad (7)$$

with

$$p(0)=p(1)=0, \qquad \text{and } p>0. \qquad (8)$$

First we recall the known constraints on the speed. The first simple bound on the speed is obtained from (7), dividing by p and integrating between 0 and 1, since f and p are positive one obtains $c>\mu/2$. In addition, linearization around the unstable point $u=0$ leads to the additional constraint, $c\geq 2\sqrt{f'(0)}$.

Now we construct the variational principle following a method previously employed for the pure reaction diffusion equation. For completeness we present some of the details here. Let g be any positive function in (0,1) such that $h = -dg/du > 0$. Multiplying equation (4a) by g/p and integrating with respect to u we find that

$$c \int_0^1 g\, du = \int_0^1 \left(h\, p + \frac{f(u)}{p}\, g \right) du + \mu \int_0^1 ug(u)du \tag{9}$$

where the first term is obtained after integration by parts. However since p, h, f, and g are positive, we have that for every fixed u

$$h\, p + \frac{f\, g}{p} \geq 2\, \sqrt{f\, g\, h}$$

so that,

$$c \geq 2\, \frac{\int_0^1 \sqrt{f\, g\, h}\, du}{\int_0^1 g\, du} + \mu \frac{\int_0^1 ugdu}{\int_0^1 g\, du} \tag{10}$$

Equality is attained for $g = \hat{g}$ such that

$$\frac{\hat{g} f}{p} = -p \hat{g}'$$

from where it follows that

$$\hat{g} = p(u) \exp\left[- \int_{u_0}^{u} \left(\frac{c}{p} \right) du' \right].$$

The analysis of the singular points $u = 0$ and $u = 1$ show that the maximizing g exists whenever $c > 2\sqrt{f'(0)}$. The main result is then, the minimal speed c_* for the existence of monotonic fronts is

$$c_* = \sup_g 2\, \frac{\int_0^1 \sqrt{f\, g\, h}\, du}{\int_0^1 g\, du} + \mu \frac{\int_0^1 udu}{\int_0^1 g\, du}.$$

2.2. Upper and Lower Bounds

The variational principle provides lower bounds with suitably chosen trial functions, which can be arbitrarily close to the exact value of the speed. We now reobtain the limit obtained from linear considerations alone, namely $c_* \geq 2\sqrt{f'(0)}$ and construct an upper bound from the variational principle.

To bring out the constraint based on the linear approach to the unstable point $u = 0$, choose as a trial function $g_\alpha(u) = u^{\alpha-1}$ for $0 < \alpha < 1$. Then, in the limit $\alpha \to 0$ we obtain $c_* \geq 2\sqrt{1-\alpha}\sqrt{f'(0)} + \alpha\mu/(1+\alpha) \to 2\sqrt{f'(0)}$ as $\alpha \to 0$, which proves the linear lower bound.

To obtain the upper bound notice that

$$2\sqrt{fh/g} \leq 1 + fh/g,$$

(which is just $2ab \leq a^2 + b^2$ with $b = 1$). Then,

$$c_* = \sup_g 2\,\frac{\int_0^1 \sqrt{f\,g\,h}\,du}{\int_0^1 g\,du} + \mu\frac{\int_0^1 ugdu}{\int_0^1 g\,du} \leq \sup_g \frac{\int_0^1 g(1+\mu u + fh/g)du}{\int_0^1 g\,du}.$$

The last term can be integrated by parts to obtain

$$c_* \leq \sup_g \frac{\int_0^1 g(1+\mu u + f')du}{\int_0^1 g\,du}. \leq \sup_u [1 + \mu u + f']. \tag{11}$$

Summarizing, we have that

$$2\sqrt{f'(0)} \leq c_* \leq \sup_u (1 + f' + \mu u). \tag{12}$$

As mentioned above, improved lower bounds are obtained with different trial functions.

2.3. Application to the Fisher reaction term

Let us now apply the above results to the case studied by Murray, namely $f(u) = u(1-u)$. For this reaction term Murray found the lower bound

$$c_* \geq \begin{cases} \frac{2}{\mu} + \frac{\mu}{2} & \text{if } \mu > 2 \\ 2\sqrt{f'(0)} = 2 & \text{if } \mu \leq 2 \end{cases}$$

We first show that this lower bound may be obtained by a suitable choice of trial function. Take the trial function

$$g(u) = \left(\frac{1-u}{u}\right)^{\lambda} \qquad \text{with} \quad 0 < \lambda < 1.$$

A straightforward integration of equation (10) leads to

$$c \geq 2\sqrt{\lambda} + \frac{\mu}{2}(1-\lambda) \equiv c(\lambda).$$

If $\mu > 2$ the maximum of $c(\lambda)$, $2/\mu + \mu/2$, occurs for $\lambda = 4/\mu^2$ which satisfies $0 < \lambda < 1$. For $\mu < 2$, however, the supremum of $c(\lambda)$ occurs as $\lambda \to 1$. We have then

$$\max c(\lambda) = \frac{2}{\mu} + \frac{\mu}{2} \qquad \text{for} \quad \mu > 2,$$

and

$$\sup c(\lambda) = 2 \qquad \text{for} \quad \mu < 2,$$

and we reobtain Murray's result.

In addition now we have the upper bound equation (11). For $f(u) = u(1-u)$, we obtain

$$\sup_{0 \leq u \leq 1} (1 + f' + \mu u) = \sup_{0 \leq u \leq 1} (2 + (\mu - 2)u) = \begin{cases} \mu \text{ if } \mu > 2 \\ 2 \text{ if } \mu \leq 2 \end{cases}$$

For $\mu < 2$ the upper and lower bounds coincide so $c_* = 2$ without any ambiguity. Thus convection has no effect on the speed of propagation for such low values of μ.

For arbitrary reaction terms better lower bounds are obtained by a judicious choice of trial functions, the upper bound given here gives a tool to establish the maximum increase in speed due to the convective term.

2.4. Results for a general density dependent advection velocity

The above results can be extended directly to more general equation

$$u_t + \mu\phi(u)_x = u_{xx} + f(u).$$

where $\phi(u)$ is analytic in u and $\phi'(0) = 0$. The calculations are straightforward, the precise nature of the convective term does not play any role in the derivation of the variational principle. We find that the minimal speed for the existence of a monotonic front is given by

$$c_* = \sup_g 2\,\frac{\int_0^1 \sqrt{f\,g\,h}\,du}{\int_0^1 g\,du} + \mu\frac{\int_0^1 \phi'(u)du}{\int_0^1 g\,du}.$$

From this expression it follows that

$$2\sqrt{f'(0)} \le c_* \le \sup_u (1 + f' + \mu\phi'(u)).$$

As an example consider the case $\phi(u) = u^3/3$ again with a Fisher reaction term $f(u) = u(1-u)$. We obtain

$$2 \le c_* \le \sup_u (2 - 2u + \mu u^2)$$

If $\mu < 2$ the supremum occurs at $u = 0$, the upper an lower bounds coincide and $c_* = 2$. If $\mu > 2$ the supremum occurs at $u = 1$ and we have that $2 \leq c_* \leq \mu$. Direct integration of the equation in phase space provides the additional lower bound $c_* \geq \mu/3$.

3. Conclusion

We have studied the effect of a convective term on the propagation of reaction-diffusion fronts. The minimal speed of the fronts derives from a variational principle, from which, upper and lower bounds can be obtained. These bounds allow one to establish the transition from a pure diffusive behavior to a convective dominated region. A general feature is that if the convective term is not sufficiently strong, it will have no effect on the minimal speed. Above a certain critical strength it can greatly increase the speed of propagation, an upper bound on its increase has also been constructed. Such behavior is found for arbitrary density dependent local convective terms.

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