BOUNDS FOR NONLINEAR EIGENVALUE PROBLEMS

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Abstract. We develop a new technique to obtain bounds on bifurcation curves of nonlinear boundary value problems defined through nonlinear elliptic partial differential equations.

Recently, we have obtained a variational characterization for the principal solution of a two point boundary value problem in the line [1]. In particular, we proved the following result.

Theorem 1. *Let the pair* (λ, u) *be the principal solution (i.e., with* $u(x) \geq 0$*) of the two point boundary value problem*

$$\frac{d^2u}{dx^2} + \lambda u = N(u) \tag{1}$$

subject to $u'(0) = u(1) = 0$*. Let* $u_m = u(0)$*, the sup-norm of the solution. Here* $N(u)$ *is a general nonlinear term, which is continuous in* $(0, u_m)$*. Then,*

$$\lambda[u_m] = \max_{g \in D} \left(\int_0^{u_m} N(u) g(u)\, du + \frac{1}{2} \left(\int_0^{u_m} g'(u)^{1/3}\, du \right)^3 \right) / \int_0^{u_m} u g(u)\, du, \tag{2}$$

where $D = \{g \mid g \in C^1(0, u_m), g' > 0, g(0) = 0\}$*. Moreover, the maximum is attained at some* $\hat{g} \in D$*, which is unique up to a multiplicative constant.*

This theorem cannot be extended, as such, to higher dimensional boundary value problems since the methods used in the proof depend heavily on the one dimensional character of (1). Nevertheless, at least for one particular three dimensional boundary value problem (namely, the Thomas–Fermi equation) we were able to obtain a, suitably modified, variational characterization of the principal solution [2]. Thus, there are hopes that at least for some boundary value problems defined through partial differential equations one can obtain a variational characterization of the principal solution. The purpose of this article is to illustrate how the methods used in [1, 2] can be extended to find bounds for the principal solution of boundary value problems defined through elliptic partial differential equations. Unfortunately, in general we fall short of obtaining a variational characterization. We will proceed through a well known example, since we believe the methods are well illustrated by it, and it is clear how to extend them to more general situations.

Consider the boundary value problem

$$-\Delta u + u^3 = \lambda u \qquad \text{in } \Omega \subset \mathbb{R}^3, \tag{3}$$

with

$$u = 0 \qquad \text{in } \partial\Omega. \tag{4}$$

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Here Ω is a bounded, smooth, domain in $\mathbb{R}^3$. Then we have,

Theorem 2. *Let the pair (λ, u) be the principal solution (i.e., with $u(x) \geq 0$) of the boundary value problem (3), (4), then*

$$4\pi u(x) \leq \frac{2}{3\sqrt{3}} \lim_{\epsilon \to 0} \int_{\Omega \setminus B_\epsilon(x)} \frac{(\Delta g + \lambda g)^{3/2}}{g^{1/2}} \, dy, \tag{5}$$

where $B_\epsilon(x) \subset \Omega$ is a ball of radius ϵ centered at x. Here the function g satisfies,

$$g \in C^2(\Omega \setminus B_\epsilon(x)) \cap C^0(\overline{\Omega \setminus B_\epsilon(x)}), \tag{6}$$

$$g = 0 \qquad \text{in } \partial\Omega, \tag{7}$$

$$g(y) \approx \frac{1}{|x-y|} \qquad \text{in the neighborhood of } x, \tag{8}$$

(i.e., $g(y)$ behaves like the fundamental solution around x),

$$g(y) > 0 \qquad \text{and} \qquad \Delta g + \lambda g > 0 \qquad \text{in } \Omega \setminus B_\epsilon(x), \tag{9}$$

but is otherwise arbitrary.

Proof. Pick any function g satisfying (6), (7), (8) and (9). If we multiply (3) by g, and integrate over $\Omega \setminus B_\epsilon(x)$ we obtain,

$$-\int_{\Omega \setminus B_\epsilon(x)} g \, \Delta u \, dy + \int_{\Omega \setminus B_\epsilon(x)} g \, u^3 \, dy = \lambda \int_{\Omega \setminus B_\epsilon(x)} g \, u \, dy. \tag{10}$$

Using Green's formula and the boundary conditions (4) and (7) satisfied by u and g repectively, we have

$$\int_{\Omega \setminus B_\epsilon(x)} (g\Delta u - u\Delta g) \, dy = -\int_{\partial B_\epsilon(x)} (g\nabla u - u\nabla g) \cdot \hat{n} \, dS, \tag{11}$$

where $\hat{n}$ is the exterior normal to the surface of the ball $B_\epsilon(x)$ and dS its surface element. Since $u \in C^2(\Omega)$, it follows from (8) that the limit of the right side of (11) as ϵ goes to zero is given by $-4\pi u(x)$. Thus,

$$\lim_{\epsilon \to 0} \int_{\Omega \setminus B_\epsilon(x)} (g\Delta u - u\Delta g) \, dy = -4\pi u(x). \tag{12}$$

Hence, using (10), (11), and (12) we have

$$4\pi u(x) = \lim_{\epsilon \to 0} \int_{\Omega \setminus B_\epsilon(x)} \left((\Delta g + \lambda g) u - g u^3 \right) \, dy. \tag{13}$$

Let us denote $h \equiv \Delta g + \lambda g$, which is positive by assumption (9). For fixed y, consider the integrand of (13) as a function of u. Maximizing the integrand with respect to u we get

$$h \, u - u^3 \, g \leq \frac{2}{3\sqrt{3}} \frac{h^{3/2}}{g^{1/2}} \tag{14}$$

and (9) follows from here. □

Remark 1. *For general facts about bifurcation problems defined through ordinary differential equations see* [5]. *For general bifurcation problems see* [6].

As an application, consider Ω to be the unit ball in $\mathbb{R}^3$. We will use Theorem 2 to find an estimate for the principal branch (λ, u_m) (with $u(x) \geq 0$) of the nonlinear eigenvalue problem (3), (4), in this case. Here u_m denotes the sup–norm of the solution, which occurs at zero (the center of the ball). In fact, for any balanced, smooth domain Ω, the sup–norm of the positive solution of (3), (4), is attained at the center of balance, in this case the origin of the ball.

For our purpose, take $g(x) = \cos(\pi r/2)/r$, with $r = |x|$. Clearly, this g satisfies the hypothesis (6), (7), and (8). An elementary computation shows that

$$\Delta g = -\frac{\pi^2}{4} g, \qquad \text{for } r \neq 0. \tag{15}$$

Thus, the corresponding h will be nonnegative as long as $\lambda \geq \pi^2/4$. It turns out that this is not a restriction, since the nonlinear eigenvalue problem (3), (4) has a nontrivial positive solution if and only if λ is larger than the first Dirichlet eigenvalue of Ω, which for the case of the unit ball is π^2. Setting $x = 0$, and using the function g picked above, we conclude from (5)

$$u_m \equiv u(0) \leq \frac{4\sqrt{3}}{9\pi}(1 - \frac{2}{\pi}) \left(\lambda - \frac{\pi^2}{4}\right)^{3/2}, \tag{16}$$

which gives the following lower bound for the *nonlinear eigenvalue* λ,

$$\lambda \geq \frac{\pi^2}{4} + \left[\frac{3\sqrt{3}}{4}\pi \frac{u_m}{(1 - 2/\pi)}\right]^{2/3} = \frac{\pi^2}{4} + c u_m^{2/3}, \tag{17}$$

with $c \approx 5.015$. A better multiplicative constant can be obtained in (17) using a different trial function g. In fact take

$$g(x) = \frac{1}{r}(1 - r), \tag{18}$$

where $r = |x|$, as before. This function g satisfies (6), (7), and (8). Moreover, $\Delta g = 0$ for $r > 0$, and $g > 0$ for $0 < r < 1$. Hence, from (5) we get, $u_m \equiv u(0) \leq \lambda^{3/2}/(9\sqrt{3})$, i.e.,

$$\lambda \geq 3^{5/3} u_m^{2/3} \approx 6.240 u_m^{2/3}, \tag{19}$$

which is better than (17) for large values of u_m.

The bound embodied in (5) is a local upper bound on the principal solution of the boundary value problem (3), (4). To end with this example, we will consider a function $g(y)$ depending parametrically on x to produce an x dependent bound in (5). What we will use as a trial g will be the fundamental solution of the Laplacian in a ball of radius 1. For the ball of radius R, the fundamental solution can be constructed using the method of images. It is given explicitly by

$$G_x(y) = \frac{1}{|y - x|} - \frac{R}{|x|} \frac{1}{|y - R^2 x/|x|^2|} \tag{20}$$

(see, e.g., [3], pp. 19–20). Clearly this particular function satisfies all the hypothesis. Moreover, $\Delta G_x(y) = 0$ for $y \neq x$. Thus, using $g(y) = G_x(y)$, with $R = 1$ in (5), together with Newton's theorem (i.e., $\int d\Omega_y (1/|y - x|) = 4\pi/\max(|x|, |y|)$, where the integral is over the sphere of radius $|y|$ and $d\Omega_y$ is the invariant measure on the sphere), we get at once

$$u(x) \leq \frac{\lambda^{3/2}}{3\sqrt{3}}(1-|x|^2). \tag{21}$$

The bound (21), with a better constant, can be obtained using comparison theorems. In fact, using the comparison function $1-|x|^2$, and the maximum principle (see e.g., [7, 4]), one can show that

$$u(x) \leq \frac{\lambda^{3/2}}{9\sqrt{3}}(1-|x|^2). \tag{22}$$

Notice that at $x=0$ the bound (22) is precisely the bound (19) obtained above.

If instead of (3), (4), one considers a more general boundary value problem

$$-\Delta u + f(u) = \lambda u \qquad \text{in } \Omega \subset \mathbb{R}^3, \tag{23}$$

with

$$u = 0 \qquad \text{in } \partial\Omega, \tag{24}$$

where Ω, as before, is a bounded, smooth, domain in $\mathbb{R}^3$, and f is a positive, continuous, increasing function, with $f(u)/u \to \infty$ as $u \to \infty$, then theorem 2 holds with (5) replaced by

$$4\pi u(x) \leq \lim_{\epsilon\to 0} \int_{\Omega\setminus B_\epsilon(x)} f_L(\frac{h}{g})\, g\, dy. \tag{25}$$

Here, f_L denotes the Legendre transform of the function f, and $h \equiv \Delta g + \lambda g$ as before.

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