Recent Developments in the Spectral Theory of Orthogonal Polynomials

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Lecture 1: Introduction and Overview

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- Lecture 2: Sum Rules and Large Deviations
- Lecture 3: Szegő-Widom asymptotics for Chebyshev Polynomials
- Lecture 4: Killip-Simon Theorems for Finite Gap Sets

References for Lecture 1

[OPUC] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series **54.1**, American Mathematical Society, Providence, RI, 2005.

[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

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OPs also enter in many application—both specific polynomials and the general theory. Indeed, my own interest came from studying discrete Schrödinger operators on $\ell^2(\mathbb{Z})$

$$(Hu)_n = u_{n+1} + u_{n-1} + V u_n$$

and the realization that when restricted to $\mathbb{Z}_+$, one had a special case of OPRL.

OPRL basics

μ will be a probability measure on $\mathbb{R}$. We'll always suppose that μ has bounded support $[a, b]$ which is not a finite set of points. (We then say that μ is non-trivial.) This implies that $1, x, x^2, \ldots$ are independent since $\int |P(x)|^2 \, d\mu = 0 \Rightarrow \mu$ is supported on the zeroes of P.

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Apply Gram Schmidt to $1, x, \ldots$ and get monic polynomials

$$P_j(x) = x^j + \alpha_{j,1} x^{j-1} + \ldots$$

and orthonormal (ON) polynomials

$$p_j = P_j / \|P_j\|$$

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If $d\mu = d\theta/2\pi$ on $\partial\mathbb{D}$, the span of $\{z^j\}_{j=0}^{\infty}$ is not dense in L^2 (but is only H^2). Perhaps, surprisingly, as one can prove using Szegő's theorem, there are measures $d\mu$ on $\partial\mathbb{D}$ for which they are dense (e.g., μ purely singular).

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More significantly, the argument we'll give for our recursion relation fails if $\mathrm{supp}(d\mu) \not\subset \mathbb{R}$.

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$\|P_N\| = a_N \, a_{N-1} \ldots a_1$ (since $\|P_0\| = 1$).

This, in turn, implies $p_n = P_n / a_1 \ldots a_n$ obeys

$$xp_n = a_{n+1} p_{n+1} + b_{n+1} p_n + a_n p_{n-1}$$

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In the orthonormal basis, $\{p_n\}_{n=0}^{\infty}$, multiplication by x has the matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

called a Jacobi matrix.

Favard's Theorem

Since

$$b_n = \int x p_{n-1}^2(x)\, d\mu, \quad a_n = \int x p_{n-1}(x) p_n(x)\, d\mu$$

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If one uses Gram-Schmidt to orthonormalize $\{J^\ell (1,0,\dots)^t\}_{\ell=0}^\infty$, one finds μ has Jacobi matrix exactly given by J.

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Favard's Theorem. *There is a one–one correspondence between bounded Jacobi parameters*

$$\{a_n, b_n\}_{n=1}^{\infty} \in \big[(0,\infty) \times \mathbb{R}\big]^{\infty}$$

and non-trivial probability measures, μ, of bounded support via:

$$\mu \Rightarrow \{a_n, b_n\} \quad \textit{(OP recursion)}$$

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There are also results for μ's with unbounded support so long as $\int x^n \, d\mu < \infty$. In this case, $\{a_n, b_n\} \Rightarrow \mu$ may not be unique because J may not be essentially self-adjoint on vectors of finite support.

OPUC basics

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if $j \geq 1$.

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In the OPUC case, we want to characterize deg $P = n$, $P \perp \{z, z^2, \dots, z^n\}$.

OPUC basics

Define * on degree n polynomials to themselves by

$$Q^*(z) = z^n \, \overline{Q\left(\frac{1}{\bar{z}}\right)}$$

(bad but standard notation!) or

$$Q(z) = \sum_{j=0}^{n} c_j z^j \Rightarrow Q^*(z) = \sum_{j=0}^{n} \bar{c}_{n-j} \, z^j$$

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Then, * is anti-unitary and so for $\deg Q = n$

$$Q \perp \{1, \ldots, z^{n-1}\} \Leftrightarrow Q = c\,\Phi_n$$

is equivalent to

$$Q \perp \{z, \ldots, z^n\} \Leftrightarrow Q = c\,\Phi_n^*$$

Szegő recursion and Verblunsky coefficients

Thus, we see, there are parameters $\{\alpha_n\}_{n=0}^{\infty}$ (called Verblunsky coefficients) so that

$$\Phi_{n+1}(z) = z\Phi_n - \overline{\alpha}_n \Phi_n^*(z)$$

This is the Szegő Recursion (History: Szegő and Geronimus in 1939; Verblunsky in 1935–36)

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The strange looking $-\bar{\alpha}_n$ rather than say $+\alpha_n$ is to have the α_n be the Schur parameter of the Schur function of μ (Geronimus); also the Verblunsky parameterization then agrees with α_n. These are discussed in [OPUC1].

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which implies $|\alpha_n| < 1$ (i.e., $\alpha_n \in \mathbb{D}$) and

$$\|\Phi_n\| = \rho_{n-1} \cdots \rho_0$$

Verblunsky's Theorem

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For Jacobi, it is similar. Instead of $\rho_{n-1} = 0$, we have $a_n = 0$ and there are again $2n-1$ free parameters $\{a_j\}_{j=1}^{n-1} \cup \{b_j\}_{j=1}^n$.

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$$d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_s \Rightarrow \lim_{n\to\infty} ||\Phi_n||^2 = \exp\left(\int \log(w(\theta))\frac{d\theta}{2\pi}\right)$$

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Less common is information in the other direction. I invented the name "gems of spectral theory" for equivalences.

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This is especially interesting because it implies there exists ℓ^2 α's with essentially arbitrary imbedded singular spectrum!

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$$D(z) = \exp\left(\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\,\log\bigl(w(\theta)\bigr)\,\frac{d\theta}{4\pi}\right)$$

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$$\int |\varphi_n^*(e^{i\theta})\,D(e^{i\theta}) - 1|^2 \frac{d\theta}{2\pi} + \int |\varphi_n^*(e^{i\theta})|^2 d\mu_s = 2\bigl(1 - \textstyle\prod_{j=n}^{\infty} \rho_j\bigr)$$

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which is called *Szegő asymptotics* for φ_n.

Szegő's Asymptotics for $[-2, 2]$

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Szegő Asymptotics translates to the existence of analytic functions, G, on $\mathbb{D}$ and, $\widetilde{G}$, on $\mathbb{C} \setminus [-2, 2]$ so that if the Szegő condition for $[-2, 2]$ holds.

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The function $x \mapsto \frac{x}{2} + \sqrt{\left(\frac{x}{2}\right) - 1}$ that replaces z when one moves Szegő Asymptotics from $\mathbb{D}$ to $[-2, 2]$ can be understood by noting that its $\log$ is the potential theorist's Green's function, The function $x \mapsto \frac{x}{2} + \sqrt{\left(\frac{x}{2}\right) - 1}$ that replaces z when one moves Szegő Asymptotics from $\mathbb{D}$ to $[-2, 2]$ can be understood by noting that its $\log$ is the potential theorist's Green's function, that is the unique function on $\mathbb{C}$ (with $\mathfrak{e} = [-2, 2]$)

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3. $G_{\mathfrak{e}}(z) - \log|z|$ is harmonic at ∞

Moreover, near ∞

$$\exp(G_{\mathfrak{e}}(z)) = \frac{|z|}{C(\mathfrak{e})} + \mathrm{O}(1)$$

where $C(\mathfrak{e})$ is the logarithmic capacity of $\mathfrak{e}$ which is 1 for $\mathfrak{e} = [-2,2]$).

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(iii) (Quasi-Szegő) $\int (x^2 - 4)^{1/2} \log\big(f(x)\big)\,dx < \infty$.

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For Schrödinger operators in 1D (and so on half line), Lieb–Thirring proved (initially for $p > \frac{1}{2}$, $p = \frac{1}{2}$ is Weidl and then Hundertmark–Lieb–Thomas)

$$\sum_{E_j,\pm} |E_j^{\pm}|^p \leq C_p \int_0^\infty |V(x)|^{p+\frac{1}{2}}$$

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Hundertmark–Simon (Killip–Simon for $p = \frac{3}{2}$)

$$\sum \left(|E_j^{\pm}| - 2\right)^p \leq \widetilde{C}_p \sum_{n=0}^{\infty} |a_j - 1|^{p+\frac{1}{2}} + |b_j|^{p+\frac{1}{2}}$$

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Quasi-Sezgő because power is $+\frac{1}{2}$, not $-\frac{1}{2}$ of Szegő condition.

P_2-Sum Rule

Define F on $\mathbb{R} \setminus [-2, 2]$ by $(|\beta| > 1)$

$$F(\beta + \beta^{-1}) = \tfrac{1}{4}\big[\beta^2 - \beta^{-2} - \log(\beta^4)\big];$$

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$$Q(\mu) + \sum F(E_n^{\pm}) = \sum_{n=1}^{\infty} \left[\tfrac{1}{4}\, b_n^2 + \tfrac{1}{2}\, G(a_n)\right]$$

if $\sigma_{\text{ess}}(\mu) = [-2,2]$

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Nazarov-Peherstorfer-Volberg-Yuditskii and then, Denisov Kupin for OPUC, did find ways of generating positive sum rules but they didn't seem to be calculationally tractable.

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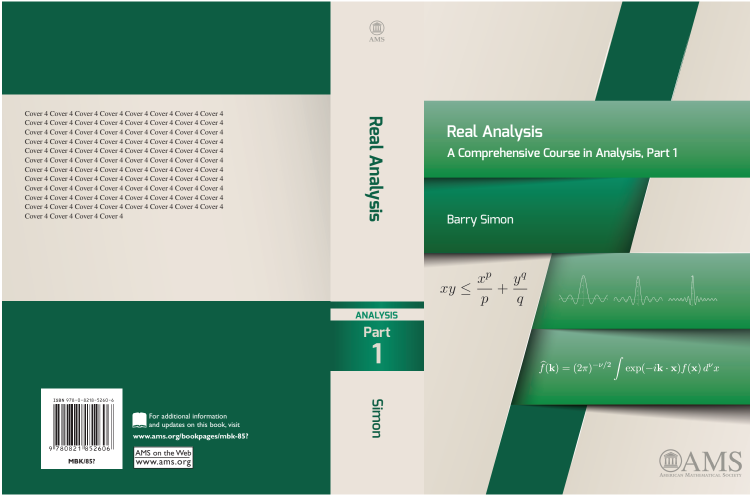

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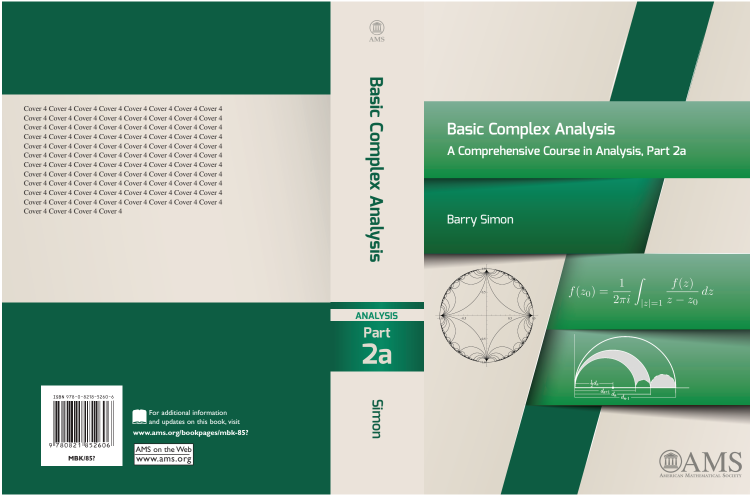

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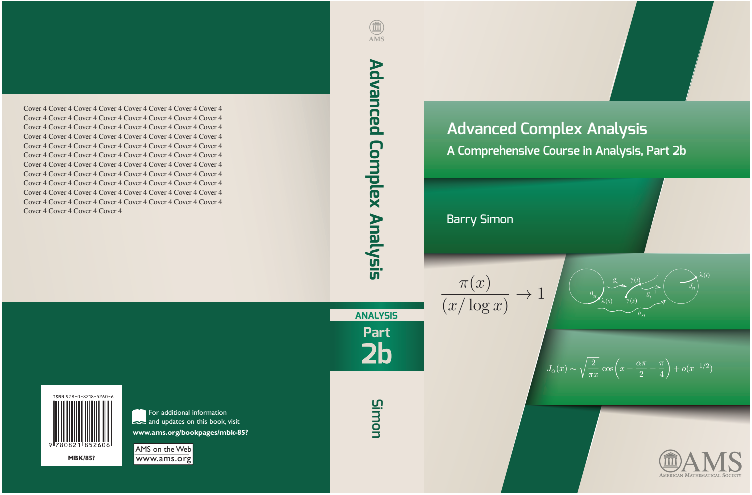

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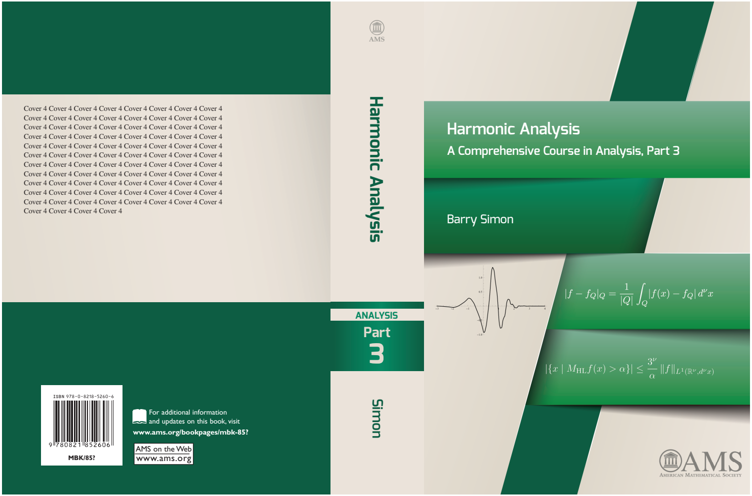

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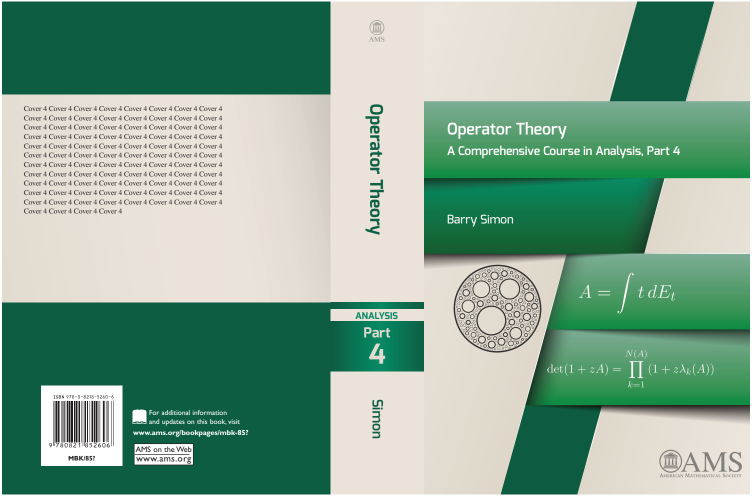

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