Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Recent Developments in the Spectral Theory of Orthogonal Polynomials

Barry Simon
IBM Professor of Mathematics and Theoretical Physics
California Institute of Technology
Pasadena, CA, U.S.A.

Lecture 2: Sum Rules and Large Deviations

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

- Lecture 1: Introduction and Overview
- Lecture 2: Sum Rules and Large Deviations
- Lecture 3: Szegő-Widom asymptotics for Chebyshev Polynomials
- Lecture 4: Killip-Simon Theorems for Finite Gap Sets

References for Lecture 2

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

[GNR] F. Gamboa, J. Nagel, and A. Rouault
Sum rules via large deviations
Preprint: arXiv:1407.1384

[DS] J.D. Deuschel, and D. Stroock,
Large deviations
Academic Press, Inc., Boston, MA, 1989.

[DZ] A. Dembo, and O. Zeitouni,
Large Deviations Techniques and Applications
Springer, 1998.

The Punchline

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

First, I'm going to make propaganda for a year-old preprint of Gamboa, Nagel and Rouault (henceforth GNR) which has not been sufficiently appreciated.

The Punchline

Introduction

Large deviations

GNR Strategy

CUE and Szego's Theorem

GUE and Killip-Simon

Higher Order Szegő Theorems

(1,0) Case

(1,1) Case

First, I'm going to make propaganda for a year-old preprint of Gamboa, Nagel and Rouault (henceforth GNR) which has not been sufficiently appreciated. Then, I'm going to discuss some recent joint work with Jonathan Breuer and Ofer Zeitouni which is so much in progress that it is perhaps premature to discuss it.

The Punchline

First, I'm going to make propaganda for a year-old preprint of Gamboa, Nagel and Rouault (henceforth GNR) which has not been sufficiently appreciated. Then, I'm going to discuss some recent joint work with Jonathan Breuer and Ofer Zeitouni which is so much in progress that it is perhaps premature to discuss it.

To jump to the punchline of Gamboa, Nagel and Rouault.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

The Punchline

First, I'm going to make propaganda for a year-old preprint of Gamboa, Nagel and Rouault (henceforth GNR) which has not been sufficiently appreciated. Then, I'm going to discuss some recent joint work with Jonathan Breuer and Ofer Zeitouni which is so much in progress that it is perhaps premature to discuss it.

To jump to the punchline of Gamboa, Nagel and Rouault.

Szegő's Theorem in Verblunsky's sum rule form is just large deviations for CUE

The Punchline

First, I'm going to make propaganda for a year-old preprint of Gamboa, Nagel and Rouault (henceforth GNR) which has not been sufficiently appreciated. Then, I'm going to discuss some recent joint work with Jonathan Breuer and Ofer Zeitouni which is so much in progress that it is perhaps premature to discuss it.

To jump to the punchline of Gamboa, Nagel and Rouault.

Szegő's Theorem in Verblunsky's sum rule form is just large deviations for CUE and

The Killip-Simon sum rule is just large deviations for GUE!!!!

The Punchline

First, I'm going to make propaganda for a year-old preprint of Gamboa, Nagel and Rouault (henceforth GNR) which has not been sufficiently appreciated. Then, I'm going to discuss some recent joint work with Jonathan Breuer and Ofer Zeitouni which is so much in progress that it is perhaps premature to discuss it.

To jump to the punchline of Gamboa, Nagel and Rouault.

Szegő's Theorem in Verblunsky's sum rule form is just large deviations for CUE and

The Killip-Simon sum rule is just large deviations for GUE!!!!

Ironically, it appears the sum rules BSZ write using large deviations are just those of the Nazarov et al method that were previously regarded as untractable.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

The LD Framework

Large deviations go back to Laplace. The modern theory was initiated by Cramer in the 1930's and made into a powerful machine by Donsker–Varadan and then Varadan alone (work for which he got the Abel prize).

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

The LD Framework

Large deviations go back to Laplace. The modern theory was initiated by Cramer in the 1930's and made into a powerful machine by Donsker–Varadan and then Varadan alone (work for which he got the Abel prize). Two standard texts are DS and DZ.

The LD Framework

Large deviations go back to Laplace. The modern theory was initiated by Cramer in the 1930's and made into a powerful machine by Donsker–Varadan and then Varadan alone (work for which he got the Abel prize). Two standard texts are DS and DZ.

We consider a sequence of probability measures, $\{\mu_n\}_{n=1}^{\infty}$, on a space, X. Naively, one has a Large Deviation Principle (aka LDP) if the μ_n–probability that x is near x_0 is $\mathrm{O}(e^{-nI(x_0)})$.

The LD Framework

Large deviations go back to Laplace. The modern theory was initiated by Cramer in the 1930's and made into a powerful machine by Donsker–Varadan and then Varadan alone (work for which he got the Abel prize). Two standard texts are DS and DZ.

We consider a sequence of probability measures, $\{\mu_n\}_{n=1}^{\infty}$, on a space, X. Naively, one has a Large Deviation Principle (aka LDP) if the μ_n–probability that x is near x_0 is $\mathrm{O}(e^{-nI(x_0)})$. To be mathematically precise, one supposes that X is a Polish space (aka complete metric space), allows multiplicative factors other than n and so speaks of the *speed*, a_n, *rate function*, $I : X \to [0, \infty]$ and requires that:

The LD Framework

1. I is lower semicontinuous

The LD Framework

1. I is lower semicontinuous
2. For all closed sets $F \subset X$
   $$\limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(F) \leq -\inf_{x\in F} I(x)$$

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

The LD Framework

1. I is lower semicontinuous
2. For all closed sets $F \subset X$
   $$\limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(F) \leq -\inf_{x\in F} I(x)$$
3. For all open sets $U \subset X$
   $$\liminf_{n\to\infty} \frac{1}{a_n} \log \mu_n(U) \geq -\inf_{x\in U} I(x)$$

The LD Framework

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

1. I is lower semicontinuous
2. For all closed sets $F \subset X$
   $$\limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(F) \leq -\inf_{x\in F} I(x)$$
3. For all open sets $U \subset X$
   $$\liminf_{n\to\infty} \frac{1}{a_n} \log \mu_n(U) \geq -\inf_{x\in U} I(x)$$

One of the simplest but also most powerful results is that of Cramer–Chernoff:

The LD Framework

1. I is lower semicontinuous
2. For all closed sets $F \subset X$
   $$\limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(F) \le -\inf_{x\in F} I(x)$$
3. For all open sets $U \subset X$
   $$\liminf_{n\to\infty} \frac{1}{a_n} \log \mu_n(U) \ge -\inf_{x\in U} I(x)$$

One of the simplest but also most powerful results is that of Cramer–Chernoff: If $\{X_j\}_{j=1}^{\infty}$ are iidrv with individual expectation $\mathbb{E}$. Let μ_n be the distribution on $\mathbb{R}$ of $\frac{1}{n}\sum_{j=1}^{n} X_j$. Then a LDP holds with speed n and rate function

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

The LD Framework

1. I is lower semicontinuous
2. For all closed sets $F \subset X$
   $\limsup_{n\to\infty} \frac{1}{a_n} \log \mu_n(F) \leq -\inf_{x\in F} I(x)$
3. For all open sets $U \subset X$
   $\liminf_{n\to\infty} \frac{1}{a_n} \log \mu_n(U) \geq -\inf_{x\in U} I(x)$

One of the simplest but also most powerful results is that of Cramer–Chernoff: If $\{X_j\}_{j=1}^{\infty}$ are iidrv with individual expectation $\mathbb{E}$. Let μ_n be the distribution on $\mathbb{R}$ of $\frac{1}{n}\sum_{j=1}^{n} X_j$. Then a LDP holds with speed n and rate function

$$I(x) = \sup_{\theta}\left[\theta x - \log\left(\mathbb{E}(e^X)\right)\right]$$

LDP and Sum Rules

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Gamboa, Nagel and Rouault had the following lovely idea. Let X be the set of probability measures on $\partial\mathbb{D}$ or on $\mathbb{R}$ (with some song and dance to handle measures which don't have compact support

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

LDP and Sum Rules

Gamboa, Nagel and Rouault had the following lovely idea. Let X be the set of probability measures on $\partial\mathbb{D}$ or on $\mathbb{R}$ (with some song and dance to handle measures which don't have compact support — I'll henceforth suppress this phrase) and suppose we have a sequence of probability measures on X with an LDP.

LDP and Sum Rules

Gamboa, Nagel and Rouault had the following lovely idea. Let X be the set of probability measures on $\partial\mathbb{D}$ or on $\mathbb{R}$ (with some song and dance to handle measures which don't have compact support — I'll henceforth suppress this phrase) and suppose we have a sequence of probability measures on X with an LDP. The Verblunsky and Jacobi maps are continuous to sequences of Verblunsky coefficients or Jacobi parameters and so one has an LDP on sequence space.

LDP and Sum Rules

Gamboa, Nagel and Rouault had the following lovely idea. Let X be the set of probability measures on $\partial\mathbb{D}$ or on $\mathbb{R}$ (with some song and dance to handle measures which don't have compact support — I'll henceforth suppress this phrase) and suppose we have a sequence of probability measures on X with an LDP. The Verblunsky and Jacobi maps are continuous to sequences of Verblunsky coefficients or Jacobi parameters and so one has an LDP on sequence space. But the rate functions are clearly the same, so we have the equality of a function of the spectral measures and of a function of the parameters

LDP and Sum Rules

Gamboa, Nagel and Rouault had the following lovely idea. Let X be the set of probability measures on $\partial\mathbb{D}$ or on $\mathbb{R}$ (with some song and dance to handle measures which don't have compact support — I'll henceforth suppress this phrase) and suppose we have a sequence of probability measures on X with an LDP. The Verblunsky and Jacobi maps are continuous to sequences of Verblunsky coefficients or Jacobi parameters and so one has an LDP on sequence space. But the rate functions are clearly the same, so we have the equality of a function of the spectral measures and of a function of the parameters *and as rate functions, these functions are automatically non-negative*!!!!!

LDP and Sum Rules

Gamboa, Nagel and Rouault had the following lovely idea. Let X be the set of probability measures on $\partial\mathbb{D}$ or on $\mathbb{R}$ (with some song and dance to handle measures which don't have compact support — I'll henceforth suppress this phrase) and suppose we have a sequence of probability measures on X with an LDP. The Verblunsky and Jacobi maps are continuous to sequences of Verblunsky coefficients or Jacobi parameters and so one has an LDP on sequence space. But the rate functions are clearly the same, so we have the equality of a function of the spectral measures and of a function of the parameters *and as rate functions, these functions are automatically non-negative*!!!!! We thus have a way to generate positive sum rules and demanding they be finite gives us a gem.

LDP and Sum Rules

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

GNR had the further idea that the measures on the spectral measures should come from random matrix measures with a cyclic vector in the limit as the matrix dimension goes to infinity.

LDP and Sum Rules

GNR had the further idea that the measures on the spectral measures should come from random matrix measures with a cyclic vector in the limit as the matrix dimension goes to infinity.

Of course, the issue becomes to effectively compute the rate function on both sides and alas, we haven't yet found a magic way to do these calculations in a general context.

CUE: Measure Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

For CUE, we first consider Haar measure on $U(n)$, the n x n unitary matrices.

CUE: Measure Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

For CUE, we first consider Haar measure on $U(n)$, the n x n unitary matrices. Any fixed vector is cyclic with probability one, so the corresponding spectral measures have the form $\sum_{j=1}^{n} w_j \delta_{\theta_j}$ where $\lambda_j \equiv e^{i\theta_j}$ are the eigenvalues.

CUE: Measure Side

For CUE, we first consider Haar measure on $U(n)$, the n x n unitary matrices. Any fixed vector is cyclic with probability one, so the corresponding spectral measures have the form $\sum_{j=1}^{n} w_j \delta_{\theta_j}$ where $\lambda_j \equiv e^{i\theta_j}$ are the eigenvalues. Haar measure induces a measure on measures which is supported on the n-point measures.

CUE: Measure Side

For CUE, we first consider Haar measure on $U(n)$, the n x n unitary matrices. Any fixed vector is cyclic with probability one, so the corresponding spectral measures have the form $\sum_{j=1}^{n} w_j \delta_{\theta_j}$ where $\lambda_j \equiv e^{i\theta_j}$ are the eigenvalues. Haar measure induces a measure on measures which is supported on the n-point measures.

As is well–known, the λ's and w's are independent, the w's are uniformly distributed on the simplex $\{\mathbf{w} | \sum_{j=1}^{n} w_j = 1\}$

CUE: Measure Side

For CUE, we first consider Haar measure on $U(n)$, the n x n unitary matrices. Any fixed vector is cyclic with probability one, so the corresponding spectral measures have the form $\sum_{j=1}^{n} w_j \delta_{\theta_j}$ where $\lambda_j \equiv e^{i\theta_j}$ are the eigenvalues. Haar measure induces a measure on measures which is supported on the n-point measures.

As is well–known, the λ's and w's are independent, the w's are uniformly distributed on the simplex $\{\mathbf{w}| \sum_{j=1}^{n} w_j = 1\}$ and by the Weyl integration formula, the θ's have distribution

$$\frac{1}{n!} \prod_{1\leq j<k\leq n} |e^{i\theta_j} - e^{\theta_k}|^2 \prod_{j=1}^{n} \frac{d\theta_j}{2\pi}$$

CUE: Measure Side

The first step in the analysis of the measure side is to analyze what probabilists call the *empirical measure* and physicists *the density of states*, namely the random measure $\frac{1}{n}\sum_{j=1}^{n}\delta_{\theta_j}$.

CUE: Measure Side

The first step in the analysis of the measure side is to analyze what probabilists call the *empirical measure* and physicists *the density of states*, namely the random measure $\frac{1}{n}\sum_{j=1}^{n}\delta_{\theta_j}$. This also defines a family of measures on measures and, in 1997, Ben Arous and Guionnet made the important discovery that this (or rather an analog on the real line with a confining potential) has an LDP with speed n^2

CUE: Measure Side

The first step in the analysis of the measure side is to analyze what probabilists call the *empirical measure* and physicists *the density of states*, namely the random measure $\frac{1}{n}\sum_{j=1}^{n}\delta_{\theta_j}$. This also defines a family of measures on measures and, in 1997, Ben Arous and Guionnet made the important discovery that this (or rather an analog on the real line with a confining potential) has an LDP with speed n^2 (note the square) and rate function the 2D Coulomb energy $-\int \log|x-y|\,d\mu(x)\,d\mu(y)$.

CUE: Measure Side

The first step in the analysis of the measure side is to analyze what probabilists call the *empirical measure* and physicists *the density of states*, namely the random measure $\frac{1}{n}\sum_{j=1}^{n}\delta_{\theta_j}$. This also defines a family of measures on measures and, in 1997, Ben Arous and Guionnet made the important discovery that this (or rather an analog on the real line with a confining potential) has an LDP with speed n^2 (note the square) and rate function the 2D Coulomb energy $-\int \log|x-y|\,d\mu(x)\,d\mu(y)$.

This is easy to understand.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

CUE: Measure Side

The first step in the analysis of the measure side is to analyze what probabilists call the *empirical measure* and physicists *the density of states*, namely the random measure $\frac{1}{n}\sum_{j=1}^{n}\delta_{\theta_j}$. This also defines a family of measures on measures and, in 1997, Ben Arous and Guionnet made the important discovery that this (or rather an analog on the real line with a confining potential) has an LDP with speed n^2 (note the square) and rate function the 2D Coulomb energy $-\int \log|x-y|\,d\mu(x)\,d\mu(y)$.

This is easy to understand. The Weyl distribution can be viewed as a discrete two dimensional Coulomb gas in the canonical ensemble

CUE: Measure Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

The first step in the analysis of the measure side is to analyze what probabilists call the *empirical measure* and physicists *the density of states*, namely the random measure $\frac{1}{n}\sum_{j=1}^{n}\delta_{\theta_j}$. This also defines a family of measures on measures and, in 1997, Ben Arous and Guionnet made the important discovery that this (or rather an analog on the real line with a confining potential) has an LDP with speed n^2 (note the square) and rate function the 2D Coulomb energy $-\int \log|x-y|\,d\mu(x)\,d\mu(y)$.

This is easy to understand. The Weyl distribution can be viewed as a discrete two dimensional Coulomb gas in the canonical ensemble (2D because $|x-y|^{-2}$ is the exponential of $-2\log|x-y|$).

CUE: Measure Side

The first step in the analysis of the measure side is to analyze what probabilists call the *empirical measure* and physicists *the density of states*, namely the random measure $\frac{1}{n}\sum_{j=1}^{n}\delta_{\theta_j}$. This also defines a family of measures on measures and, in 1997, Ben Arous and Guionnet made the important discovery that this (or rather an analog on the real line with a confining potential) has an LDP with speed n^2 (note the square) and rate function the 2D Coulomb energy $-\int \log|x-y|\,d\mu(x)\,d\mu(y)$.

This is easy to understand. The Weyl distribution can be viewed as a discrete two dimensional Coulomb gas in the canonical ensemble (2D because $|x-y|^{-2}$ is the exponential of $-2\log|x-y|$). The $n\to\infty$ limit is a high density limit and due to repulsion, there is a strong tendancy towards equal spacing.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

CUE: Measure Side

To get a significant difference from equal spacing, one has $\mathrm{O}(n^2)$ smaller distances and so the speed is n^2. The optimal spacing will still be locally equal and the discrete Coulomb energy will converge to the continuum.

CUE: Measure Side

To get a significant difference from equal spacing, one has $\mathrm{O}(n^2)$ smaller distances and so the speed is n^2. The optimal spacing will still be locally equal and the discrete Coulomb energy will converge to the continuum.

The fact that n^2 is much larger than n implies that for a measure to have finite rate at speed n, it has to have points close to uniformly distributed

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

CUE: Measure Side

To get a significant difference from equal spacing, one has $\mathrm{O}(n^2)$ smaller distances and so the speed is n^2. The optimal spacing will still be locally equal and the discrete Coulomb energy will converge to the continuum.

The fact that n^2 is much larger than n implies that for a measure to have finite rate at speed n, it has to have points close to uniformly distributed and the large deviations comes from entirely from the lack of a uniform weight.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

CUE: Measure Side

To get a significant difference from equal spacing, one has $\mathrm{O}(n^2)$ smaller distances and so the speed is n^2. The optimal spacing will still be locally equal and the discrete Coulomb energy will converge to the continuum.

The fact that n^2 is much larger than n implies that for a measure to have finite rate at speed n, it has to have points close to uniformly distributed and the large deviations comes from entirely from the lack of a uniform weight.

The weights are close to independent (except for the normalization they are)

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

CUE: Measure Side

To get a significant difference from equal spacing, one has $\mathrm{O}(n^2)$ smaller distances and so the speed is n^2. The optimal spacing will still be locally equal and the discrete Coulomb energy will converge to the continuum.

The fact that n^2 is much larger than n implies that for a measure to have finite rate at speed n, it has to have points close to uniformly distributed and the large deviations comes from entirely from the lack of a uniform weight.

The weights are close to independent (except for the normalization they are) – a slick way to see this is to note if Y_j are positive expoentially distributed iidrv,

CUE: Measure Side

To get a significant difference from equal spacing, one has $\mathrm{O}(n^2)$ smaller distances and so the speed is n^2. The optimal spacing will still be locally equal and the discrete Coulomb energy will converge to the continuum.

The fact that n^2 is much larger than n implies that for a measure to have finite rate at speed n, it has to have points close to uniformly distributed and the large deviations comes from entirely from the lack of a uniform weight.

The weights are close to independent (except for the normalization they are) – a slick way to see this is to note if Y_j are positive expoentially distributed iidrv, then $w_j = Y_j / \sum_{j=1}^{N} Y_j$.

CUE: Measure Side

To get a significant difference from equal spacing, one has $\mathrm{O}(n^2)$ smaller distances and so the speed is n^2. The optimal spacing will still be locally equal and the discrete Coulomb energy will converge to the continuum.

The fact that n^2 is much larger than n implies that for a measure to have finite rate at speed n, it has to have points close to uniformly distributed and the large deviations comes from entirely from the lack of a uniform weight.

The weights are close to independent (except for the normalization they are) – a slick way to see this is to note if Y_j are positive expoentially distributed iidrv, then $w_j = Y_j / \sum_{j=1}^N Y_j$. This allows one (using the Chernoff-Cramer theorem on small blocks) to prove an LDP for the spectral measure with speed n and rate function the Szegő integral $-\int \log(w(\theta)) \frac{d\theta}{2\pi}$.

CUE: Coefficient Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

In 2004, Killip and Nenciu wrote down the distribution of $\{\alpha_j\}_{j=0}^{n-1}$ induced by restricting Haar measure on a fixed vector as we are.

CUE: Coefficient Side

In 2004, Killip and Nenciu wrote down the distribution of $\{\alpha_j\}_{j=0}^{n-1}$ induced by restricting Haar measure on a fixed vector as we are. The α's are independent with α_{n-1} (which lies on $\partial\mathbb{D}$) uniformly distributed on $\partial\mathbb{D}$ and for $j = 0 \ldots n-2$, α_j has density on $\mathbb{D}$

$$\frac{n-j-1}{\pi}(1-|z|^2)^{n-j-2}\, d^2 z$$

CUE: Coefficient Side

In 2004, Killip and Nenciu wrote down the distribution of $\{\alpha_j\}_{j=0}^{n-1}$ induced by restricting Haar measure on a fixed vector as we are. The α's are independent with α_{n-1} (which lies on $\partial\mathbb{D}$) uniformly distributed on $\partial\mathbb{D}$ and for $j = 0 \ldots n-2$, α_j has density on $\mathbb{D}$

$$\frac{n-j-1}{\pi}(1-|z|^2)^{n-j-2}\,d^2z$$

which says that α_j is distributed as the first complex component of a unit vector in $\mathbb{C}^{n-j}$.

CUE: Coefficient Side

In 2004, Killip and Nenciu wrote down the distribution of $\{\alpha_j\}_{j=0}^{n-1}$ induced by restricting Haar measure on a fixed vector as we are. The α's are independent with α_{n-1} (which lies on $\partial\mathbb{D}$) uniformly distributed on $\partial\mathbb{D}$ and for $j = 0 \ldots n-2$, α_j has density on $\mathbb{D}$

$$\frac{n-j-1}{\pi}(1-|z|^2)^{n-j-2}\,d^2z$$

which says that α_j is distributed as the first complex component of a unit vector in $\mathbb{C}^{n-j}$.

α_0 is U_{11}.

CUE: Coefficient Side

In 2004, Killip and Nenciu wrote down the distribution of $\{\alpha_j\}_{j=0}^{n-1}$ induced by restricting Haar measure on a fixed vector as we are. The α's are independent with α_{n-1} (which lies on $\partial\mathbb{D}$) uniformly distributed on $\partial\mathbb{D}$ and for $j = 0 \ldots n-2$, α_j has density on $\mathbb{D}$

$$\frac{n-j-1}{\pi}(1-|z|^2)^{n-j-2}\, d^2 z$$

which says that α_j is distributed as the first complex component of a unit vector in $\mathbb{C}^{n-j}$.

α_0 is U_{11}. Under Haar measure, each row is clearly uniformly distributed on the unit sphere in $\mathbb{C}^n$ so we understand where the distribution of α_0 comes from.

CUE: Coefficient Side

In 2004, Killip and Nenciu wrote down the distribution of $\{\alpha_j\}_{j=0}^{n-1}$ induced by restricting Haar measure on a fixed vector as we are. The α's are independent with α_{n-1} (which lies on $\partial\mathbb{D}$) uniformly distributed on $\partial\mathbb{D}$ and for $j = 0 \ldots n-2$, α_j has density on $\mathbb{D}$

$$\frac{n-j-1}{\pi}(1-|z|^2)^{n-j-2}\, d^2 z$$

which says that α_j is distributed as the first complex component of a unit vector in $\mathbb{C}^{n-j}$.

α_0 is U_{11}. Under Haar measure, each row is clearly uniformly distributed on the unit sphere in $\mathbb{C}^n$ so we understand where the distribution of α_0 comes from. Intuitively, the rest of U which is a unitary map of ${\delta_1}^\perp$ to $U{\delta_1}^\perp$ is just Haar measure on this set of unitaries,

CUE: Coefficient Side

In 2004, Killip and Nenciu wrote down the distribution of $\{\alpha_j\}_{j=0}^{n-1}$ induced by restricting Haar measure on a fixed vector as we are. The α's are independent with α_{n-1} (which lies on $\partial\mathbb{D}$) uniformly distributed on $\partial\mathbb{D}$ and for $j = 0 \dots n-2$, α_j has density on $\mathbb{D}$

$$\frac{n-j-1}{\pi}(1-|z|^2)^{n-j-2}\,d^2z$$

which says that α_j is distributed as the first complex component of a unit vector in $\mathbb{C}^{n-j}$.

α_0 is U_{11}. Under Haar measure, each row is clearly uniformly distributed on the unit sphere in $\mathbb{C}^n$ so we understand where the distribution of α_0 comes from. Intuitively, the rest of U which is a unitary map of ${\delta_1}^\perp$ to $U{\delta_1}^\perp$ is just Haar measure on this set of unitaries, so it is reasonable that the other α's are independent and distributed according to CUE$_{n-1}$.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

CUE: Coefficient Side

The Killip–Nenciu proof is more involved but there should be a proof along these lines.

CUE: Coefficient Side

The Killip–Nenciu proof is more involved but there should be a proof along these lines.

$\prod \rho_j^2$ appears to the nth power so the rate function is $-\sum_{j=1}^{\infty} \log(1 - |\alpha_j|^2)$.

CUE: Coefficient Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

The Killip–Nenciu proof is more involved but there should be a proof along these lines.

$\prod \rho_j^2$ appears to the nth power so the rate function is $-\sum_{j=1}^{\infty} \log(1 - |\alpha_j|^2)$. In this calculation, one makes use of the theory of LDP projective limits to handle the technicalities of going from finite to infinite support.

CUE: Coefficient Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

The Killip–Nenciu proof is more involved but there should be a proof along these lines.

$\prod \rho_j^2$ appears to the nth power so the rate function is $-\sum_{j=1}^{\infty} \log(1 - |\alpha_j|^2)$. In this calculation, one makes use of the theory of LDP projective limits to handle the technicalities of going from finite to infinite support. So, voilá, a new proof of Szegő's Theorem!!!!!

CUE: Coefficient Side

The Killip–Nenciu proof is more involved but there should be a proof along these lines.

$\prod \rho_j^2$ appears to the nth power so the rate function is $-\sum_{j=1}^{\infty} \log(1-|\alpha_j|^2)$. In this calculation, one makes use of the theory of LDP projective limits to handle the technicalities of going from finite to infinite support. So, voilá, a new proof of Szegő's Theorem!!!!!

My OPUC book has something like four other proofs of Szegő's Theorem, but until what I'll discuss next, there was really only one proof of the Killip-Simon theorem.

GUE: Coefficient Side

The argument for GUE, normalized so the limiting density is the semicircle law on $[-2, 2]$, is similar.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

GUE: Coefficient Side

The argument for GUE, normalized so the limiting density is the semicircle law on $[-2, 2]$, is similar. Instead of results of Killip-Nenciu for the distribution of α's, one has earlier results of Dumitriu and Edelman (2002) for the Jacobi parameters.

GUE: Coefficient Side

The argument for GUE, normalized so the limiting density is the semicircle law on $[-2, 2]$, is similar. Instead of results of Killip-Nenciu for the distribution of α's, one has earlier results of Dumitriu and Edelman (2002) for the Jacobi parameters. The calculation is made easier by the independence of the Jacobi parameters (which leads to sums of terms that depend only a single a or b).

GUE: Coefficient Side

The argument for GUE, normalized so the limiting density is the semicircle law on $[-2, 2]$, is similar. Instead of results of Killip-Nenciu for the distribution of α's, one has earlier results of Dumitriu and Edelman (2002) for the Jacobi parameters. The calculation is made easier by the independence of the Jacobi parameters (which leads to sums of terms that depend only a single a or b). The b's are Gaussian which leads to a simple b^2 in the sum rule but the a's have chi–squared distributions which leads to $G(a)$ terms in the sum rule.

GUE: Measure Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

One thing that this proof illuminates is why the Q term involves the quasi-Szegő $(4 - x^2)^{1/2}$ rather than the Szegő $(4 - x^2)^{-1/2}$.

GUE: Measure Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

One thing that this proof illuminates is why the Q term involves the quasi-Szegő $(4 - x^2)^{1/2}$ rather than the Szegő $(4 - x^2)^{-1/2}$. The Szegő form is related to the equilibrium measure for $[-2, 2]$ which is the density of states for Jacobi matrices whose parameters go to the free ones.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

GUE: Measure Side

One thing that this proof illuminates is why the Q term involves the quasi-Szegő $(4-x^2)^{1/2}$ rather than the Szegő $(4-x^2)^{-1/2}$. The Szegő form is related to the equilibrium measure for $[-2,2]$ which is the density of states for Jacobi matrices whose parameters go to the free ones. Quasi-Szegő is the density of states for GUE – the celebrated Wigner semi–circle law.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

GUE: Measure Side

One thing that this proof illuminates is why the Q term involves the quasi-Szegő $(4-x^2)^{1/2}$ rather than the Szegő $(4-x^2)^{-1/2}$. The Szegő form is related to the equilibrium measure for $[-2,2]$ which is the density of states for Jacobi matrices whose parameters go to the free ones. Quasi-Szegő is the density of states for GUE – the celebrated Wigner semi–circle law. For GUE, it is the appropriate limiting emprical measure which enters in the $\log(w(\theta))$ integral.

GUE: Measure Side

One thing that this proof illuminates is why the Q term involves the quasi-Szegő $(4 - x^2)^{1/2}$ rather than the Szegő $(4 - x^2)^{-1/2}$. The Szegő form is related to the equilibrium measure for $[-2, 2]$ which is the density of states for Jacobi matrices whose parameters go to the free ones. Quasi-Szegő is the density of states for GUE – the celebrated Wigner semi–circle law. For GUE, it is the appropriate limiting emprical measure which enters in the $\log(w(\theta))$ integral.

One needs to make some additional arguments going back to Ben Arous-Dembo-Guionnet (2001) to deal with eigenvalues outside the essential support. The positions of these eigenvalues matters and we get $F(E)$ terms.

GUE: Measure Side

One thing that this proof illuminates is why the Q term involves the quasi-Szegő $(4 - x^2)^{1/2}$ rather than the Szegő $(4 - x^2)^{-1/2}$. The Szegő form is related to the equilibrium measure for $[-2, 2]$ which is the density of states for Jacobi matrices whose parameters go to the free ones. Quasi-Szegő is the density of states for GUE – the celebrated Wigner semi–circle law. For GUE, it is the appropriate limiting emprical measure which enters in the $\log(w(\theta))$ integral.

One needs to make some additional arguments going back to Ben Arous-Dembo-Guionnet (2001) to deal with eigenvalues outside the essential support. The positions of these eigenvalues matters and we get $F(E)$ terms.

What results is a new proof of the Killip-Simon sum rule.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Higher Order Szegő Theorems

Killip-Simon was followed up by a variety of authors looking for other applications of sum rules. Among the authors who looked at this are Denisov, Golinskii, Kupin, Laptev, Lukic, Naboko, Nazarov, Novitskii, Peherstorfer, Safronov, Simon, Vainberg, Volberg, Yuditskii and Zlatos.

Higher Order Szegő Theorems

Killip-Simon was followed up by a variety of authors looking for other applications of sum rules. Among the authors who looked at this are Denisov, Golinskii, Kupin, Laptev, Lukic, Naboko, Nazarov, Novitskii, Peherstorfer, Safronov, Simon, Vainberg, Volberg, Yuditskii and Zlatos. Many of the results produce what I'd call flawed gems – instead of spectral data $\Longleftrightarrow$ coefficient data, they prove that only under some additional apriori condition on the coefficients.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Higher Order Szegő Theorems

Killip-Simon was followed up by a variety of authors looking for other applications of sum rules. Among the authors who looked at this are Denisov, Golinskii, Kupin, Laptev, Lukic, Naboko, Nazarov, Novitskii, Peherstorfer, Safronov, Simon, Vainberg, Volberg, Yuditskii and Zlatos. Many of the results produce what I'd call flawed gems – instead of spectral data $\Longleftrightarrow$ coefficient data, they prove that only under some additional apriori condition on the coefficients. I will focus on a possible set of unflawed gems for OPUC.

Higher Order Szegő Theorems

In writing my OPUC books, I realized that there was an analog of the combination of zeroth and second order that Killip-Simon used which led to the equality of

Higher Order Szegő Theorems

In writing my OPUC books, I realized that there was an analog of the combination of zeroth and second order that Killip-Simon used which led to the equality of

$$\exp\left(\int_0^{2\pi} (1-\cos\theta)\log w(\theta)\frac{d\theta}{2\pi}\right)$$

and

$$\exp\left(\tfrac{1}{2}-\tfrac{1}{2}|\alpha_0+1|^2\right)\prod_{n=0}^{\infty} e^{-\frac{1}{2}|\alpha_{n+1}-\alpha_n|^2}\prod_{n=0}^{\infty}(1-|\alpha_n|^2)e^{|\alpha_n|^2}$$

Higher Order Szegő Theorems

In writing my OPUC books, I realized that there was an analog of the combination of zeroth and second order that Killip-Simon used which led to the equality of

$$\exp\left(\int_0^{2\pi} (1 - \cos\theta) \log w(\theta) \frac{d\theta}{2\pi}\right)$$

and

$$\exp(\tfrac{1}{2} - \tfrac{1}{2}|\alpha_0 + 1|^2) \prod_{n=0}^{\infty} e^{-\frac{1}{2}|\alpha_{n+1} - \alpha_n|^2} \prod_{n=0}^{\infty} (1 - |\alpha_n|^2) e^{|\alpha_n|^2}$$

Notice the $|\alpha_0 + 1|^2$ term.

Higher Order Szegő Theorems

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

In writing my OPUC books, I realized that there was an analog of the combination of zeroth and second order that Killip-Simon used which led to the equality of

$$\exp\left(\int_0^{2\pi}(1-\cos\theta)\log w(\theta)\frac{d\theta}{2\pi}\right)$$

and

$$\exp(\tfrac{1}{2}-\tfrac{1}{2}|\alpha_0+1|^2)\prod_{n=0}^{\infty}e^{-\frac{1}{2}|\alpha_{n+1}-\alpha_n|^2}\prod_{n=0}^{\infty}(1-|\alpha_n|^2)e^{|\alpha_n|^2}$$

Notice the $|\alpha_0+1|^2$ term. It is irrelevant for deducing gems and is a finite boundary term.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Higher Order Szegő Theorems

In writing my OPUC books, I realized that there was an analog of the combination of zeroth and second order that Killip-Simon used which led to the equality of

$$\exp\left(\int_0^{2\pi}(1-\cos\theta)\log w(\theta)\frac{d\theta}{2\pi}\right)$$

and

$$\exp\left(\tfrac{1}{2}-\tfrac{1}{2}|\alpha_0+1|^2\right)\prod_{n=0}^{\infty}e^{-\frac{1}{2}|\alpha_{n+1}-\alpha_n|^2}\prod_{n=0}^{\infty}(1-|\alpha_n|^2)e^{|\alpha_n|^2}$$

Notice the $|\alpha_0+1|^2$ term. It is irrelevant for deducing gems and is a finite boundary term. As we go further, these terms become more involved and we will throw them in an "always finite" basket and ignore them.

Higher Order Szegő Theorems

Because the last exponential cancels the first term in the expansion of $\log(1-|\alpha_n|^2)$, we get the gem

$$\exp\left(\int_0^{2\pi}(1-\cos\theta)\log w(\theta)\frac{d\theta}{2\pi}\right) > -\infty \iff \sum_{n=0}^{\infty}(|\alpha_{n+1}-\alpha_n|^2+|\alpha_n|^4) < \infty$$

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Higher Order Szegő Theorems

Because the last exponential cancels the first term in the expansion of $\log(1 - |\alpha_n|^2)$, we get the gem

$$\exp\left(\int_0^{2\pi} (1 - \cos\theta) \log w(\theta) \frac{d\theta}{2\pi}\right) > -\infty \iff \sum_{n=0}^{\infty} (|\alpha_{n+1} - \alpha_n|^2 + |\alpha_n|^4) < \infty$$

Basically, w is allowed to have a higher order zero at $\theta = 0$ but one loses the ℓ^2 property of the α's which are allowed to decay more slowly.

Simon's Conjecture

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Motivated by this example, I made the following conjecture:

Simon's Conjecture

Motivated by this example, I made the following conjecture: Let $\theta_1, \ldots, \theta_k \in [0, 2\pi)$ be distinct and $m_1, \ldots, m_k$ strictly positive integers, let $q = 1 + \max_{j=1,\ldots,k} m_j$ and let S be the operator $(S\alpha)_k = \alpha_{k+1}$. Then

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Simon's Conjecture

Motivated by this example, I made the following conjecture: Let $\theta_1, \ldots, \theta_k \in [0, 2\pi)$ be distinct and $m_1, \ldots, m_k$ strictly positive integers, let $q = 1 + \max_{j=1,\ldots,k} m_j$ and let S be the operator $(S\alpha)_k = \alpha_{k+1}$. Then

$$\int \prod_{j=1}^{k} [1 - \cos(\theta - \theta_j)]^{m_j} \log w(\theta) \frac{d\theta}{2\pi} > -\infty \iff$$

Simon's Conjecture

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Motivated by this example, I made the following conjecture: Let $\theta_1, \ldots, \theta_k \in [0, 2\pi)$ be distinct and $m_1, \ldots, m_k$ strictly positive integers, let $q = 1 + \max_{j=1,\ldots,k} m_j$ and let S be the operator $(S\alpha)_k = \alpha_{k+1}$. Then

$$\int \prod_{j=1}^{k} [1 - \cos(\theta - \theta_j)]^{m_j} \log w(\theta) \frac{d\theta}{2\pi} > -\infty \iff$$

$$\prod_{j=1}^{k} (S - e^{-i\theta_j})^{m_j} \alpha \in \ell^2 \quad \text{and} \quad \alpha \in \ell^{2q}$$

Simon's Conjecture

Motivated by this example, I made the following conjecture: Let $\theta_1, \dots, \theta_k \in [0, 2\pi)$ be distinct and $m_1, \dots, m_k$ strictly positive integers, let $q = 1 + \max_{j=1,\dots,k} m_j$ and let S be the operator $(S\alpha)_k = \alpha_{k+1}$. Then

$$\int \prod_{j=1}^{k} [1 - \cos(\theta - \theta_j)]^{m_j} \log w(\theta) \frac{d\theta}{2\pi} > -\infty \iff$$

$$\prod_{j=1}^{k} (S - e^{-i\theta_j})^{m_j} \alpha \in \ell^2 \quad \text{and} \quad \alpha \in \ell^{2q}$$

Above, we discussed the case $k = 1, m_1 = 1$. Simon-Zlatos proved this for $k = 1, m_1 = 2$ and $k = 2, m_1 = m_2 = 1$.

Lukic's Conjecture

By using a flawed gem, Lukic was able to find a counterexample to the next case, viz: $k = 2, \theta_1 = 0$, $\theta_2 = \pi, m_2 = 2, m_1 = 1$!!!

Lukic's Conjecture

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

By using a flawed gem, Lukic was able to find a counterexample to the next case, viz: $k = 2, \theta_1 = 0,$ $\theta_2 = \pi, m_2 = 2, m_1 = 1$!!!

But he came up with a modified conjecture:

$$\int_0^{2\pi} \left[\prod_{j=1}^{n} (1 - \cos(\theta - \theta_j))^{m_j} \right] \log(w(\theta)) d\theta > -\infty \iff$$

$$\alpha = \beta^{(1)} + \cdots + \beta^{(n)} \quad (S - e^{-i\theta_j})^{m_j} \beta^{(j)} \in \ell^2 \quad \beta^{(j)} \in \ell^{2m_j+2}$$

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Lukic's Conjecture

By using a flawed gem, Lukic was able to find a counterexample to the next case, viz: $k = 2, \theta_1 = 0$, $\theta_2 = \pi, m_2 = 2, m_1 = 1$!!!

But he came up with a modified conjecture:

$$\int_0^{2\pi} \left[\prod_{j=1}^{n} (1 - \cos(\theta - \theta_j))^{m_j} \right] \log(w(\theta)) d\theta > -\infty \iff$$

$$\alpha = \beta^{(1)} + \cdots + \beta^{(n)} \quad (S - e^{-i\theta_j})^{m_j} \beta^{(j)} \in \ell^2 \quad \beta^{(j)} \in \ell^{2m_j+2}$$

In some sense the α's are non-local moments, aka Fourier coefficients, so the measure is sort of the non-linear Fourier sum of α's,

Lukic's Conjecture

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

By using a flawed gem, Lukic was able to find a counterexample to the next case, viz: $k = 2, \theta_1 = 0$, $\theta_2 = \pi, m_2 = 2, m_1 = 1$!!!

But he came up with a modified conjecture:

$$\int_0^{2\pi} \left[\prod_{j=1}^{n} (1 - \cos(\theta - \theta_j))^{m_j} \right] \log(w(\theta)) d\theta > -\infty \iff$$

$$\alpha = \beta^{(1)} + \cdots + \beta^{(n)} \quad (S - e^{-i\theta_j})^{m_j} \beta^{(j)} \in \ell^2 \quad \beta^{(j)} \in \ell^{2m_j+2}$$

In some sense the α's are non-local moments, aka Fourier coefficients, so the measure is sort of the non-linear Fourier sum of α's, so the β's are the localized pieces.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Lukic's Conjecture

While the decomposition is conceptually useful, it is not clear how to realize it as part of a sum rule,

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Lukic's Conjecture

While the decomposition is conceptually useful, it is not clear how to realize it as part of a sum rule, so motivated by a remark of Lukic for the case $k = 2, \theta_1 = 0, \theta_2 = \pi$, $m_2 = 2, m_1 = 1$, Breuer, Zeitouni and I noted that Lukic's condition is equivalent to, for $k = 1, \ldots, n$

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Lukic's Conjecture

While the decomposition is conceptually useful, it is not clear how to realize it as part of a sum rule, so motivated by a remark of Lukic for the case $k = 2, \theta_1 = 0, \theta_2 = \pi$, $m_2 = 2, m_1 = 1$, Breuer, Zeitouni and I noted that Lukic's condition is equivalent to, for $k = 1, \dots, n$

$$\prod_{j=1}^{n} (S - e^{-i\theta_j})^{m_j} \alpha \in \ell^2 \qquad \prod_{j \neq k} (S - e^{-i\theta_j})^{m_j} \alpha \in \ell^{2m_k+2}$$

Lukic's Conjecture

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

While the decomposition is conceptually useful, it is not clear how to realize it as part of a sum rule, so motivated by a remark of Lukic for the case $k = 2, \theta_1 = 0, \theta_2 = \pi$, $m_2 = 2, m_1 = 1$, Breuer, Zeitouni and I noted that Lukic's condition is equivalent to, for $k = 1, \dots, n$

$$\prod_{j=1}^{n} (S - e^{-i\theta_j})^{m_j} \alpha \in \ell^2 \qquad \prod_{j \neq k} (S - e^{-i\theta_j})^{m_j} \alpha \in \ell^{2m_k+2}$$

The project that BSZ has in process is to use the insights of GNR to approach Lukic's conjecture. We are focusing, for now, on the cases where $k \leq 3, \theta_1 = 0, \theta_2 = \pi$.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Lukic's Conjecture

While the decomposition is conceptually useful, it is not clear how to realize it as part of a sum rule, so motivated by a remark of Lukic for the case $k = 2, \theta_1 = 0, \theta_2 = \pi$, $m_2 = 2, m_1 = 1$, Breuer, Zeitouni and I noted that Lukic's condition is equivalent to, for $k = 1, \dots, n$

$$\prod_{j=1}^{n} (S - e^{-i\theta_j})^{m_j} \alpha \in \ell^2 \qquad \prod_{j \neq k} (S - e^{-i\theta_j})^{m_j} \alpha \in \ell^{2m_k+2}$$

The project that BSZ has in process is to use the insights of GNR to approach Lukic's conjecture. We are focusing, for now, on the cases where $k \leq 3, \theta_1 = 0, \theta_2 = \pi$. We have recovered the known cases where (m_1, m_2) is $(1, 0)$ or $(1, 1)$ or $(2, 0)$.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Lukic's Conjecture

While the decomposition is conceptually useful, it is not clear how to realize it as part of a sum rule, so motivated by a remark of Lukic for the case $k = 2, \theta_1 = 0, \theta_2 = \pi$, $m_2 = 2, m_1 = 1$, Breuer, Zeitouni and I noted that Lukic's condition is equivalent to, for $k = 1, \dots, n$

$$\prod_{j=1}^{n} (S - e^{-i\theta_j})^{m_j} \alpha \in \ell^2 \qquad \prod_{j \neq k} (S - e^{-i\theta_j})^{m_j} \alpha \in \ell^{2m_k+2}$$

The project that BSZ has in process is to use the insights of GNR to approach Lukic's conjecture. We are focusing, for now, on the cases where $k \leq 3, \theta_1 = 0, \theta_2 = \pi$. We have recovered the known cases where (m_1, m_2) is $(1, 0)$ or $(1, 1)$ or $(2, 0)$. Our next goal will be to get $(2, 1)$ where Lukic's conjecture is different from my (incorrect) conjecture.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Lukic's Conjecture

While the decomposition is conceptually useful, it is not clear how to realize it as part of a sum rule, so motivated by a remark of Lukic for the case $k=2, \theta_1=0, \theta_2=\pi$, $m_2=2, m_1=1$, Breuer, Zeitouni and I noted that Lukic's condition is equivalent to, for $k=1,\dots,n$

$$\prod_{j=1}^{n}(S-e^{-i\theta_j})^{m_j}\alpha \in \ell^2 \qquad \prod_{j\neq k}(S-e^{-i\theta_j})^{m_j}\alpha \in \ell^{2m_k+2}$$

The project that BSZ has in process is to use the insights of GNR to approach Lukic's conjecture. We are focusing, for now, on the cases where $k\le 3, \theta_1=0, \theta_2=\pi$. We have recovered the known cases where (m_1,m_2) is $(1,0)$ or $(1,1)$ or $(2,0)$. Our next goal will be to get $(2,1)$ where Lukic's conjecture is different from my (incorrect) conjecture. To the extent that time allows, I'll say something about the details of our calculations.

Measure Side

One thing that made the calculations for CUE and GUE easy was independence of almost everything in sight (eigenvalues, α's, and almost the weights).

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Measure Side

One thing that made the calculations for CUE and GUE easy was independence of almost everything in sight (eigenvalues, α's, and almost the weights). The occurrence of difference operators says we'll lose independence but, by looking at multiplicative perturbations of CUE, that will be tractable.

Measure Side

One thing that made the calculations for CUE and GUE easy was independence of almost everything in sight (eigenvalues, α's, and almost the weights). The occurrence of difference operators says we'll lose independence but, by looking at multiplicative perturbations of CUE, that will be tractable. So we'll look at measures on random unitary matrices of the form

$$d\nu_n = Z_n^{-1} \exp(-n \sum_{j=1}^{n} V(\lambda_j)) d\nu_n^{(0)}$$

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Measure Side

One thing that made the calculations for CUE and GUE easy was independence of almost everything in sight (eigenvalues, α's, and almost the weights). The occurrence of difference operators says we'll lose independence but, by looking at multiplicative perturbations of CUE, that will be tractable. So we'll look at measures on random unitary matrices of the form

$$d\nu_n = Z_n^{-1} \exp(-n \sum_{j=1}^{n} V(\lambda_j)) d\nu_n^{(0)}$$

where $d\nu_n^{(0)}$ is CUE, λ_j are the eigenvalues and Z_n is a normalizing factor.

Introduction

Large deviations

GNR Strategy

CUE and Szego's Theorem

GUE and Killip-Simon

Higher Order Szegő Theorems

(1,0) Case

(1,1) Case

Measure Side

We are interested in getting a limiting empirical measure of the form $d\eta(\theta)$.

Measure Side

We are interested in getting a limiting empirical measure of the form $d\eta(\theta)$. Prior work says that we get this if V is picked to be (confusing $\lambda = e^{i\theta}$ and θ):

$$V(\theta) = 2 \int \log |e^{i\theta} - e^{i\psi}| d\eta(\psi)$$

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Measure Side

We are interested in getting a limiting empirical measure of the form $d\eta(\theta)$. Prior work says that we get this if V is picked to be (confusing $\lambda = e^{i\theta}$ and θ):

$$V(\theta) = 2\int \log|e^{i\theta} - e^{i\psi}|d\eta(\psi)$$

In that case, the rate function for the spectral measure is just a relative entropy,

Measure Side

We are interested in getting a limiting empirical measure of the form $d\eta(\theta)$. Prior work says that we get this if V is picked to be (confusing $\lambda = e^{i\theta}$ and θ):

$$V(\theta) = 2 \int \log |e^{i\theta} - e^{i\psi}| d\eta(\psi)$$

In that case, the rate function for the spectral measure is just a relative entropy, so if $d\eta$ is absolutely continuous, up to an additive constant, the rate function is $-\int \log w(\theta) d\eta(\theta)$.

Measure Side

We are interested in getting a limiting empirical measure of the form $d\eta(\theta)$. Prior work says that we get this if V is picked to be (confusing $\lambda = e^{i\theta}$ and θ):

$$V(\theta) = 2\int \log|e^{i\theta} - e^{i\psi}|d\eta(\psi)$$

In that case, the rate function for the spectral measure is just a relative entropy, so if $d\eta$ is absolutely continuous, up to an additive constant, the rate function is $-\int \log w(\theta) d\eta(\theta)$. Thus, to explore cases of Lukic's conjecture, we need to compute

$$V(\theta) = 2\frac{\int \log|e^{i\theta} - e^{i\psi}| \left[\prod_{j=1}^{n}(1-\cos(\psi-\theta_j))^{m_j}\right] d\psi}{\int \left[\prod_{j=1}^{n}(1-\cos(\psi-\theta_j))^{m_j}\right] d\psi}$$

Measure Side

It is useful to know that

$$\begin{aligned}\int_0^{2\pi} \log[1-\cos(\psi)]\cos(n\psi)\frac{d\psi}{2\pi} &= \int_0^{2\pi} \log[1-\cos(\psi)]e^{in\psi}\frac{d\psi}{2\pi} \\ &= \frac{1}{n}\end{aligned}$$

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Measure Side

It is useful to know that

$$\int_0^{2\pi} \log[1-\cos(\psi)]\cos(n\psi)\frac{d\psi}{2\pi} = \int_0^{2\pi} \log[1-\cos(\psi)]e^{in\psi}\frac{d\psi}{2\pi} = \frac{1}{n}$$

and that

$$|1-e^{i\theta}|^2 = 2(1-\cos(\theta))$$

Measure Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

It is useful to know that

$$\int_0^{2\pi} \log[1-\cos(\psi)]\cos(n\psi)\frac{d\psi}{2\pi} = \int_0^{2\pi} \log[1-\cos(\psi)]e^{in\psi}\frac{d\psi}{2\pi} = \frac{1}{n}$$

and that

$$|1-e^{i\theta}|^2 = 2(1-\cos(\theta))$$

This is used inside a $\log$ so the 2 gives an irrelevant additive constant but the square gives a multiplicative factor of 2 which is important.

Measure Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

It is useful to know that

$$\int_0^{2\pi} \log[1-\cos(\psi)]\cos(n\psi)\frac{d\psi}{2\pi} = \int_0^{2\pi} \log[1-\cos(\psi)]e^{in\psi}\frac{d\psi}{2\pi}$$
$$= \frac{1}{n}$$

and that

$$|1-e^{i\theta}|^2 = 2(1-\cos(\theta))$$

This is used inside a $\log$ so the 2 gives an irrelevant additive constant but the square gives a multiplicative factor of 2 which is important. These formulae show that for the $(1,0)$ case, $V(\theta) = \cos(\theta)$.

Coefficient Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

$\cos(\theta) = \mathrm{Re}(e^{i\theta})$, so $\sum V(\lambda_j) = \mathrm{Re}(\mathrm{Tr}(U))$.

Coefficient Side

$\cos(\theta) = \mathrm{Re}(e^{i\theta})$, so $\sum V(\lambda_j) = \mathrm{Re}(\mathrm{Tr}(U))$. The trace can be computed in any basis and it is convenient to pick the basis in which U is a so–called CMV matrix which is five diagonal with $(\bar{\alpha}_0, -\bar{\alpha}_1\alpha_0, \dots, -\bar{\alpha}_{n+1}\alpha_n, \dots)$ along the diagonal so

$$\mathrm{Re}(\mathrm{Tr}(U)) = \frac{1}{2}\left[\bar{\alpha}_0 + \alpha_0 - \sum_{n=0}^{\infty}(\bar{\alpha}_{n+1}\alpha_n + \alpha_{n+1}\bar{\alpha}_n)\right]$$

Coefficient Side

Introduction

Large deviations

GNR Strategy

CUE and Szego's Theorem

GUE and Killip-Simon

Higher Order Szegő Theorems

(1,0) Case

(1,1) Case

$\cos(\theta) = \mathrm{Re}(e^{i\theta})$, so $\sum V(\lambda_j) = \mathrm{Re}(\mathrm{Tr}(U))$. The trace can be computed in any basis and it is convenient to pick the basis in which U is a so–called CMV matrix which is five diagonal with $(\bar{\alpha}_0, -\bar{\alpha}_1\alpha_0, \dots, -\bar{\alpha}_{n+1}\alpha_n, \dots)$ along the diagonal so

$$\mathrm{Re}(\mathrm{Tr}(U)) = \frac{1}{2}\left[\bar{\alpha}_0 + \alpha_0 - \sum_{n=0}^{\infty}(\bar{\alpha}_{n+1}\alpha_n + \alpha_{n+1}\bar{\alpha}_n)\right]$$

The CUE rate function has this added to it, so completing the square, one recovers the sum rule that I used to do the $(1,0)$ case.

Coefficient Side

$\cos(\theta) = \mathrm{Re}(e^{i\theta})$, so $\sum V(\lambda_j) = \mathrm{Re}(\mathrm{Tr}(U))$. The trace can be computed in any basis and it is convenient to pick the basis in which U is a so–called CMV matrix which is five diagonal with $(\bar\alpha_0, -\bar\alpha_1\alpha_0, \dots, -\bar\alpha_{n+1}\alpha_n, \dots)$ along the diagonal so

$$\mathrm{Re}(\mathrm{Tr}(U)) = \frac{1}{2}\left[\bar\alpha_0 + \alpha_0 - \sum_{n=0}^{\infty}(\bar\alpha_{n+1}\alpha_n + \alpha_{n+1}\bar\alpha_n)\right]$$

The CUE rate function has this added to it, so completing the square, one recovers the sum rule that I used to do the $(1,0)$ case. What is striking is that I had to be clever to get the exact sum rule while here is is automated.

Measure Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

The potential is, up to an additive constant (which doesn't matter for gems), $\frac{1}{2}\cos(2\theta)$.

Measure Side

The potential is, up to an additive constant (which doesn't matter for gems), $\frac{1}{2}\cos(2\theta)$. In this calculation, one uses that the normalized measure (which for the $(1,0)$ case was $d\eta = (2\pi)^{-1}(1-\cos(\theta))d\theta$) is now $d\eta = \pi^{-1}(1-\cos^2(\theta))d\theta$.

Coefficient Side

$\cos(2\theta)$ looks like an innocuous 2 but on the coefficient side, it corresponds to $\mathrm{Re}\mathrm{Tr}(U^2)$ which is more complicated than $\mathrm{Tr}(U)$ since U is 5 diagonal.

Coefficient Side

$\cos(2\theta)$ looks like an innocuous 2 but on the coefficient side, it corresponds to $\mathrm{Re}\mathrm{Tr}(U^2)$ which is more complicated than $\mathrm{Tr}(U)$ since U is 5 diagonal. It turns out that there are only three terms and two are equal, so up to finite boundary terms,

$$\mathrm{Tr}(U^2) = \sum_{j=0}^{\infty} (\bar{\alpha}_j \alpha_{j+1})^2 - 2 \sum_{j=0}^{\infty} \rho_j^2 \bar{\alpha}_{j+1} \alpha_{j-1}$$

Coefficient Side

$\cos(2\theta)$ looks like an innocuous 2 but on the coefficient side, it corresponds to $\mathrm{Re}\,\mathrm{Tr}(U^2)$ which is more complicated than $\mathrm{Tr}(U)$ since U is 5 diagonal. It turns out that there are only three terms and two are equal, so up to finite boundary terms,

$$\mathrm{Tr}(U^2) = \sum_{j=0}^{\infty} (\bar{\alpha}_j \alpha_{j+1})^2 - 2 \sum_{j=0}^{\infty} \rho_j^2 \bar{\alpha}_{j+1} \alpha_{j-1}$$

One expands ρ and also the sum of logs and combines all second order terms to get that the rate function is (up to finite terms)

$$\frac{1}{2}\left[\sum_{j=0}^{\infty} |\alpha_{j+1} - \alpha_{j-1}|^2 + E_1 + E_2\right] + \sum_{k=2}^{\infty} k^{-1} \sum_{j=0}^{\infty} |\alpha_j|^{2k}$$

Coefficient Side

where

$$E_1 = \sum_{j=0}^{\infty} (|\alpha_j|^2 |\alpha_{j+1}|^2 + \mathrm{Re}[(\bar{\alpha}_j \alpha_{j+1})^2])$$

Coefficient Side

where

$$E_1 = \sum_{j=0}^{\infty} (|\alpha_j|^2 |\alpha_{j+1}|^2 + \mathrm{Re}[(\bar{\alpha}_j \alpha_{j+1})^2])$$

$$E_2 = \sum_{j=1}^{\infty} |\alpha_j|^2 \mathrm{Re}[\bar{\alpha}_j (\alpha_{j-1} - \alpha_{j+1})]$$

Coefficient Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

where

$$E_1 = \sum_{j=0}^{\infty}(|\alpha_j|^2|\alpha_{j+1}|^2 + \mathrm{Re}[(\bar{\alpha}_j\alpha_{j+1})^2])$$

$$E_2 = \sum_{j=1}^{\infty}|\alpha_j|^2\mathrm{Re}[\bar{\alpha}_j(\alpha_{j-1} - \alpha_{j+1})]$$

The goal is to prove that:

$$\sum_{j=1}^{\infty}|\alpha_{j+1} - \alpha_{j-1}|^2 + |\alpha_j|^4 < \infty \iff \text{rate function finite}$$

Coefficient Side

where

$$E_1 = \sum_{j=0}^{\infty} (|\alpha_j|^2 |\alpha_{j+1}|^2 + \mathrm{Re}[(\bar{\alpha}_j \alpha_{j+1})^2])$$
$$E_2 = \sum_{j=1}^{\infty} |\alpha_j|^2 \mathrm{Re}[\bar{\alpha}_j (\alpha_{j-1} - \alpha_{j+1})]$$

The goal is to prove that:

$$\sum_{j=1}^{\infty} |\alpha_{j+1} - \alpha_{j-1}|^2 + |\alpha_j|^4 < \infty \iff \text{rate function finite}$$

Since E_1 and E_2 are fourth order, LHS and Hölder's inequality proves that the LHS $\Rightarrow$ RHS.

Coefficient Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

On the other hand, suppose that the rate function is finite.

Coefficient Side

On the other hand, suppose that the rate function is finite. By using the measure side, we see that the a.c. part of the spectrum is the entire unit circle, so by a Theorem of Rakhmanov, $\alpha_j \to 0$.

Coefficient Side

On the other hand, suppose that the rate function is finite. By using the measure side, we see that the a.c. part of the spectrum is the entire unit circle, so by a Theorem of Rakhmanov, $\alpha_j \to 0$. Since $E_1 \geq 0$ and the $|\alpha_j|^{2k}, k > 2$ are positive, we are finite if we drop them. By the Schwarz inequality,

$$|E_2| \leq \left[\sum_{j=1}^{\infty} |\alpha_{j-1} - \alpha_{j+1}|^2\right]^{1/2} \left[\sum_{j=1}^{\infty} |\alpha_j|^6\right]^{1/2}$$

Coefficient Side

On the other hand, suppose that the rate function is finite. By using the measure side, we see that the a.c. part of the spectrum is the entire unit circle, so by a Theorem of Rakhmanov, $\alpha_j \to 0$. Since $E_1 \geq 0$ and the $|\alpha_j|^{2k}, k > 2$ are positive, we are finite if we drop them. By the Schwarz inequality,

$$|E_2| \leq \left[\sum_{j=1}^{\infty} |\alpha_{j-1} - \alpha_{j+1}|^2\right]^{1/2} \left[\sum_{j=1}^{\infty} |\alpha_j|^6\right]^{1/2}$$

Since $\alpha_j \to 0$, this can be controlled by a small amount of $\sum_{j=1}^{\infty} |\alpha_{j+1} - \alpha_{j-1}|^2 + |\alpha_j|^4$ so we conclude that this latter sum is finite showing that RHS $\Rightarrow$ LHS.

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

Coefficient Side

This bare hands approach gets harder and harder as orders increase but we think we have $(2,0)$ (where $V(\theta) = \frac{4}{3}\cos(\theta) - \frac{1}{6}\cos(2\theta) + \text{constant}$) and hope to do $(2,1)$.

Coefficient Side

Introduction
Large deviations
GNR Strategy
CUE and Szego's Theorem
GUE and Killip-Simon
Higher Order Szegő Theorems
(1,0) Case
(1,1) Case

This bare hands approach gets harder and harder as orders increase but we think we have $(2,0)$ (where $V(\theta) = \frac{4}{3}\cos(\theta) - \frac{1}{6}\cos(2\theta) + \text{constant}$) and hope to do $(2,1)$. It also is plausible there is some clever way of avoiding too many explicit calculations.

And Now a Word from Our Sponsor

Introduction

Large deviations

GNR Strategy

CUE and Szego's Theorem

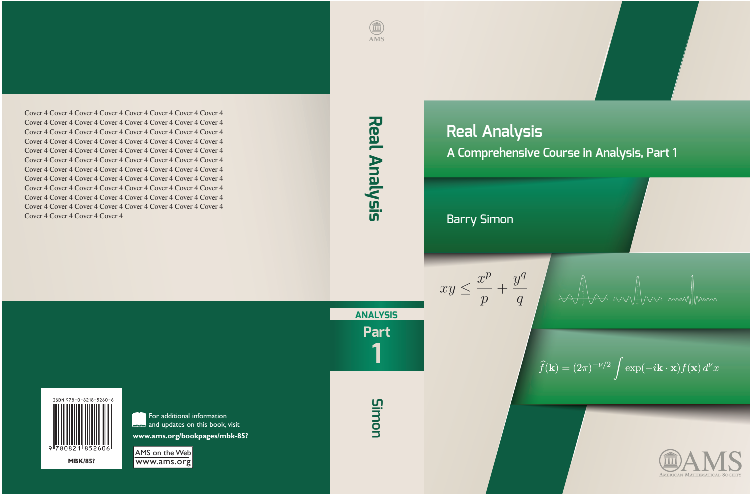
GUE and Killip-Simon

Higher Order Szegő Theorems

(1,0) Case

(1,1) Case

And Now a Word from Our Sponsor



Google: *Simon Comprehensive Course*

And Now a Word from Our Sponsor

Introduction

Large deviations

GNR Strategy

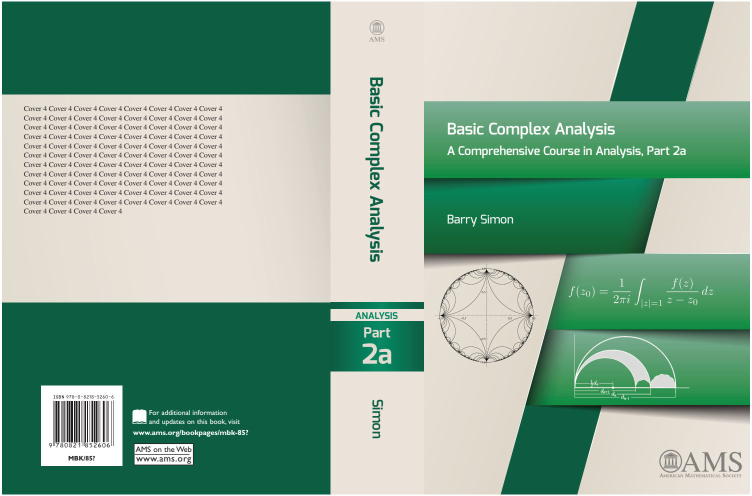
CUE and Szego's Theorem

GUE and Killip-Simon

Higher Order Szegő Theorems

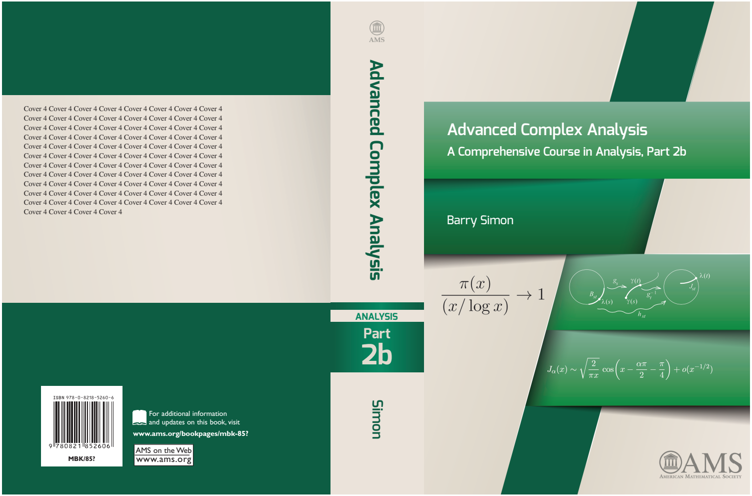
(1,0) Case

(1,1) Case



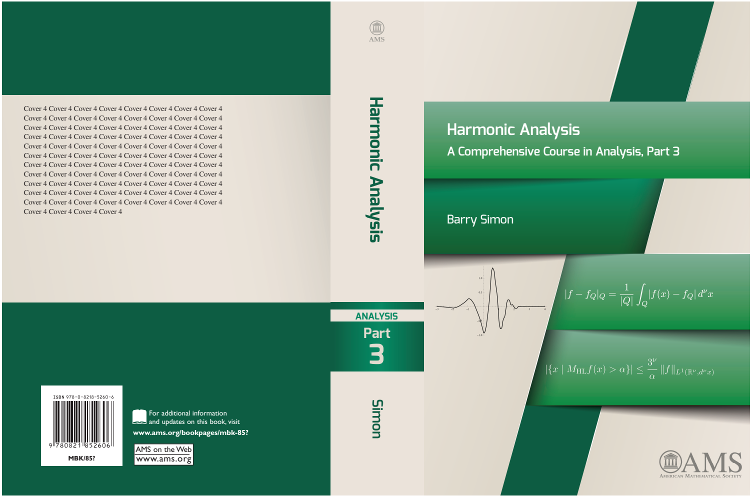
Google: *Simon Comprehensive Course*

And Now a Word from Our Sponsor



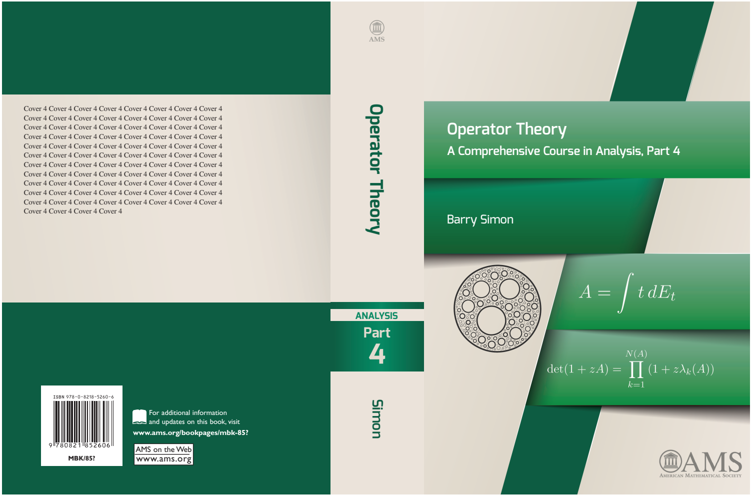
Google: *Simon Comprehensive Course*

And Now a Word from Our Sponsor



Google: *Simon Comprehensive Course*

And Now a Word from Our Sponsor



Google: *Simon Comprehensive Course*