Recent Developments in the Spectral Theory of Orthogonal Polynomials

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Lecture 3: Szegő-Widom asymptotics for Chebyshev Polynomials

- Lecture 1: Introduction and Overview
- Lecture 2: Sum Rules and Large Deviations
- Lecture 3: Szegő-Widom asymptotics for Chebyshev Polynomials
- Lecture 4: Killip-Simon Theorems for Finite Gap Sets

References for Lecture 4

[W1969] H. Widom,
Extremal polynomials associated with a system of curves in the complex plane
Adv. in Math. **3** (1969), 127–232.

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Asymptotics of Chebyshev Polynomials,I. Subsets of $\mathbb{R}$
Preprint: arXiv: 1505.02604

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$$\|f\|_{\mathfrak{e}} = \sup \{|f(z)| \mid z \in \mathfrak{e}\}$$

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$$\|f\|_{\mathfrak{e}} = \sup \{|f(z)| \mid z \in \mathfrak{e}\}$$

The *Chebyshev polynomial of degree* n is the monic polynomial, T_n, with

$$\|T_n\|_{\mathfrak{e}} = \inf \{\|P\|_{\mathfrak{e}} \mid \deg(P) = n \text{ and } P \text{ is monic}\}$$

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I assume that I am not the only one who does not understand the interest in and significance of these strange problems on maxima and minima studied by Chebysbev in memoirs whose titles often begin with, "On functions deviating least from zero ...". Could it be that one must have a Slavic soul to understand the great Russian Scholar?

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We say that P_n, a degree n polynomial, has an *alternating set* in $\mathfrak{e} \subset \mathbb{R}$ if there exists $\{x_j\}_{j=0}^n \subset \mathfrak{e}$ with

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While the basic idea of the following theorem goes back to Chebyshev, the result itself is due to Borel and Markov, independently, around 1905.

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Conversely, let P_n be a degree n monic polynomial with an alternating set and suppose that $\|T_n\|_{\mathfrak{e}} < \|P_n\|_{\mathfrak{e}}$.

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At the alternating points for Q, we must have $T_n = S_n$, so they must be equal polynomials since there are $n+1$ points and their difference has degree at most $n-1$.

Alternation and Zeros

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Here, $\mathrm{cvh}(\mathfrak{e})$ is the convex hull of $\mathfrak{e}$ and that result follows from $x_0, x_n \in \mathfrak{e}$.

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$$\mathfrak{e}_n \equiv T_n^{-1}([-\|T_n\|_{\mathfrak{e}}, \|T_n\|_{\mathfrak{e}}]) \subset \mathrm{cvh}(\mathfrak{e})$$

Coulomb Energies and All That

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$$u_f(\infty) = \int_{\mathfrak{e}} f(x) d\rho_{\mathfrak{e}}(x)$$

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$$G_{\mathfrak{e}}(z) = \log|z| + R(\mathfrak{e}) + \mathrm{O}(1/|z|)$$

Green's Function

The function $\Phi_{\mathfrak{e}}(z) = \int_{\mathfrak{e}} d\rho_{\mathfrak{e}}(x) \, \log|x - z|^{-1}$ is called the *equilibrium potential*. The *Green's function*, $G_{\mathfrak{e}}(z)$, of a compact subset, $\mathfrak{e} \subset \mathbb{C}$, is defined by

$$G_{\mathfrak{e}}(z) = R(\mathfrak{e}) - \Phi_{\mathfrak{e}}(z)$$

It is the unique function harmonic on $\mathbb{C} \setminus \mathfrak{e}$ with q.e. boundary value 0 on $\mathfrak{e}$ and so that $G_{\mathfrak{e}}(z) - \log|z|$ is harmonic at ∞. Moreover, $G_{\mathfrak{e}}(z) \geq 0$ everywhere and near ∞

$$G_{\mathfrak{e}}(z) = \log|z| + R(\mathfrak{e}) + \mathrm{O}(1/|z|)$$

equivalently,

$$\exp(G_{\mathfrak{e}}(z)) = \frac{|z|}{C(\mathfrak{e})} + \mathrm{O}(1)$$

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As we'll see shortly, there are sets where this is optimal but for $\mathfrak{e} \subset \mathbb{R}$, there is a lower bound of $2\,C(\mathfrak{e})^n$, which we'll see somewhat later. This is also optimal.

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$$T_n(\cos(\theta)) = 2^{-n+1}\cos(n\theta); \quad \|T_n\|_{\mathfrak{e}} = 2^{-n+1} = 2\, C(\mathfrak{e})^n$$

The Discriminant

The final preliminary we need concerns the spectra of periodic Jacobi matrices. So $\{a_n, b_n\}_{n=-\infty}^{\infty}$ are two-sided sequences with $a_n > 0$, $b_n \in \mathbb{R}$ and some $p > 0$ in $\mathbb{Z}$ so that

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We define doubly infinite tridiagonal matrices, J, with b_n along the diagonal and a_n on the principle subdiagonals (so that row j has non-zero elements $a_{j-1}\, b_j\, a_j$ with b_j in column j.).

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For $z \in \mathbb{C}$ fixed, we are interested in solutions, $\{u_n\}_{n=-\infty}^{\infty}$, of

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We study the p-step transfer (aka update) matrix.

$$M_p(z)\begin{pmatrix} u_1 \\ a_0 u_0 \end{pmatrix} = \begin{pmatrix} u_{p+1} \\ a_p u_p \end{pmatrix}$$

The Discriminant

We put a's in the bottom component so the one step matrix $\frac{1}{a_j}\begin{pmatrix} z-b_j & -1 \\ a_j^2 & 0 \end{pmatrix}$ has determinant 1 and thus $\det(M_p(z)) = 1$.

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Since it is known that the spectrum of J is the closure of the set of z for which there are polynomially bounded solutions (Schnol's Theorem), we conclude that

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spec$(J) = \Delta^{-1}([-2, 2])$ so we have that $\Delta^{-1}([-2, 2]) \subset \mathbb{R}$

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Therefore, between successive points where $\Delta(x_0) = \pm 2$ and where $\Delta(x_1) = \mp 2$, $\Delta(x)$ is strictly monotone and Δ is a bijection. It follows there are real numbers $\alpha_1 < \beta_1 \leq \alpha_2 < \dots \beta_{p-1} \leq \alpha_p < \beta_p$ so that

$$\Delta^{-1}([-2,2]) = \bigcup_{j=1}^{p} [\alpha_j, \beta_j] = \text{spec}(J)$$

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Theorem Let J be a period p periodic Jacobi matrix with Jacobi parameters $\{a_n, b_n\}_{n=-\infty}^{\infty}$ and let $\Delta(z)$ be its discriminant.

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Notice that since the Jacobi parameters are also periodic with period $2p, 3p, \ldots$, for $\mathfrak{e} =$ spec(J), we have that $T_n^{-1}[-\|T_n\|_{\mathfrak{e}}, \|T_n\|_{\mathfrak{e}}] = \mathfrak{e}$ for $n = kp$ with $k = 1, 2, \ldots$

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The $[\alpha_j, \beta_j]$ are the *bands* and (β_{j-1}, α_j) are the *gaps*. If $\beta_{j-1} = \alpha_j$, we say that the gap is *closed*; otherwise it is *open*. Notice that $\Delta(\alpha_j) \neq \Delta(\beta_j) = \Delta(\alpha_{j+1})$ so that $\alpha_1, \beta_1, \beta_2, \ldots, \beta_p$ is an alternating set for $\Delta(x)$ on $\text{Spec}(J)$. We have thus proven

Theorem Let J be a period p periodic Jacobi matrix with Jacobi parameters $\{a_n, b_n\}_{n=-\infty}^{\infty}$ and let $\Delta(z)$ be its discriminant. Then $(a_1 \ldots a_p)\Delta(z)$ is the pth Chebyshev polynomial for $\text{spec}(J)$.

Notice that since the Jacobi parameters are also periodic with period $2p, 3p, \ldots$, for $\mathfrak{e} = \text{spec}(J)$, we have that $T_n^{-1}[-\|T_n\|_{\mathfrak{e}}, \|T_n\|_{\mathfrak{e}}] = \mathfrak{e}$ for $n = kp$ with $k = 1, 2, \ldots$ The idea of exploiting the fact that $\text{spec}(J)$ is a polynomial inverse image goes back to Geronimo-van Assche and was raised to high art by Totik.

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The conjugate function to $\log|q(z)|$ is $\arg q(z)$ which on $\mathfrak{e}$ is given by $\arccos(\frac{\Delta(x)}{2})$. From this we conclude that each band obeys $\rho_{\mathfrak{e}}([\alpha_j, \beta_j]) = \frac{1}{p}$ so, taking into account possible closed gaps, each connected component of $\mathfrak{e}$ has harmonic measure $\frac{k}{p}$ with $k \in \mathbb{Z}_+$.

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If these harmonic measures are all $\frac{k}{p}$ with $k \in \mathbb{Z}_+$, the Jacobi matrices in the isospectral torus are periodic of period p. Thus

Theorem A subset $\mathfrak{e} \subset \mathbb{R}$ is the spectrum of a period p Jacobi matrix if and only if it has no more than p connected components where each such component has harmonic measure $\frac{k}{p}$ with $k \in \mathbb{Z}_+$.

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Fact 2 For all period n sets $\tilde{\mathfrak{e}} \supset \mathfrak{e}$, we have $C(\tilde{\mathfrak{e}}) \geq C(\mathfrak{e}_n)$ with equality only if $\tilde{\mathfrak{e}} = \mathfrak{e}_n$.

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Fekete's work on transfinite diameters and its connection to capacity for some special cases is from 1923. Szegő had the full theorem in a 1924 paper whose title started "Comments on a paper by Mr. M. Fekete".

Character Automorphic Functions

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Put differently, $B_\mathfrak{e}(z)$ can be continued along any curve in Ω and there is a map from the fundamental group of Ω to $\partial\mathbb{D}$, which is a character (i.e. group homomorphism), so that after continuation around a closed curve, $B_\mathfrak{e}(z)$ is multiplied by the character applied to that curve.

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If $T_n(z)B_{\mathfrak{e}}(z)^n C(\mathfrak{e})^{-n}$ had a limit, that limit cannot be n independent since the character is n dependent. Widom had the idea that there should be functions $F_\chi(z)$ defined for each χ in the character group and continuous in χ so the limit is the F_χ, call it F_n, associated to the character of $B_{\mathfrak{e}}(z)^n$. As a function of n, the limit will be almost periodic!

Widom's Minimizers

He even found a candidate for the functions! Let $F_\chi(z)$ be that function among all character automorphic functions, $A(z)$, on Ω with character χ and with $A(\infty) = 1$, that minimizes $\sup_{z\in\Omega}\{|A(z)|\}$.

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Widom proved uniqueness of the minimizer and found a formula for it (in terms of some theta functions and solutions of some implicit equations). He also proved that $\|F_\chi\|_\Omega$ is continuous in χ. Because of the uniqueness, one can prove that the functions, $F_\chi(z)$, defined for $z \in \Omega$, are continuous in χ on the compact set of characters, uniformly locally in z (but as functions on the covering space not uniformly in all z).

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Since $|B_{\mathfrak{e}}(z)| \to 1$ and $\|F_n\|_\Omega$ is taken as $z \to \mathfrak{e}$, the z asymptotics and norm limit fit together.

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Theorem (*Widom*) Let $\mathfrak{e}$ be a finite gap subset of $\mathbb{R}$. Let $F_n(z)$ be as above. Then

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Note: This is Widom's conjecture for $\mathfrak{e} \subset \mathbb{R}$; he made the conjecture for more general cases of $\mathfrak{e} \subset \mathbb{C}$.

Back to $[-1, 1]$

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It was this example that led Widom to his conjecture.

Outer Approximation for General Sets

In order to extend Markov and other polynomial inequalities to general sets, Totik proved that:

Theorem (*Totik's Approximation Theorem*) For any compact set $\mathfrak{e} \subset \mathbb{R}$, there exist period n sets $\tilde{\mathfrak{e}}_n \supset \mathfrak{e}$ so that $C(\tilde{\mathfrak{e}}_n) \to C(\mathfrak{e})$

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Totik published his approximation theorem in 2001. In 2009, he published an improvement for finite gap case:

Totik–Widom bounds

Theorem (Totik's $1/n$ bound) If $\mathfrak{e}$ is a finite gap set, the period n sets $\tilde{\mathfrak{e}}_n \supset \mathfrak{e}$ can be chosen so that $C(\tilde{\mathfrak{e}}_n) \leq C(\mathfrak{e}) \left(1 + \frac{E}{n}\right)$ for some constant E.

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Recall that we say that $\mathfrak{e} \subset \mathbb{R}$ obeys a Totik-Widom bound if there is a D with $\|T_n\|_{\mathfrak{e}} \leq DC(\mathfrak{e})^n$ and that this was only known for finite gap sets.

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a notion introduced by Carleson in his study of sets for which the Corona Theorem holds. A positive measure Cantor set (i.e. [0,1] with the middle $\frac{1}{n_j}$th removed at step j where $\sum_1^\infty \frac{1}{n_j} < \infty$) is homogeneous.

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Recall that we say that $\mathfrak{e} \subset \mathbb{R}$ obeys a Totik-Widom bound if there is a D with $\|T_n\|_{\mathfrak{e}} \leq DC(\mathfrak{e})^n$ and that this was only known for finite gap sets. A compact subset $\mathfrak{e} \subset \mathbb{R}$ is called *homogeneous* if there exist Q and $c \in (0, \frac{1}{2})$ so that

$$\forall_{x \in \mathfrak{e}} \forall_{0<\delta<Q} \ |\mathfrak{e} \cap (x-\delta, x+\delta)| > c\delta$$

a notion introduced by Carleson in his study of sets for which the Corona Theorem holds. A positive measure Cantor set (i.e. [0,1] with the middle $\frac{1}{n_j}$th removed at step j where $\sum_1^\infty \frac{1}{n_j} < \infty$) is homogeneous.

Theorem Every homogeneous set obeys a Totik Widom bound.

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Interesting Open Question Does potential theory regularity + Parreau-Widom $\Rightarrow$ Totik-Widom bound for general $\mathfrak{e} \subset \mathbb{C}$ (our proof is only for $\mathfrak{e} \subset \mathbb{R}$).

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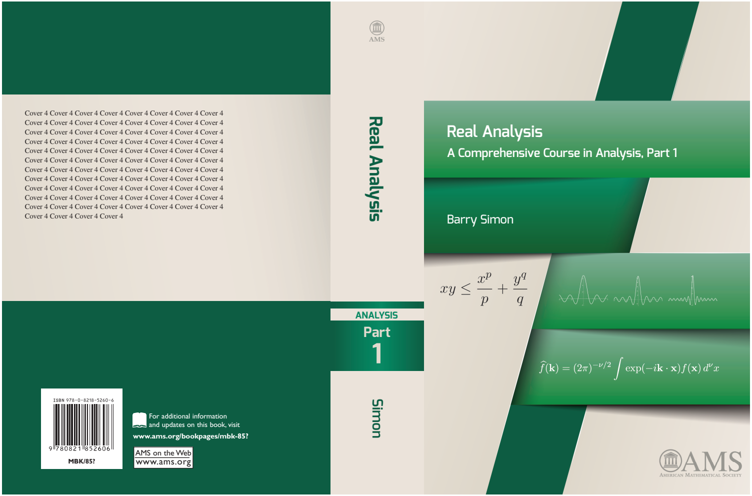

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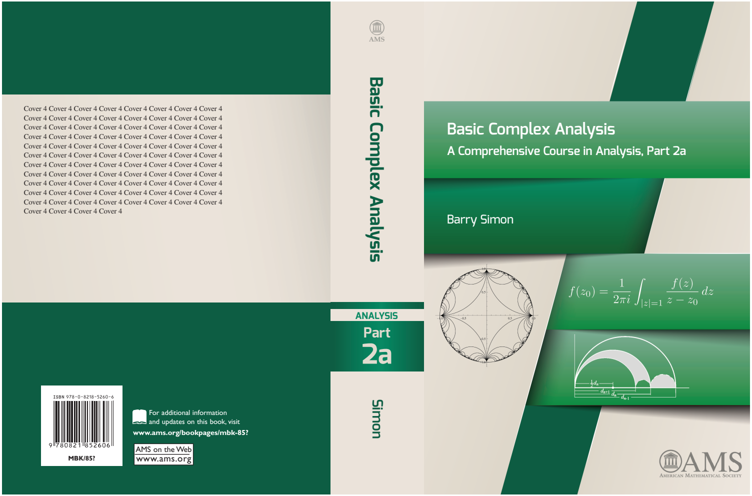

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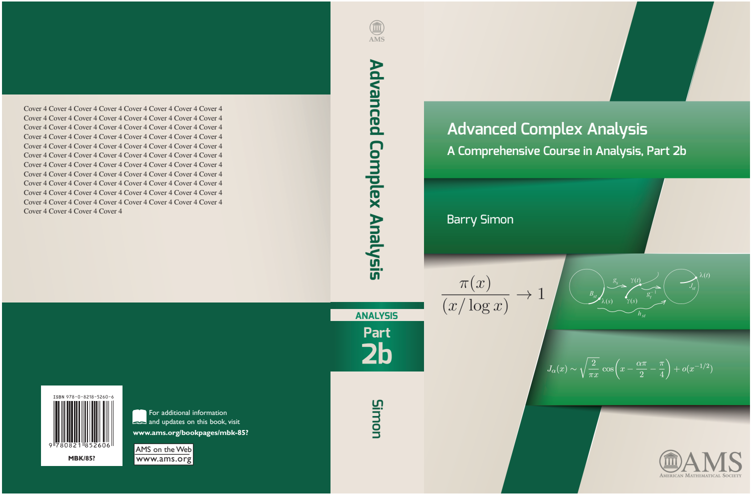

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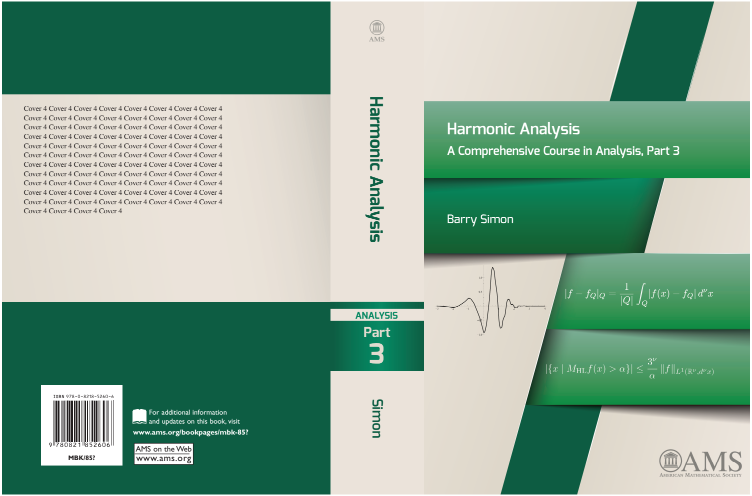

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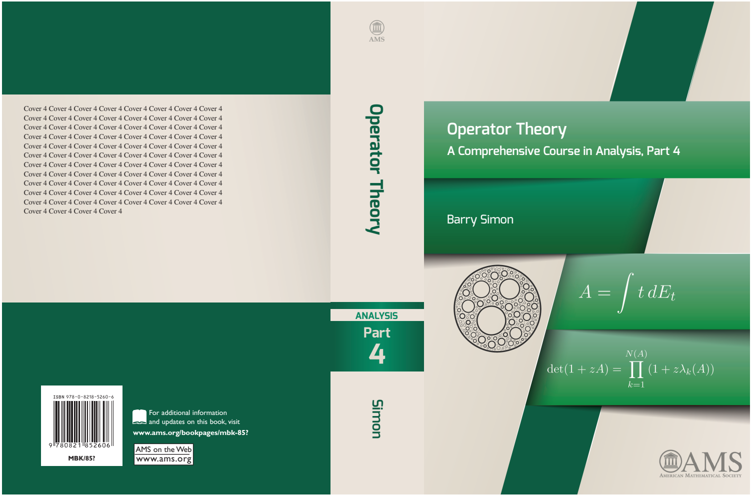

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