Recent Developments in the Spectral Theory of Orthogonal Polynomials

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Lecture 4: Killip-Simon Theorems for Finite Gap Sets

- Lecture 1: Introduction and Overview
- Lecture 2: Sum Rules and Large Deviations
- Lecture 3: Szegő-Widom asymptotics for Chebyshev Polynomials
- Lecture 4: Killip-Simon Theorems for Finite Gap Sets

References for Lecture 4

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2. **Minimal Herglotz Functions** which is discussed by CSZ. Points are associated to half line Jacobi matrices and the data are the poles on a two sheeted branched Riemann surface. The branch points are the edges of the gaps, so the inverse image of each gap closure is a circle.

3. **Character Automorphic H^2 Spaces** The is due to Sodin-Yuditskii and they call it the *functional model*. The labels are characters for the fundamental group of $(\mathbb{C} \cup \{\infty\}) \setminus \mathfrak{e}$. One looks at the character automorphic functions with that character in a suitable Hardy space and finds a natural basis and Jacobi operator in that basis. pause The character group is, of course, a torus.

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We'll discuss the first two briefly now, and, if time allows the third in more detail later since it is an important element of Yuditskii's work.

Reflectionless Operators

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Given a finite gap set, $\mathfrak{e}$, a whole Jacobi matrix is called *reflectionless* on it spectrum, $\mathfrak{e}$, if for all n and a.e. $x \in \mathfrak{e}$, we have that $\lim_{\epsilon \downarrow 0} \mathrm{Re}(G_{nn}(x+i\epsilon)) = 0$.

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Killip–Simon for Period-n Sets
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if and only if

(i) (Blumental–Weyl) $\sigma_{\text{ess}}(J) = \operatorname{ess\,supp}(d\mu) = \mathfrak{e}$,

(ii) (Lieb–Thirring) $\sum_{E \in \sigma(J)\setminus\mathfrak{e}} (dist(E,\mathfrak{e}))^{3/2} < \infty$.

(iii) (Quasi-Szegő) $\int (dist(x,\mathbb{R}\setminus\mathfrak{e}))^{1/2} \log\big(f(x)\big)\,dx < \infty$.

"Direct Integral" Viewpoint

Let S be the operator on two sided ℓ^2 sequences $(Su)_n = u_{n-1}$. If J is a two sided period-n Jacobi matrix, $[J, S^n] = 0$ so they can be simultaneously diagonalized.

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One formal proof uses the theory of direct integrals. This is one half of the *Magic Formula* that for a two sided bounded J:

$$\Delta(J) = S^n + S^{-n} \iff J \text{ has period } n \text{ and } J \in \mathcal{T}_{\mathfrak{e}}$$

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If $\widetilde{\Delta}$ is the discriminant for J, we see that $\widetilde{\Delta}(J) = \Delta(J)$ by the other direction of the magic formula formula for $\sigma(J)$. A little lemma shows for any Jacobi matrix, J, if $p(J) = q(J)$ for two polynomials, then the two polynomials are equal. Since Δ is the discriminant for J, we have that $\sigma(J) = \Delta^{-1}[-2, 2] = \mathfrak{e}$ so $J \in \mathcal{T}_{\mathfrak{e}}$.

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It is easy to get a Killip–Simon theorem on the equivalence of $\Delta(J) - S^n - S^{-n}$ in Hilbert Schmidt class. The hard work in the DKS result is translate that into information about J and that was only possible if all gaps were open.

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$$\Delta(z) = \psi(z) + \psi(z)^{-1} = \frac{2+G(z)^2}{1-G(z)^2}$$

has the required properties of the Yuditskii discriminant. Moreover, for $A>0$, $B \in \mathbb{R}$, $c_j \in (\beta_j, \alpha_{j+1})$, $d_j > 0$

$$\Delta(z) = Az + B + \sum_{j=1}^{\ell} \frac{d_j}{c_j - z}$$

One-Sided GMP Matrices

If $J \in \mathcal{T}_{\mathfrak{e}}$ and Δ is the Yuditskii discriminant of $\mathfrak{e}$, then $\Delta(J)$ clearly has spectrum $[-2, 2]$ with multiplicity $N \equiv \ell + 1$ and it is can be shown to be purely a.c. so it is unitarily equivalent to $S^N + S^{-N}$.

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It also arose earlier in the study of the strong moment problem – i.e. finding a measure $d\mu$ with specified values of $\int x^n d\mu(x)$ where now n runs from $-\infty$ to ∞. The key again is to orthonormalize $\{1, x, x^{-1}, x^2, x^{-2}, \dots\}$ which yields a 5-diagonal matrix M where M^{-1} is also 5-diagonal.

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Given real numbers $\{c_j\}_{j=1}^{\ell} \equiv C$, and a half line Jacobi matrix with associated measure, $d\mu$ of compact support in $\mathbb{R} \setminus C$, we orthonormalize the set $\{1, (x - c_1)^{-1}, \dots, (x - c_\ell)^{-1}, x, (x - c_1)^{-2}, \dots\}$ in $L^2(d\mu)$ and write J in this new basis. The resulting matrix, $A(J)$, is called a GMP (for generalized moment problem) matrix by Yuditskii.

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Two-Sided GMP Matrices

Given a two sided Jacobi matrix, J, if J^+ is half line Jacobi matrix as above, then Yuditskii constructs a two sided $2N+1$-diagonal matrix, $A(J)$, unitarily equivalent to J, with $(A(J)-c_j)^{-1}$ also $2N+1$ and so that its half line piece if $A(J^+)$.

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One fact we mention immediately is that if $J \in \mathcal{T}_{\mathfrak{e}}$, then $A(J)$ is periodic with period-N.

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That elements of the isospectral torus obey the magic formula follows from looking at the functional model. The converse is similar to the DKS argument – the magic formula implies $A(J)$ is periodic and that is enough to get $J \in \mathcal{T}_{\mathfrak{e}}$.

Isospectral Torus Flow

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Yuditskii Theorem *Let* $d\mu(x) = f(x)\,dx + d\mu_s$ *with Jacobi parameters* $\{a_n, b_n\}_{n=1}^\infty$. *Then*

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The Free Jacobi Matrix

We begin by describing the functional model for the case $\mathfrak{e} = [-2, 2]$ where the isospectral torus has one element with $a_n \equiv 1$ and $b_n \equiv 0$.

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By allowing z^n for $n < 0$, we get a basis for L^2 and in this basis, multiplication by $\mathsf{x}(z)$ is the two sided free Jacobi matrix.

The Fuchsian Group Picture

We owe to Sodin–Yuditskii [SY1997] a lovely way of organizing the multivalued analytic functions we saw were critical to Widom's view of Szegő asymptotics. Let $\Omega \equiv \mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$.

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Multivalued character automorphic functions, g, lift to functions, f on $\mathbb{D}$ so that for some $\chi \in \Gamma^*$, we have that $f(\gamma(z)) = \chi(\gamma) f(z)$. Here Γ^* is the character group of Γ. Since Γ is the free group on ℓ-generators, its abelianization is $\mathbb{Z}^n$ and thus Γ^* is an ℓ dimension torus.

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It is a basic fact about these groups that $\sum_{\gamma\in\Gamma} 1-|\gamma(w)|<\infty$ for any $w\in\mathbb{D}$ so if $b(z,w)$ is a standard Blaschke factor, one can form $B_w(z)=\prod_{\gamma\in\Gamma} b(z,\gamma(w))$.

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This means that for y fixed $G(x,y)$ is harmonic on $\Omega\setminus\{y\}$ with a logarithmic pole at y. It is positive on Ω and goes to zero on $\mathfrak{e}$. It is thus what is known as the (potential theoretic) Green's function with pole at y.

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When one looks at the functional model for $A(J)$ with $J \in \mathcal{T}_{\mathfrak{e}}$ (which we won't have time for!), the formula above implies that $\psi(A(J)) = S^n$ which shows that all $J \in \mathcal{T}_{\mathfrak{e}}$ obey the magic formula.

Functional Model for Isospectral Torus

For this last topic, let's shift to an additive view of Γ^* and use μ for the character of $B \equiv B_0$. Now let $\mathcal{H}_\alpha$ be the set of elements in H^2 which are character automorphic with character α.

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Similarly, if we define e_n^α for negative n also and $\mathcal{L}_\alpha$ as the character automorphic functions in $L^2(\partial\mathbb{D})$ (using the fact that the $\gamma \in \Gamma$ have a meromorphic or anti-meromorphic extension to $\mathbb{C}$),

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By construction, e_n^α is orthogonal to all functions in $\mathcal{H}_\alpha$ that vanish to order $n+1$ at zero, so for any $m > n$ and $g \in \mathcal{H}_{\alpha - m\mu}$, we have that $\langle e_n^\alpha, B^m g\rangle = 0$ and that $\langle e_n^\alpha, B^n g\rangle = g(0)(k_\alpha(0))^{-1/2}$.

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$$\langle e_{n-1}^\alpha, \mathsf{x}(z) e_n^\alpha \rangle = a_{n-1}^\alpha \equiv C(\mathfrak{e}) \sqrt{\frac{k_{\alpha-(n-1)\mu}(0)}{k_{\alpha-n\mu}(0)}}$$

Functional Model for Isospectral Torus

Since multiplication by $\mathsf{x}(z)$ is self-adjoint on $\mathcal{L}_\alpha$ (since the function is real on $\partial\mathbb{D}$), we see that multiplication by $\mathsf{x}(z)$ is tridiagonal in the $\{e_n^\alpha\}_{n=-\infty}^\infty$ basis.

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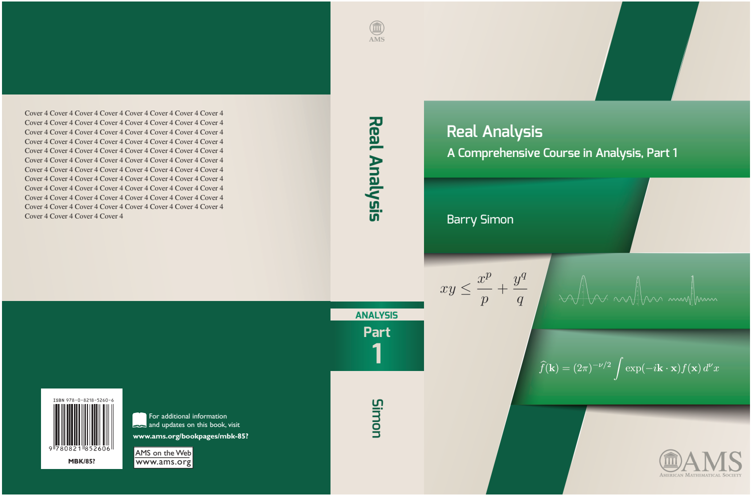

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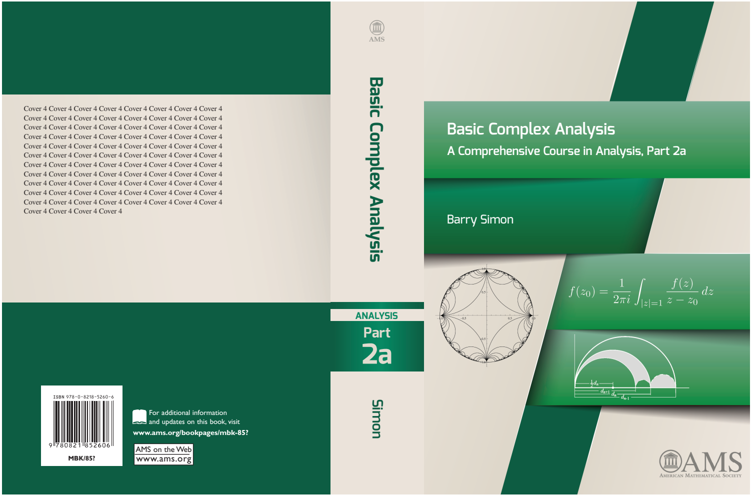

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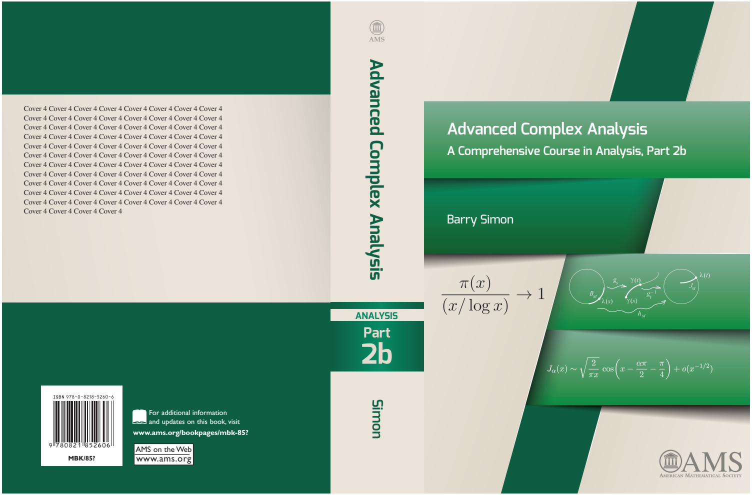

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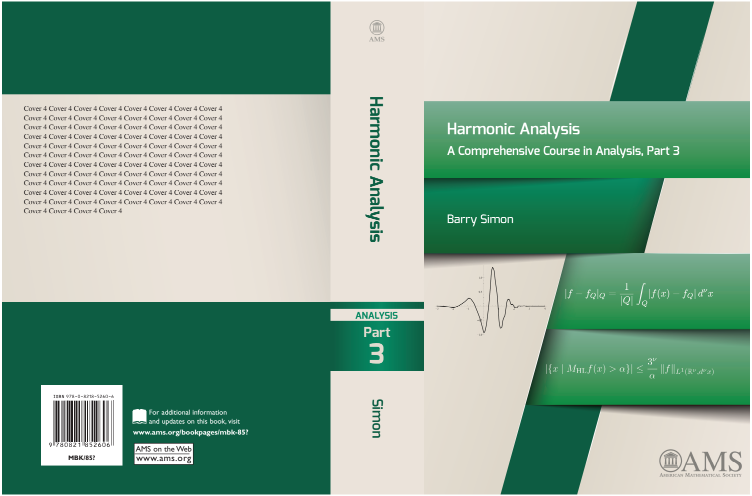

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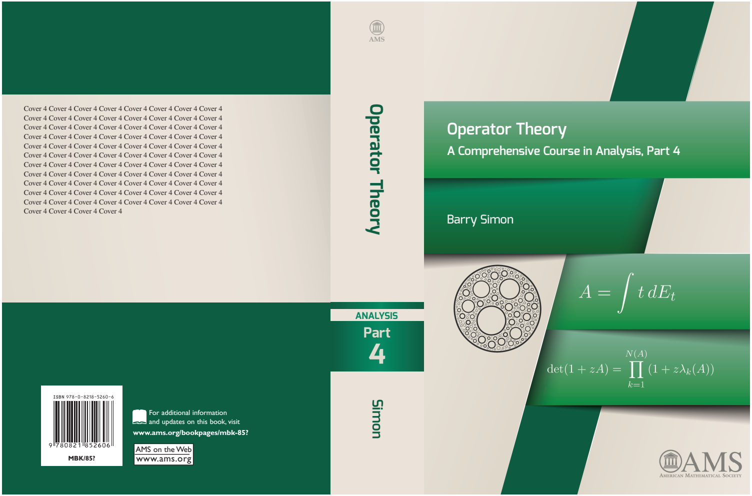

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