Summer School on Mathematical Physics

Inverse Problems: Visibility and Invisibility

Lecture I

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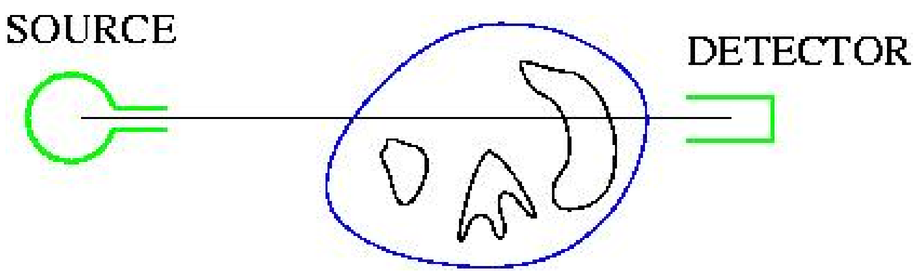
Inverse Boundary Problems

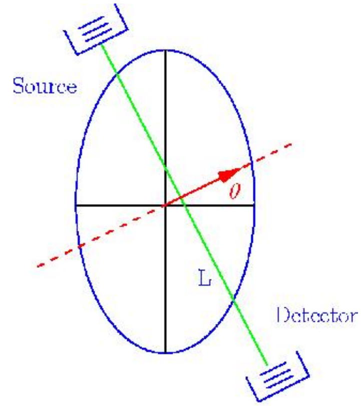
Can one determine the internal properties of a medium by making measurements outside the medium (non-invasive)?

X-ray tomography (CAT-scans)



Problem: Can we recover the density from attenuation of X-rays?

Radon (1917) $n = 2$



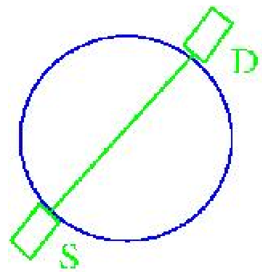
$f(x) =$ Unknown function

$$I_{detector} = e^{-\int_L f} I_{source}$$

$$Rf(s,\theta) = g(s,\theta) = \int_{\langle x,\theta\rangle = s} f(x)dH = \int_L f$$

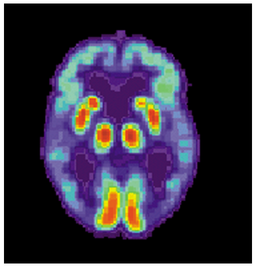
$$\boxed{f(x) = \frac{1}{4\pi^2}\ p.v. \int_{S^1} d\theta \int \frac{\frac{d}{ds} g(s,\theta)ds}{\langle x,\theta\rangle - s}}$$

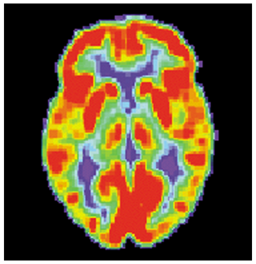
LINEAR (No Scattering)



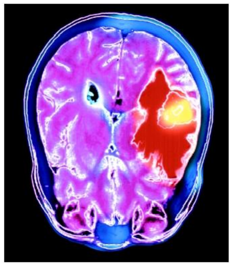
X-ray tomography (CT)

PET

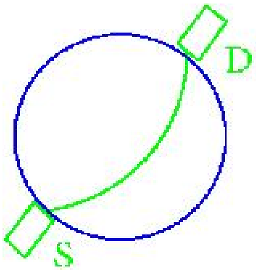




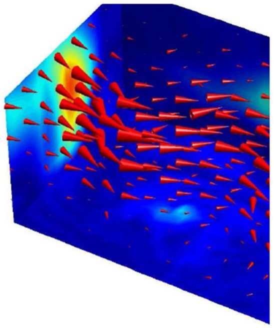
MRI



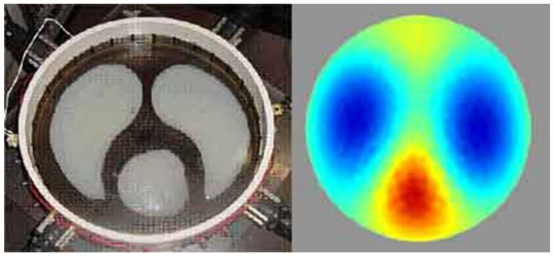
$\boxed{\text{NONLINEAR}}$ (Scattering)



Ultrasound



Electrical
Impedance
Tomography
(EIT)



Hybrid Methods

Superposition of 2 images each obtained with a single wave

One single wave in sensitive only to a given contrast

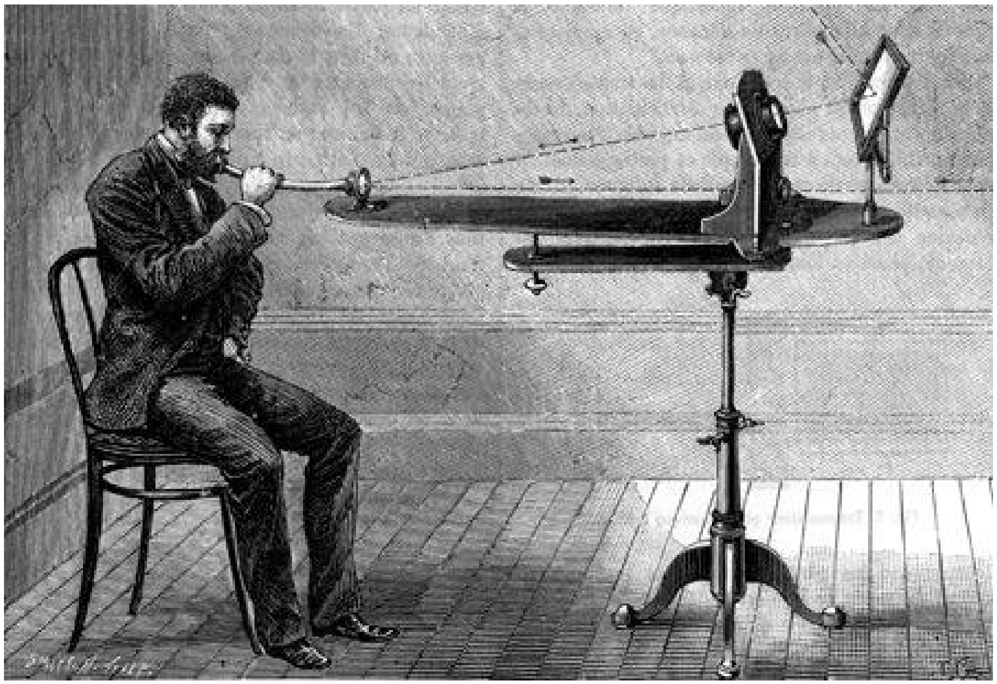
Ultrasound to bulk compressibility

$\boxed{\text{Photoacoustic Imaging}}$ Optical wave to dielectric permittivity

$\boxed{\text{Thermoacoustic Imaging}}$ LF Electromagnetic wave to electrical impedance, conductivity.

Photoacoustic Tomography

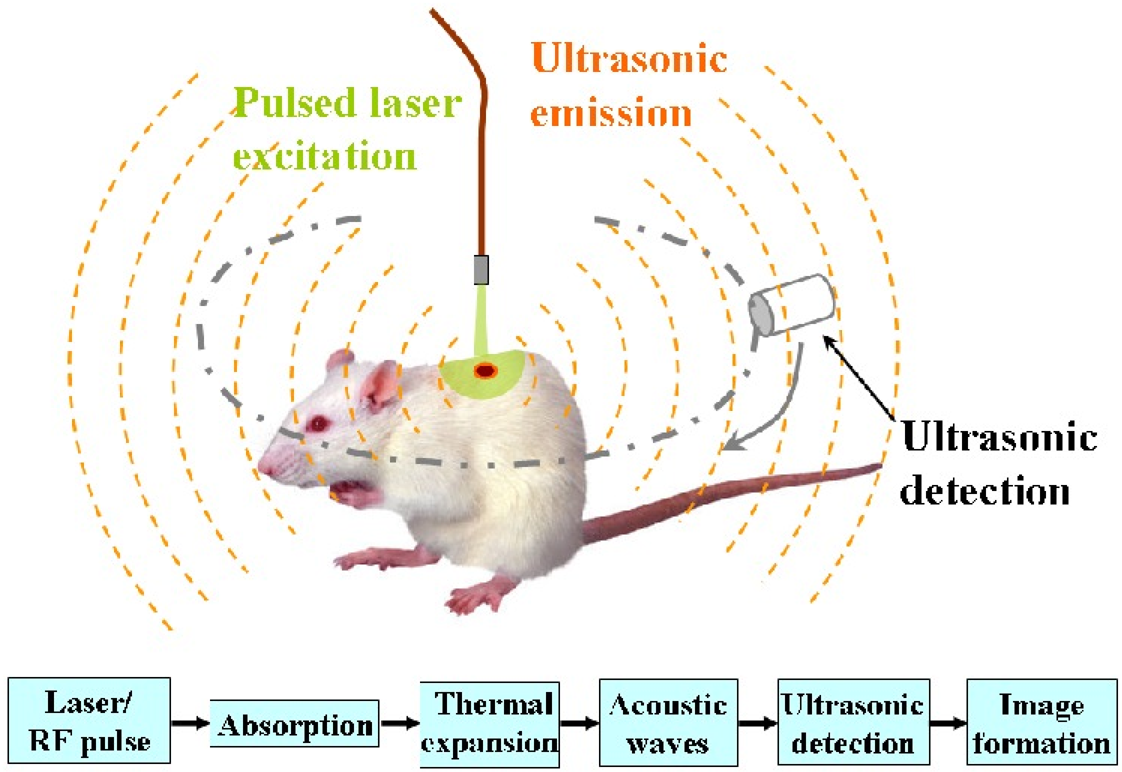
Photoacoustic Effect: **The sound of light**



Picture from Economist
(The sound of light)

Graham Bell: When rapid pulses of light are incident on a sample of matter they can be absorbed and the resulting energy will then be radiated as heat. This heat causes detectable sound waves due to pressure variation in the surrounding medium.

Thermoacoustic Tomography



Wikipedia

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Lihong Wang (Washington U.)

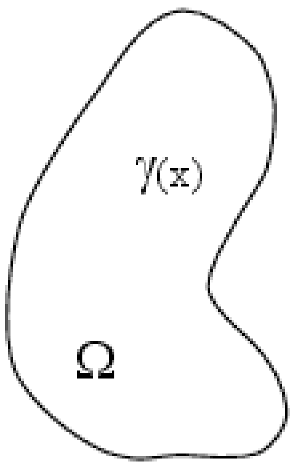
Mathematical Model

$\boxed{\text{First Step}}$: in PAT and TAT is to reconstruct $H(x)$ from $u(x,t)|_{\partial\Omega\times(0,T)}$, where u solves

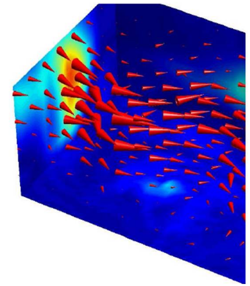
$$\begin{aligned}(\partial_t^2 - c^2(x)\Delta)u &= 0 \quad \text{on } \mathbb{R}^n \times \mathbb{R}^+ \\ u|_{t=0} &= \beta H(x) \\ \partial_t u|_{t=0} &= 0\end{aligned}$$

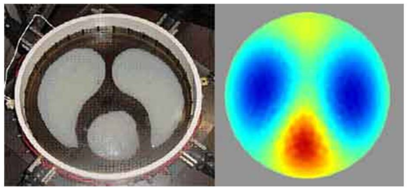
$\boxed{\text{Second Step}}$: in PAT and TAT is to reconstruct the optical or electrical properties from $H(x)$ (internal measurements).

CALDERÓN'S PROBLEM and EIT



$\Omega \subset \mathbb{R}^n$
$(n = 2, 3)$

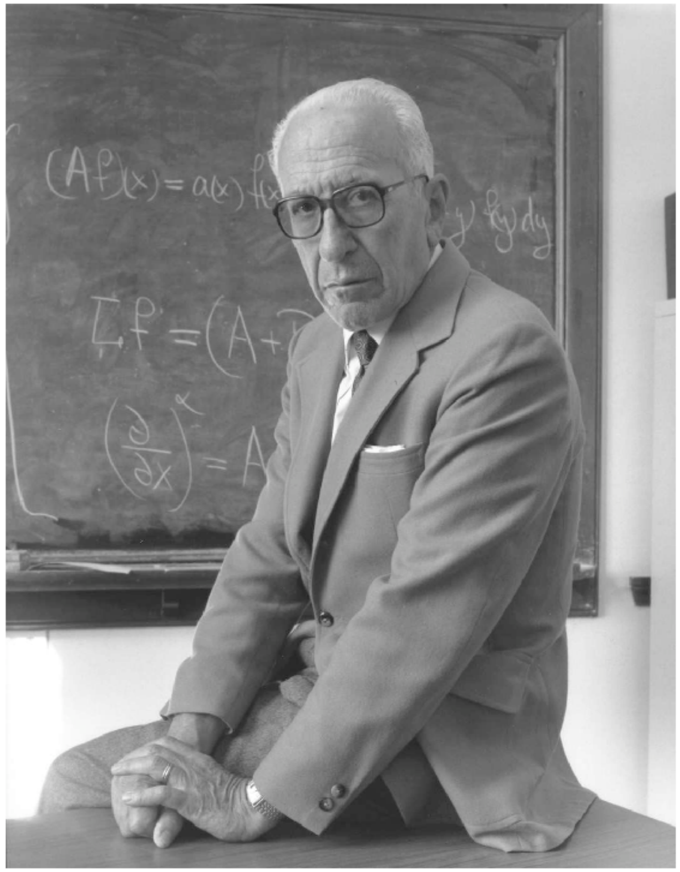




Can one determine the electrical conductivity of $\Omega, \gamma(x)$, by making voltage and current measurements at the boundary?
(Calderón; Geophysical prospection)

Early breast cancer detection

Normal breast tissue	0.3 mho
Cancerous breast tumor	2.0 mho



REMINISCENCIA DE MI VIDA MATEMATICA

Speech at Universidad Autónoma de Madrid accepting the 'Doctor Honoris Causa':

> *My work at "Yacimientos Petroliferos Fiscales" (YPF) was very interesting, but I was not well treated, otherwise I would have stayed there.*

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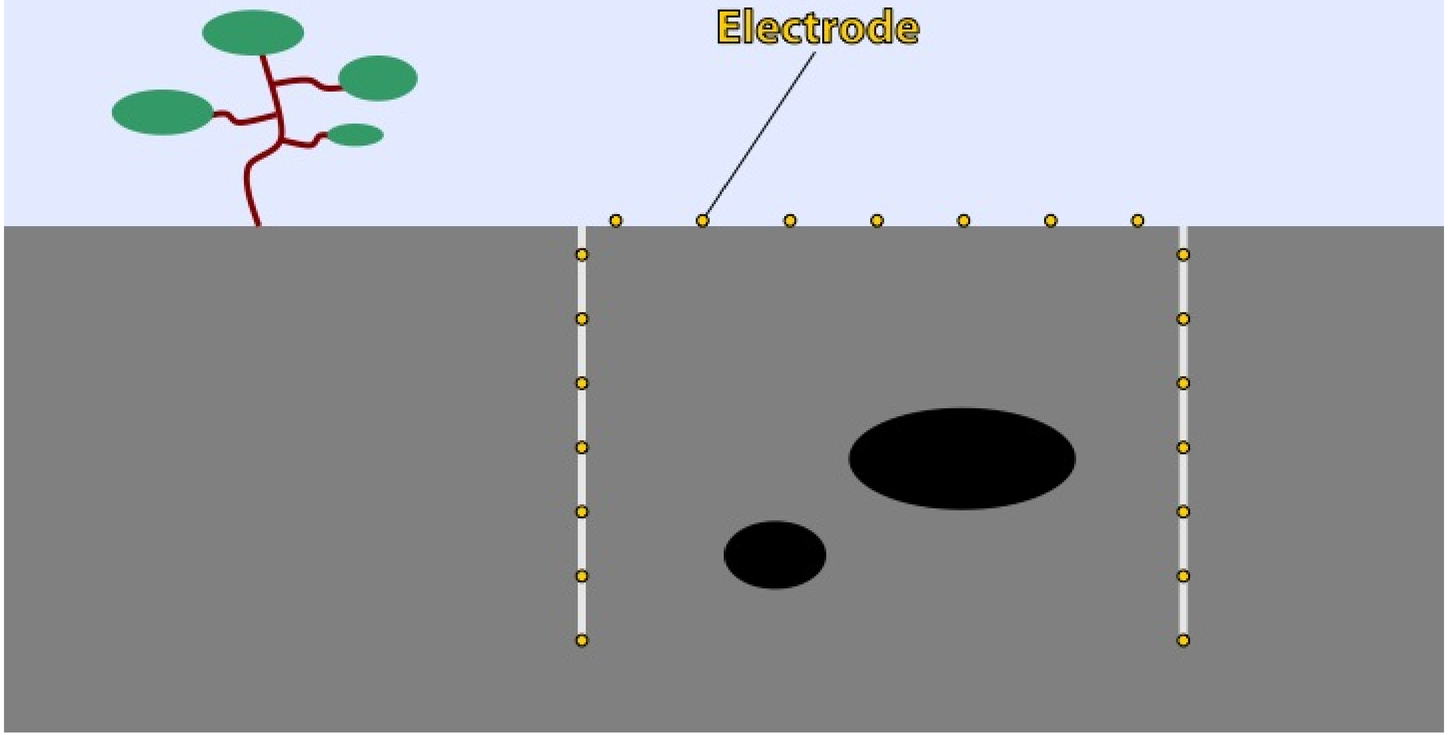
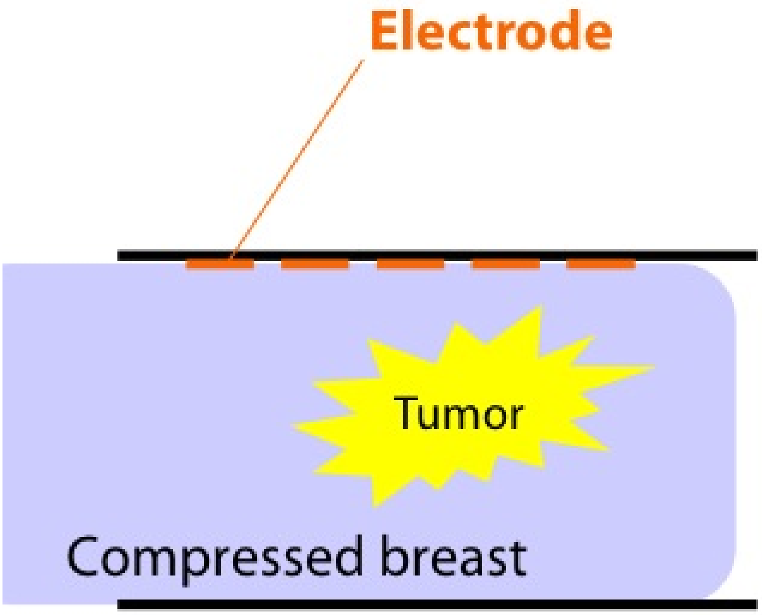
Geological underground probing is the application of EIT considered by Calderón



Early detection of breast cancer is effective using combined X-ray mammography and EIT

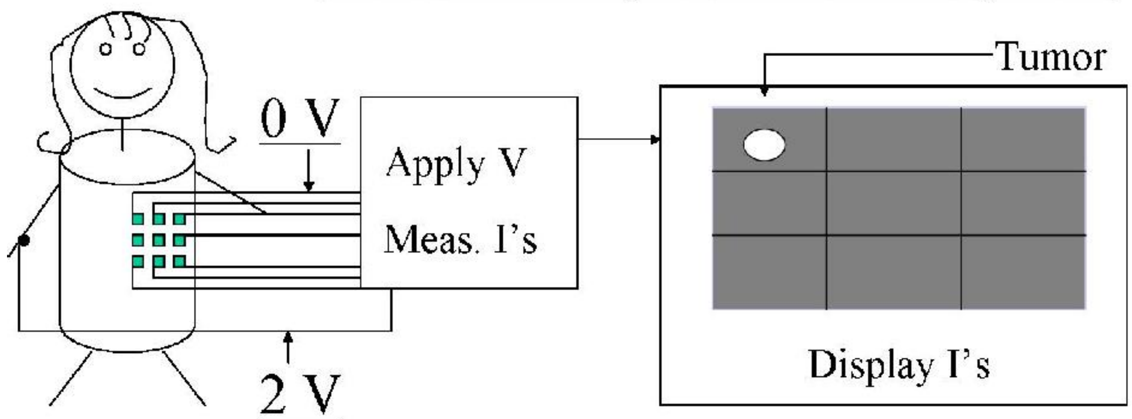


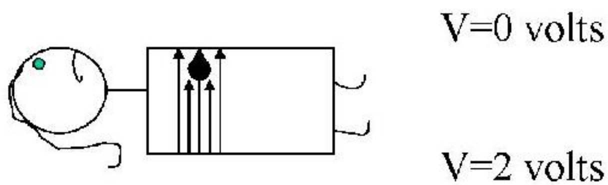
Cancerous tissue is up to four times more conductive than healthy tissue. [**Jossinet** -98]

X-ray attenuation is almost the same in cancerous and healthy tissue.

David Isaacson and his team have achieved good results in early detection of breast cancer using EIT.

T-Scan = Only Commercial System





Current I's in ma.

Combine with mammography for early detection?

RPI Group (D. Isaacson)

$$\text{Sensitivity} = \frac{\#\text{ predicted to have cancer}}{\text{Total }\#\text{ that have cancer}} \times 100$$

$$\text{Specificity} = \frac{\#\text{ predicted NOT to have cancer}}{\text{Total }\#\text{ that do NOT have cancer}} \times 100$$

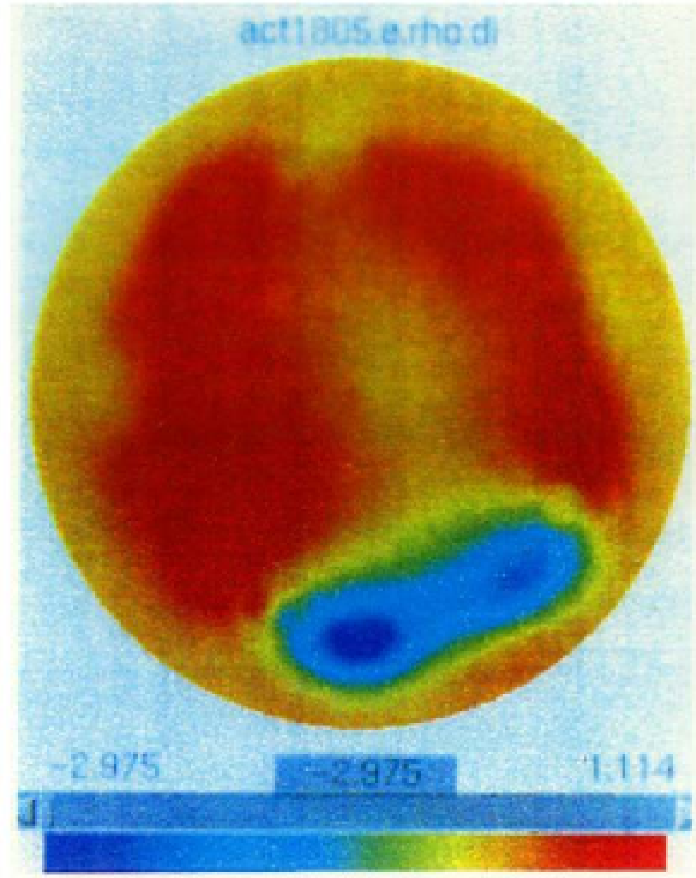
Results for Equivocal Mammograms ($N = 273$)

	Mamm. alone	T-Scan Adjunctive
Sensitivity (Biopsy ps.=50)	60%	82%
Specificity (Biopsy neg.=223)	41%	57%

Other Applications:

- Non-destructive testing (corrosion, cracks)
- Seepage of groundwater pollutants
- Medical Imaging (EIT)

Tissue	Conductivity (mho)
Blood	6.7
Liver	2.8
Cardiac muscle	6.3 (longitudinal)
	2.3 (transversal)
Grey matter	3.5
White matter	1.5
Lung	1.0 (expiration)
	0.4 (inspiration)



ACT3 imaging blood as it leaves the heart (blue) and fills the lungs (red) during systole.

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Thanks to D. Issacson

CALDERÓN'S PROBLEM (EIT)

Consider a body $\Omega \subset \mathbb{R}^n$. An electrical potential $u(x)$ causes the current

$$\boxed{I(x) = \gamma(x)\nabla u}$$

The conductivity $\gamma(x)$ can be isotropic, that is, scalar, or anisotropic, that is, a matrix valued function. If the current has no sources or sinks, we have

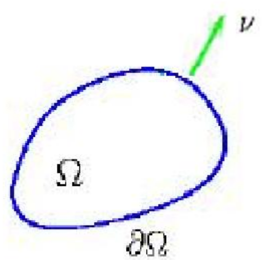
$$\boxed{\operatorname{div}(\gamma(x)\nabla u) = 0} \text{ in } \Omega$$

$$\boxed{\begin{aligned} \mathrm{div}(\gamma(x)\nabla u(x)) &= 0 \\ u\big|_{\partial\Omega} &= f \end{aligned}}$$

$\gamma(x)$ = conductivity,

f = voltage potential at $\partial\Omega$

Current flux at $\partial\Omega = (\nu \cdot \gamma \nabla u)\big|_{\partial\Omega}$ were ν is the unit outer normal.



Information is encoded in map

$$\boxed{\Lambda_\gamma(f) = \nu \cdot \gamma \nabla u\big|_{\partial\Omega}}$$

EIT (Calderón's inverse problem)

Does $\boxed{\Lambda_\gamma}$ determine $\boxed{\gamma}$?

Λ_γ = Dirichlet-to-Neumann map

$$\boxed{\begin{aligned} \text{div}(\gamma(x)\nabla u(x)) &= 0 \\ u\big|_{\partial\Omega} &= f \end{aligned}}$$

Dirichlet Integral:

$$Q_\gamma(f) := \int_\Omega \gamma(x)|\nabla u(x)|^2 dx$$

$$Q_\gamma(f,g) := \int_\Omega \gamma(x)\nabla u \cdot \nabla v dx$$

$$\boxed{\begin{aligned} \text{div}(\gamma(x)\nabla v(x)) &= 0 \\ v\big|_{\partial\Omega} &= g \end{aligned}}$$

$$\boxed{\begin{array}{rl} \operatorname{div}(\gamma \nabla u) & = 0 \\ u|_{\partial\Omega} & = f \end{array}, \quad \begin{array}{rl} \operatorname{div}(\gamma \nabla v) & = 0 \\ v|_{\partial\Omega} & = g \end{array}} \qquad \boxed{\Lambda_\gamma(f) = \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}}$$

$$Q_\gamma(f,g) = \int_\Omega \gamma \nabla u \cdot \nabla v dx.$$

A. P. Calderón: On an inverse boundary value problem, in *Seminar on Numerical Analysis and its Applications to Continuum Physics*, RÍo de Janeiro, 1980.

$$\boxed{Q_\gamma(f) = \int_\Omega \gamma |\nabla u(x)|^2 dx = \int_{\partial\Omega} \Lambda_\gamma(f) f dS.}$$

Linearization:

$$\boxed{\lim_{\varepsilon \to 0^+} \frac{Q_{\gamma+\varepsilon h}(f,g) - Q_\gamma(f,g)}{\varepsilon} = \int_\Omega h \nabla u \cdot \nabla v dx.}$$

Case $\gamma = 1$: $\quad \mathrm{div}(\gamma \nabla u) = \Delta u = 0$

Linearized Problem: Suppose we know

$$\boxed{\int_\Omega h \nabla u \cdot \nabla v dx} \qquad \forall\ \Delta u = \Delta v = 0.$$

Can we recover $\boxed{h}$?

Linearized problem at $\gamma = 1$:

$$\boxed{\int_\Omega h \nabla u \cdot \nabla v dx} \quad \text{data} \quad \forall\ \Delta u = \Delta v = 0.$$

Can we recover h?

$$\begin{array}{l} u = e^{x\cdot\rho} \\ v = e^{-x\cdot\overline{\rho}} \end{array}, \quad \rho \in \mathbb{C}^n,\ \rho\cdot\rho = 0.$$

$$\rho = \frac{\eta - i\xi}{2}, \quad \rho\cdot\rho = 0 \Leftrightarrow |\eta| = |\xi|, \eta\cdot\xi = 0.$$

$$\boxed{|\xi|^2 \int_\Omega h e^{-ix\cdot\xi} dx} \quad \text{known}$$

we can recover $\boxed{\widehat{\chi_\Omega h}(\xi)}$, therefore h on Ω.

Theorem (Kohn-Vogelius, 1984)

Assume $\gamma \in C^\infty(\overline{\Omega})$. From $\boxed{\Lambda_\gamma}$ we can determine $\boxed{\partial^\alpha \gamma\big|_{\partial\Omega}}$, $\forall\, \alpha$.

Proof (Sylvester-U, Lee-U)

$\boxed{\Lambda_\gamma}$ is a pseudodifferential operator of order 1 (Calderón).

$$\partial\Omega = \{x^n = 0\} \text{ locally.}$$

$$\text{Coordinates } x = (x', x^n),\ x' \in \mathbb{R}^{n-1}$$

$$\boxed{\Lambda_\gamma f(x') = \int e^{ix'\cdot\xi'} \lambda_\gamma(x', \xi') \widehat{f}(\xi') d\xi'}$$

$$\boxed{\Lambda_\gamma f(x') = \int e^{ix'\cdot\xi'} \lambda_\gamma(x',\xi') \widehat{f}(\xi') d\xi'}$$

$$\boxed{\lambda_\gamma(x',\xi') = \gamma(0,x')|\xi'| + a_0(x',\xi') + \cdots + a_j(x',\xi') + \cdots}$$

with $a_j(x',\xi')$ pos. homogeneous of degree $-j$ in ξ':

$$a_j(x',\lambda\xi') = \lambda^{-j} a_j(x',\xi'), \quad \lambda > 0.$$

$\boxed{\textbf{Result}}$ From a_j, we can determine $\left.\frac{\partial^j \gamma}{\partial \nu^j}\right|_{x^n=0}$.

$\boxed{\textbf{Theorem}}$ $n \geq 3$ (Sylvester-U, 1987)

$$\gamma \in C^2(\overline{\Omega}), \qquad 0 < C_1 \leq \gamma(x) \leq C_2 \quad \text{on } \overline{\Omega}$$
$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2$$

- Extended to $\gamma \in C^{3/2}(\overline{\Omega})$ (Päivarinta-Panchenko-U, Brown-Torres, 2003)
- $\gamma \in C^{1+\epsilon}(\overline{\Omega}), \gamma$ conormal (Greenleaf-Lassas-U, 2003)
- $\gamma \in C^1(\overline{\Omega})$, (Haberman-Tataru, 2012).

$\boxed{\text{Complex-Geometrical Optics Solutions (CGO)}}$

- Reconstruction A. Nachman (1988)
- Stability G. Alessandrini (1988)
- Numerical Methods (D. Issacson, J. Müller, S. Siltanen)

Reduction to Schrödinger equation

$$\boxed{\mathrm{div}(\gamma \nabla w) = 0}$$

$$u = \sqrt{\gamma} w$$

Then the equation is transformed into:

$$\boxed{(\Delta - q)u = 0, q = \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}} \qquad \left(\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \right)$$

$$\boxed{\begin{aligned} (\Delta - q)u &= 0 \\ u\Big|_{\partial\Omega} &= f \end{aligned}}$$

Define $\boxed{\Lambda_q(f) = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}}$

$\nu =$ unit-outer normal to $\partial\Omega$.

$\boxed{\text{IDENTITY}}$

$$\boxed{\int_\Omega (q_1 - q_2) u_1 u_2 = \int_{\partial\Omega} \Big((\Lambda_{q_1} - \Lambda_{q_2}) u_1 \big|_{\partial\Omega} \Big) u_2 \big|_{\partial\Omega} dS}$$

$$(\Delta - q_i) u_i = 0$$

If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \Lambda_{q_1} = \Lambda_{q_2}$ and

$$\boxed{\int_\Omega (q_1 - q_2) u_1 u_2 = 0}$$

GOAL: Find **MANY** solutions of $(\Delta - q_i) u_i = 0$.

$\boxed{\text{CGO SOLUTIONS}}$

Calderón: Let $\rho \in \mathbb{C}^n$, $\boxed{\rho \cdot \rho = 0}$

$$\boxed{\rho = \eta + ik} \qquad \eta, k \in \mathbb{R}^n, |\eta| = |k|, \eta \cdot k = 0$$

$$\boxed{u = e^{x\cdot\rho}} = e^{x\cdot\eta}e^{ix\cdot k}$$

$$\Delta u = 0, \qquad u = \begin{cases} \underline{\text{exponentially decreasing}}, & x \cdot \eta < 0 \\ \underline{\text{oscillating}}, & x \cdot \eta = 0 \\ \underline{\text{exponentially increasing}}, & x \cdot \eta > 0 \end{cases}$$

$\boxed{\text{COMPLEX GEOMETRICAL OPTICS}}$

(Sylvester-U) $n \geq 2, \; q \in L^\infty(\Omega)$
Let $\rho \in \mathbb{C}^n$ $(\rho = \eta + ik, \eta, k \in \mathbb{R}^n)$ such that $\boxed{\rho \cdot \rho = 0}$
$(|\eta| = |k|, \eta \cdot k = 0)$.

Then for $|\rho|$ sufficiently large we can find solutions of

$$\boxed{(\Delta - q)w_\rho = 0 \text{ on } \Omega}$$

of the form

$$\boxed{w_\rho = e^{x\cdot\rho}(1 + \Psi_q(x, \rho))}$$

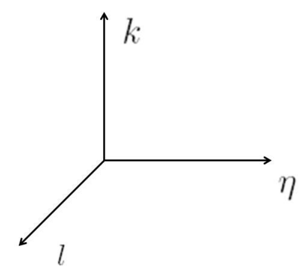
with $\Psi_q \to 0$ in Ω as $|\rho| \to \infty$.

Proof $\boxed{\Lambda_{q_1} = \Lambda_{q_2} \Rightarrow q_1 = q_2}$

$$\boxed{\int_\Omega (q_1 - q_2) u_1 u_2 = 0}$$

$$u_1 = e^{x\cdot\rho_1}(1 + \Psi_{q_1}(x,\rho_1)), \quad u_2 = e^{x\cdot\rho_2}(1 + \Psi_{q_2}(x,\rho_2))$$

$$\rho_1 \cdot \rho_1 = \rho_2 \cdot \rho_2 = 0, \qquad \begin{aligned} \rho_1 &= \eta + i(k+l) \\ \rho_2 &= -\eta + i(k-l) \end{aligned}$$



$$\eta \cdot k = \eta \cdot l = l \cdot k = 0, \quad |\eta|^2 = |k|^2 + |l|^2$$

$$\int_\Omega (q_1 - q_2) e^{2ix\cdot k}(1 + \Psi_{q_1} + \Psi_{q_2} + \Psi_{q_1}\Psi_{q_2}) = 0$$

Letting $|l| \to \infty$ $\quad \boxed{\int_\Omega (q_1 - q_2) e^{2ix\cdot k} = 0 \quad \forall k \Longrightarrow q_1 = q_2}$

$\boxed{\text{APPLICATIONS}}$

$n \geq 3 \quad (\Delta - q) = 0$, $\boxed{\Lambda_q}$ determines $\boxed{q}$

- EIT $\boxed{\Lambda_\gamma}$ determines $\boxed{\gamma}$
- Optical Tomography (Diffusion Approximation)

$$\boxed{i\omega U - \nabla \cdot D(x)\nabla U + \sigma_a(x)U = 0} \text{ in } \Omega$$

$U=$ Density of photons, D=Diffusion Coefficient, $\sigma_a(x)=$ optical absorption.

$\boxed{\text{RESULT}}$

- If $\omega \neq 0$ we can recover both $D(x)$ and $\sigma_a(x)$.
- If $\omega = 0$ we can recover either $D(x)$ or $\sigma_a(x)$.

OTHER APPLICATIONS (Fixed energy)

- Optics $(\Delta - k^2 n(x))u = 0$, $n(x)$ isotropic index of refraction $(q(x) = k^2 n(x))$.

- Acoustic $\mathrm{div}(\frac{1}{\rho(x)}\nabla p) + \omega^2 \kappa(x) p = 0$, ρ density, κ compressibility (need two frequencies ω).

- Inverse quantum scattering at fixed energy $(\Delta - q - \lambda^2)u = 0$, q potential.

- Maxwell's Equation (Isotropic) (Ola-Somersalo): Reduction to $(\Delta - Q)$, Q an 8×8 matrix.

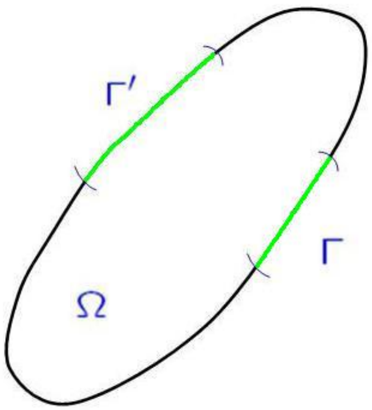
- Quantitative Photoacoustic Tomography (Bal-U)

$\boxed{\text{PARTIAL DATA PROBLEM}}$

Suppose we measure

$$\boxed{\Lambda_\gamma(f)|_\Gamma, \quad \text{supp} f \subseteq \Gamma'}$$

Γ, Γ' open subsets of $\partial\Omega$



Can one recover γ?

Important case $\Gamma = \Gamma'$.

$\boxed{\text{EXTENSION OF CGO SOLUTIONS}}$

$$\boxed{u = e^{x\cdot\rho}(1 + \Psi_q(x,\rho))}$$

$\rho \in \mathbb{C}^n, \rho \cdot \rho = 0$

(Not helpful for localizing)

Kenig-Sjöstrand-U (2007),

$$\boxed{u = e^{\tau(\varphi(x)+i\psi(x))}(a(x) + R(x,\tau))}$$

$\tau \in \mathbb{R}$, φ, ψ real-valued, $R(x,\tau) \to 0$ as $\tau \to \infty$.

φ limiting Carleman weight,

$$\nabla\varphi \cdot \nabla\psi = 0, \quad |\nabla\varphi| = |\nabla\psi|$$

Example: $\boxed{\varphi(x) = \ln|x - x_0|,}$ $\quad x_0 \notin \overline{ch(\Omega)}$

CGO SOLUTIONS

$$\boxed{\begin{aligned} u = e^{\tau(\varphi(x)+i\psi(x))}&(a_0(x) + R(x,\tau)) \\ R(x,\tau) &\overset{\tau\to\infty}{\longrightarrow} 0 \text{ in } \Omega \end{aligned}}$$

$$\boxed{\varphi(x) = \ln|x - x_0|}$$

Complex Spherical Waves

Theorem (Kenig-Sjöstrand-U) Ω strictly convex.

$$\Lambda_{q_1}\Big|_\Gamma = \Lambda_{q_2}\Big|_\Gamma, \quad \Gamma \subseteq \partial\Omega, \quad \Gamma \text{ arbitrary}$$

$$\Rightarrow q_1 = q_2$$

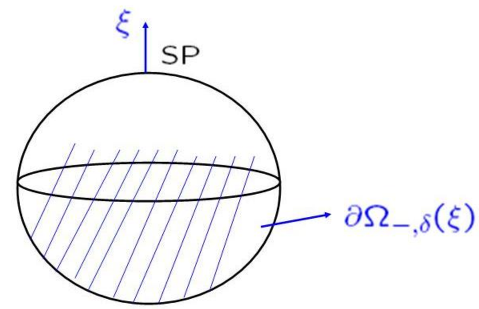
$\boxed{\text{Earlier result}}$ (Bukhgeim-U, 2002)

$n \geq 3$. Let $\xi \in \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$. We define

$$\partial\Omega_\pm = \left\{x \in \partial\Omega; \langle \nu, \xi \rangle \begin{array}{l} > 0 \\ < 0 \end{array} \right\}$$

(ν is the unit outer normal). Let $\delta > 0$

$$\partial\Omega_{+,\delta}(\xi) = \{x \in \partial\Omega; \langle \nu, \xi \rangle > \delta\}$$
$$\partial\Omega_{-,\delta}(\xi) = \{x \in \partial\Omega; \langle \nu, \xi \rangle < \delta\}$$



Theorem (Bukhgeim-U) Suppose we know

$$\boxed{\Lambda_q(f)|_{\partial\Omega_{-,\delta}(\xi)}} \qquad \text{supp } f \subseteq \partial\Omega,$$

Then we can recover $\boxed{q}$.

$\boxed{\textbf{Carleman estimate}}$ $\quad \xi \in \mathbb{S}^{n-1}$

Let $q \in L^\infty(\Omega)$, $u \in C^2(\bar{\Omega})$, $u|_{\partial\Omega} = 0$. For $\tau \geq \tau_0$

$$\tau^2 \int_\Omega |e^{-\tau\langle x,\xi\rangle} u|^2 \, dx + \underline{\tau} \int_{\underline{\partial\Omega_+}} \langle \xi, \nu(x)\rangle |e^{-\tau\langle x,\xi\rangle} \frac{\partial u}{\partial \nu}|^2 dS(x)$$

$$\leq C \left(\int_\Omega |e^{-\tau\langle x,\xi\rangle}(\Delta - q)u|^2 dx - \underline{\tau} \int_{\underline{\partial\Omega_-}} \underline{\langle \xi, \nu(x)\rangle} |e^{-\tau\langle x,\xi\rangle} \frac{\partial u}{\partial \nu}|^2 dS \right)$$

Remarks $\Delta_\rho u = e^{-x\cdot\rho}\Delta(e^{x\cdot\rho}u)$

- Carleman estimate for domains with boundary for $\boxed{\Delta_\rho = \Delta + 2\rho\cdot\nabla}$

- Weight is linear: $\langle x, \xi\rangle$

Corollary $u = 0$ on $\partial\Omega$, $\frac{\partial u}{\partial \nu}|_{\partial\Omega_-} = 0$

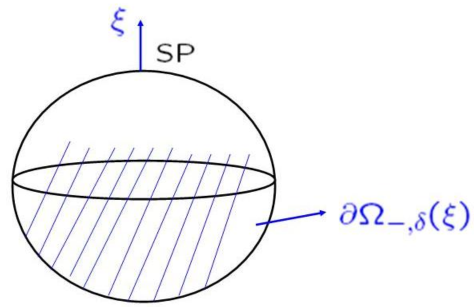
$$\boxed{\tau\int_{\partial\Omega_+}\langle x, \nu(x)\rangle|e^{-\tau\langle x,\xi\rangle}\frac{\partial u}{\partial\nu}|^2 dS(x) \le C\int_\Omega |e^{-\tau\langle x,\xi\rangle}(\Delta - q)u|^2 dx}$$

We need $\boxed{\langle \nu(x), \xi\rangle \ge \delta > 0}$

Bukhgeim-U $(n \geq 3)$

$$\Lambda_{q_1}(f)|_{\partial\Omega_{-,\delta}(\xi)} = \Lambda_{q_2}(f)|_{\partial\Omega_{-,\delta}(\xi)} \quad \forall f \quad \Longrightarrow \quad q_1 = q_2$$

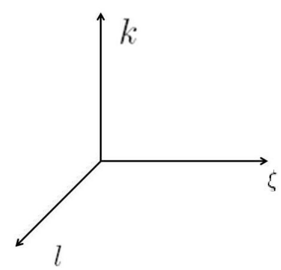
$$\partial\Omega_{-,\delta}(\xi) = \{x \in \partial\Omega; \langle \nu, \xi \rangle < \delta\}$$



Sketch of proof

Choose $\boxed{u_2 = e^{x\cdot\rho_2}(1 + \Psi_{q_2}(x, \rho_2))}$
solution of $(\Delta - q_2)u_2 = 0$

$$\rho_2 = \tau\xi + i(k + l)$$
$$|k|^2 + |l|^2 = \tau^2$$



Let u_1 be such that $\boxed{u_1|_{\partial\Omega} = u_2|_{\partial\Omega}, \dfrac{\partial u_1}{\partial \nu}|_{\partial\Omega_{-,\delta}(\xi)} = \dfrac{\partial u_2}{\partial \nu}|_{\partial\Omega_{-,\delta}(\xi)}}$

$$\boxed{u_2 = e^{x\cdot\rho_2}(1+\Psi_{q_2}(x,\rho_2))}$$

$(\Delta - q_2)u_2 = 0,\ \rho_2 = \tau\xi + i(k+l)$

$$\boxed{u_1|_{\partial\Omega} = u_2|_{\partial\Omega},\ \frac{\partial u_1}{\partial\nu_1}|_{\partial\Omega_{-,\delta}(\xi)} = \frac{\partial u_2}{\partial\nu_1}|_{\partial\Omega_{-,\delta}(\xi)}}$$

$u = u_1 - u_2, \quad q = q_1 - q_2$

$$\boxed{v_1 = e^{x\cdot\rho_1}(1+\Psi_{q_1}(x,\rho_1))} \quad \rho_1 = -\tau\xi + i(k-l)$$

solution of $(\Delta - q_1)v_1 = 0$.

$$\boxed{\int_\Omega q u_2 v_1 dx = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}v_1 dS}$$

Note that $u|_{\partial\Omega} = 0$, $\frac{\partial u}{\partial\nu}|_{\partial\Omega_{-,\delta}(\xi)} = 0$

$q = q_1 - q_2$

$$(*) \qquad \boxed{\int_\Omega q u_2 v_1 dx = \int_{\partial\Omega_{+,\delta}(\xi)} \frac{\partial u}{\partial \nu} v_1 dS}$$

$$\boxed{\begin{array}{l} u_2 = e^{x\cdot\rho_1}(1+\Psi_{q_1}) \\ v_1 = e^{x\cdot\rho_2}(1+\Psi_{q_2}) \end{array}}$$

Fix $k \in \mathbb{R}^n$, $\rho_1 + \rho_2 = 2ik$

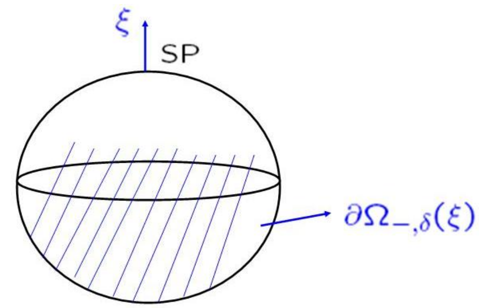
Carleman estimate

$$\tau \int_{\partial\Omega_+} \langle \xi, \nu(x)\rangle |e^{-\tau\langle x,\xi\rangle} \frac{\partial u}{\partial \nu}|^2 dS(x) \le C \int_\Omega |(\Delta - q_1) u e^{-\tau\langle x,\xi\rangle}|^2 dS(x)$$

Need $\langle x, \xi\rangle \ge \delta > 0$. $|\text{RHS}| \le C$ as $|\rho| \to \infty$, LHS$\to \boxed{\int_\Omega q e^{2ix\cdot k} dx}$

We get $\boxed{\int_\Omega e^{2ix\cdot k} q(x)dx = 0}$
$k \perp \xi$. But we can move ξ a little bit

$\xi \in \mathbb{S}^{n-1}$



$$\boxed{\partial\Omega_{-,\delta}(\xi) = \{x \in \partial\Omega; \langle \nu(x), \xi \rangle < \delta\}}$$

We obtain $\widehat{\chi_\Omega q}(-2k) = 0$ in an open cone $\Longrightarrow$ $q = 0$.

Carleman estimate $\Longrightarrow$ control of $\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega_{+,\delta}}$ (with appropriate $\boxed{\text{linear}}$ weights) (Stability estimates, Heck-Wang)

Theorem (Kenig-Sjöstrand-U) Ω strictly convex.

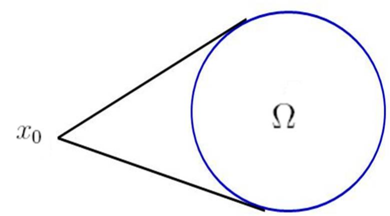
$$\Lambda_{q_1}\Big|_{\Gamma} = \Lambda_{q_2}\Big|_{\Gamma}, \quad \Gamma \subseteq \partial\Omega, \quad \Gamma \text{ arbitrary}$$

$$\Rightarrow q_1 = q_2$$

$\boxed{u_\tau = e^{\tau(\varphi + i\psi)} a_\tau}$ $\boxed{\varphi(x) = \ln|x - x_0|, x_0 \notin \overline{ch(\Omega)}}$

Eikonal: $\boxed{\nabla\varphi \cdot \nabla\psi = 0, |\nabla\varphi| = |\nabla\psi|}$

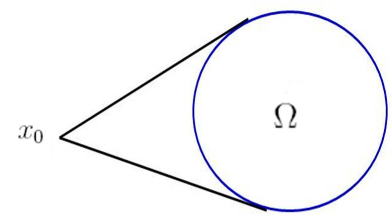
$\psi(x) = d(\frac{x - x_0}{|x - x_0|}, \omega), \omega \in S^{n-1}$: smooth for $x \in \bar{\Omega}$.



Transport: $\boxed{(\nabla\varphi + i\nabla\psi) \cdot \nabla a_\tau = 0}$

(Cauchy-Riemann equation in plane generated by $\nabla\varphi, \nabla\psi$)

$\varphi(x) = \ln|x - x_0|, \quad x_0 \notin \overline{ch(\Omega)}$



Carleman Estimates

$u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega_-} = 0 \qquad \partial\Omega_\pm = \{x \in \partial\Omega; \nabla\varphi \cdot \nu \gtrless 0\}$

$$\boxed{\int_{\partial\Omega_+} < \nabla\varphi, \nu > |e^{-\tau\varphi(x)}\frac{\partial u}{\partial \nu}|^2 ds \le \frac{C}{\tau}\int_\Omega |(\Delta - q)ue^{-\tau\varphi(x)}|^2 ds}$$

This gives control of $\frac{\partial u}{\partial \nu}|_{\partial\Omega_{+,\delta}}$,

$$\boxed{\partial\Omega_{+,\delta} = \{x \in \partial\Omega, \nabla\varphi \cdot \nu \ge \delta\}}$$

More general CGO solutions

$$\boxed{u_\tau = e^{\tau(\varphi + i\psi)} a_\tau,}$$

$\tau \gg 0$, $\tau = 1/h$ (semicl.), φ, ψ real-valued

- φ is a limiting Carleman weight

$$e^{\frac{\varphi}{h}} h^2(-\Delta + q) e^{-\frac{\varphi}{h}}$$

has semiclassical principal symbol

$$P_\varphi(x, \xi) = \xi^2 - (\nabla\varphi)^2 + 2i\nabla\varphi \cdot \xi$$

Hörmander's condition:

$$\{\mathrm{Re}\ P_\varphi, \mathrm{Im}\ P_\varphi\} \leq 0 \quad \text{on} \quad P_\varphi = 0$$

We need $\varphi, -\varphi$ to be phase of solutions.

$$\text{LCW}: \quad \boxed{\{\mathrm{Re}\ P_\varphi, \mathrm{Im}\ P_\varphi\} = 0}$$

$\nabla\varphi \neq 0$ in an open neighborhood of $\bar{\Omega}$.

CGO solutions $\quad u_h = e^{\frac{1}{h}(\varphi + i\psi)} a_h$

- φ LCW, φ real-valued

$$\{\mathrm{Re} P_\varphi, \mathrm{Im} P_\varphi\} = 0 \quad \text{on} \quad P_\varphi = 0$$

$\nabla \varphi \neq 0$ on an open neighborhood of $\bar{\Omega}$.

Examples (Dos Santos Ferreira-Kenig-Salo-U, 2009)

(a) $\varphi(x) = x \cdot \xi$, $\xi \in \mathbb{R}^n$, $|\xi| = 1$

(b) $\varphi(x) = a \ln |x - x_0| + b$, $(a, b$ constants$)$, $x_0 \notin \overline{ch(\Omega)}$

(c) $\varphi(x) = \dfrac{a\langle x - x_0, \xi\rangle}{|x - x_0|^2} + b$, $\xi \in \mathbb{R}^n$

(d) $\varphi(x) = a \arctan \frac{2\langle x - x_0, \xi\rangle}{|x - x_0|^2 - |\xi|^2} + b$

(e) $\varphi(x) = a \operatorname{arctanh} \frac{2\langle x - x_0, \xi\rangle}{|x - x_0|^2 - |\xi|^2} + b$

(f) $n = 2$, φ is a harmonic function

Instead of

$$\boxed{\int_\Omega e^{2ix\cdot k} q(x)\,dx = 0}$$

$k \perp \xi$ ($\xi \in \mathbb{S}^{n-1}$) as in Bukhgeim-U argument we get

$$\boxed{\int_\Omega e^{i\lambda f(x)} \boxed{q(x)}\, a_1 a_2\, dx = 0}$$

λ any real number, $a_1, a_2 \neq 0$, $f(x)$ real-analytic, a_1, a_2 real analytic

Analytic microlocal analysis $\Longrightarrow q = 0$ (like inversion of real-analytic Radon

$\boxed{\textbf{Linearization}}$ (Analog of Calderón)

<u>Theorem</u> (Dos Santos Ferreira, Kenig, Sjöstrand-U)

$$\boxed{\int_\Omega h u v = 0}$$

$$\Gamma \subseteq \partial\Omega,\ \Gamma \text{ open},$$

$$\Delta u = \Delta v = 0, \quad u, v \in C^\infty(\overline{\Omega}),$$

$$\text{supp } u|_{\partial\Omega},\ \text{supp } v|_{\partial\Omega} \subseteq \Gamma,$$

$$\Rightarrow \boxed{h = 0}.$$

$\boxed{\textbf{Complex Spherical Waves}}$

$$\boxed{u_\tau = e^{\tau(\varphi + i\psi)} a_\tau}$$

$$\varphi(x) = \ln|x - x_0|, \; x_0 \notin \overline{\mathrm{ch}(\Omega)}$$

Also used to determine inclusions, obstacles, etc.

a) Conductivity Ide-Isozaki-Nakata-Siltanen-U

b) Helmholtz Nakamura-Yosida

c) Elasticity J.-N. Wang-U

d) 2D Systems J.-N. Wang-U

e) Maxwell T. Zhou

$\boxed{\text{Complex Spherical Waves}}$

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$\boxed{\text{The Two Dimensional Case}}$

Theorem $(n = 2)$ Let $\gamma_j \in C^2(\overline{\Omega})$, $j = 1, 2$.

Assume $\boxed{\Lambda_{\gamma_1} = \Lambda_{\gamma_2}}$. Then $\boxed{\gamma_1 = \gamma_2}$.

- Nachman (1996)
- Brown-U (1997) Improved to γ_j Lipschitz
- Astala-Päivärinta (2006) Improved to $\gamma_j \in L^\infty(\Omega)$

Recall

$$\begin{gathered} \operatorname{div}(\gamma \nabla u) = 0, \quad \gamma \in L^\infty(\Omega) \\ u|_{\partial\Omega} = f \end{gathered}$$

$$Q_\gamma(f) = \int_\Omega \gamma |\nabla u|^2 dx = \langle \Lambda_\gamma f, f \rangle_{L^2(\partial\Omega)}.$$

This follows from more general result

Theorem ($n = 2$, Bukhgeim, 2008) Let $q_j \in L^\infty(\Omega)$, $j = 1, 2$.

Assume $\boxed{\Lambda_{q_1} = \Lambda_{q_2}}$. Then $\boxed{q_1 = q_2}$.

Recall

$$\begin{gathered}(\Delta - q)u = 0, \\ u|_{\partial\Omega} = f.\end{gathered} \qquad \Lambda_q(f) = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$$

with ν-unit outer normal.

$$\boxed{\Lambda_{q_1} = \Lambda_{q_2}} \;\Rightarrow\; \boxed{q_1 = q_2}$$

Sketch of proof New class of CGO solutions

$$\boxed{\begin{array}{c} u_1(z,\tau) = e^{\tau z^2}(1 + r_1(z,\tau)) \\ u_2(z,\tau) = e^{-\tau \bar{z}^2}(1 + r_2(z,\tau)) \end{array}} \qquad \tau \gg 1$$

solve $(\Delta - q_j)u_j = 0$ with $r_j(z,\tau) \to 0$ on Ω sufficiently fast.

Notation $z = x_1 + ix_2$

Remark $z^2 = x_1^2 - x_2^2 + 2ix_1x_2 = \varphi + i\psi$

$$\nabla\varphi \cdot \nabla\psi = 0, \quad |\nabla\varphi| = |\nabla\psi|$$

φ harmonic, ψ conjugate harmonic.

$$\Lambda_{q_1} = \Lambda_{q_2} \ \Rightarrow \ \boxed{\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0}$$

$$(\Delta - q_j) u_j = 0$$

$$u_1 = e^{\tau z^2}\left(1 + r_1(z,\tau)\right), \quad u_2 = e^{-\tau \bar{z}^2}\left(1 + r_2(z,\tau)\right)$$

Substituting

$$\int_\Omega (q_1 - q_2) e^{4i\tau x_1 x_2}(1 + r_1 + r_2 + r_1 r_2) dx = 0.$$

Letting $\tau \to \infty$ and using stationary phase

$$(q_1 - q_2)(0) = 0.$$

Changing z to $z - z_0$ we get

$$(q_1 - q_2)(z_0) = 0.$$

$\boxed{\text{Partial data}}$

Let $\Gamma \subseteq \partial\Omega$, Γ open.
Let $q_j \in C^{1+\varepsilon}(\Omega)$, $\varepsilon > 0$, $j = 1, 2$.

Theorem (Imanuvilov-U-Yamamoto 2010) n=2. Assume

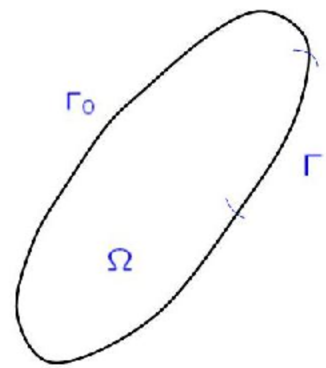
$$\boxed{\Lambda_{q_1}(f)\Big|_{\Gamma} = \Lambda_{q_2}(f)\Big|_{\Gamma}}$$

$\forall\, f$, supp$f \subseteq \Gamma$. Then

$$\boxed{q_1 = q_2}.$$

- Riemann Surfaces: Guillarmou-Tzou (2011)

$\boxed{\text{Partial data}}$ $\Gamma_0 = \partial\Omega - \Gamma$



Construct CGO solutions

$$\boxed{\begin{array}{c} \Delta u_j - q_j u_j = 0 \quad \text{in } \Omega \\ u_j|_{\Gamma_0} = 0 \end{array}}$$

In this case

$$\boxed{\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0}$$

if $\Lambda_{q_1}(f)|_\Gamma = \Lambda_{q_2}(f)|_\Gamma$, $\text{supp} f \subseteq \Gamma$.

$$\int_\Omega (q_1 - q_2) u_1 u_2 = 0$$

$$u_j|_{\Gamma_0} = u_j|_{\partial\Omega - \Gamma} = 0$$

$$\boxed{u_1(x) = e^{\tau\Phi(z)}(a(z) + \frac{a_0(z)}{\tau}) + \overline{e^{\tau\Phi(z)}(a(z) + \frac{a_1(z)}{\tau})} + e^{\tau\varphi} R_\tau^{(1)}}$$

$$\boxed{u_2(x) = e^{-\tau\bar{\Phi}(z)}(\bar{a}(z) + \frac{b_0(z)}{\tau}) + \overline{e^{-\tau\bar{\Phi}(z)}(\bar{a}(z) + \frac{b_1(z)}{\tau})} + e^{\tau\varphi} R_\tau^{(2)}}$$

$\Phi = \varphi + i\psi$ holomorphic

$$u_1 = \text{Re } e^{\tau\Phi(z)}(a(z) + \cdots), \quad u_2 = \text{Re } e^{-\tau\bar{\Phi}(z)}(\bar{a}(z) + \cdots)$$

$$\boxed{\Phi(z) = \varphi + i\psi} \quad \text{holomorphic}$$

$$\boxed{u_j|_{\partial\Omega - \Gamma} = 0}$$

$p \in \Omega$, Φ has non-degenerate critical point at p (Morse function)

$$\bar{\partial} a = 0 \quad \text{Re } a|_{\partial\Omega-\Gamma} = 0$$

$a = 0$ at other critical points

Stationary phase in

$$\boxed{\int_\Omega (q_1 - q_2) u_1 u_2 = 0}$$

$$\boxed{\begin{gathered} u_1 = \text{Re } e^{\tau\Phi(z)}(a(z) + \cdots) \\ u_2 = \text{Re } e^{-\tau\bar{\Phi}(z)}(\bar{a}(z) + \cdots) \\ u_j|_{\partial\Omega - \Gamma} = 0 \end{gathered}}$$

$\Phi(z)$ Morse function with non-degenerate critical point at p.

$$\int_\Omega (q_1 - q_2)u_1 u_2 = 0$$

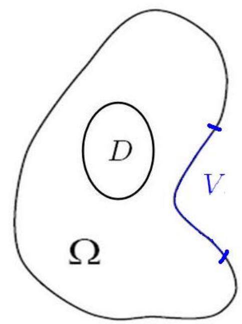
Stationary phase

$$\Longrightarrow \boxed{(q_1 - q_2)(p) = 0}$$

$\boxed{\textbf{Corollary:}}$ Obstacle Problem

$\boxed{\Omega, D \subset \mathbb{R}^2}$: smooth boundary such that $\bar{D} \subset \Omega$.

$\boxed{V \subset \partial\Omega}$: open set.



Let $\boxed{q_j \in C^{2+\alpha}(\overline{\Omega \backslash D})}$ for some $\alpha > 0$, $j = 1, 2$.

$$\widetilde{C}_{q_j} := \Big\{ (u|_V, \partial_\nu u|_V); (\Delta - q_j)u = 0 \text{ in } \Omega\backslash\bar{D} \\ \text{supp } u|_{\partial\Omega} \subset V, u \in H^1(\Omega\backslash\bar{D}) \Big\}$$

Then $\boxed{\widetilde{C}_{q_1} = \widetilde{C}_{q_2} \Longrightarrow q_1 = q_2}$.

Carleman Estimate With Degenerate Weights

Lemma 1

Let $\partial\Omega - \Gamma = \{x \in \partial\Omega;\ \nu \cdot \nabla\varphi = 0\}$. Then for τ sufficiently large, $\exists$ solution of

$$\boxed{\begin{array}{c} \Delta u - qu = f \quad \text{in} \quad \Omega \\ u|_{\partial\Omega-\Gamma} = g \end{array}}$$

such that

$$\|ue^{-\tau\varphi}\|_{L^2(\Omega)} \le C\left(|\tau|^{-1/2}\|fe^{-\tau\varphi}\|_{L^2(\Omega)} + \|ge^{-\tau\varphi}\|_{L^2(\Omega-\Gamma)}\right)$$

Phase Function

Lemma 2 (Vekua) Given points $x_1, ...x_N$ in Ω and constants $b_i, i = 1, 2$, $C_{0,j}, C_{1,j}, C_{2,j}, j = 1, ..., N$, there exists an open and dense set

$$\Theta \subseteq \overline{C^2(\partial\Omega - \Gamma)} \times \overline{C^2(\partial\Omega - \Gamma)} \times \mathbb{C}^{3N}$$

solution of

$$\phi + i\psi \text{ holomorphic in } \Omega,$$

$$\boxed{\begin{array}{c} (\phi, \psi)|_{\partial\Omega - \Gamma} = (b_1, b_2) \\ (\phi + i\psi)(x_j) = C_{0,j} \\ \frac{\partial}{\partial z}(\phi + i\psi)(x_j) = C_{1,j} \\ \frac{\partial^2}{\partial z^2}(\phi + i\psi)(x_j) = C_{2,j} \end{array}}$$

$\boxed{\textbf{CGO Solutions } (n=2)}$

$$\boxed{\partial_{\bar{z}}^{-1} g(z) := -\frac{1}{\pi}\int_{\Omega} \frac{g(\xi_1,\xi_2)}{\xi_1 + i\xi_2 - z}\, d\xi_1 d\xi_2, \qquad \partial_z^{-1} g := \overline{\partial_{\bar{z}}^{-1} \bar{g}}}$$

$$\boxed{R_{\Phi,\tau} g := e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_{\bar{z}}^{-1} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})})}$$

$$\boxed{\widetilde{R}_{\Phi,\tau} g := e^{\tau(\overline{\Phi(z)} - \Phi(z))} \partial_{z}^{-1} (g e^{\tau(\Phi(z) - \overline{\Phi(z)})})}$$

$\mathcal{H} =$ non-degenerate critical points of Φ.

CGO Solutions ($n = 2$)

$$\Delta u_1 - q_1 u_1 = 0 \quad \text{in } \Omega$$

$$u_1\Big|_{\partial\Omega\setminus\Gamma} = 0.$$

Let Φ be a holomorphic Morse function, such that $\mathrm{Im}\Phi = 0$ on $\partial\Omega\setminus\Gamma$. Let $\Phi = \varphi + i\psi$.

$$\begin{aligned} u_1(x) \;&=\; e^{\tau\Phi(z)}(a(z) + a_0(z)/\tau) + e^{\tau\overline{\Phi z}}\overline{(a(z) + a_0(z)/\tau)} \\ &\quad + e^{\tau\varphi}u_{11} + e^{\tau\varphi}u_{12}. \end{aligned}$$

$$u_1(x) = e^{\tau\Phi(z)}(a(z)+a_0(z)/\tau) + e^{\tau\overline{\Phi z}}\overline{(a(z)+a_0(z)/\tau)} + e^{\tau\varphi}u_{11} + e^{\tau\varphi}u_{12}.$$

Choice of a, a_0, a_1:

$$a, a_0, a_1 \in C^2(\bar{\Omega}), \quad \partial_{\bar{z}} a = \partial_{\bar{z}} a_0 = \partial_{\bar{z}} a_1 \equiv 0$$

$$\operatorname{Re} a\Big|_{\partial\Omega\setminus\Gamma} = 0, \quad a = \partial_z a = 0 \quad \text{on } \mathcal{H} \cap \partial\Omega.$$

$$(a_0(z) + \overline{a_1(z)})\Big|_{\partial\Omega\setminus\Gamma} = \frac{\partial_{\bar{z}}^{-1}(aq_1) - M_1(z)}{4\partial_z\Phi} + \frac{\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\bar{z})}{4\overline{\partial_z\Phi}}$$

The polynomials $M_1(z)$ and $M_3(z)$ satisfy

$$\boxed{\begin{array}{l} \partial_z^j\Big(\partial_{\bar z}^{-1}(aq_1) - M_3(z)\Big) = 0, \;\; z \in \mathcal{H}, j = 0,1,2 \\ \partial_{\bar z}^j\Big(\partial_z^{-1}(\bar a q_1) - M_3(\bar z)\Big) = 0, \;\; z \in \mathcal{H}, j = 0,1,2. \end{array}}$$

Let $e_i, i = 1,2$ be smooth, $e_1 + e_2 = 1$ on $\overline{\Omega}$ with $e_1 = 0$ in a neighborhood of $\mathcal{H} - \partial\Omega$ and $e_2 = 1$ in a neighborhood of $\partial\Omega$.

Remainder term

Choice of u_{11}:

$$
\begin{aligned}
u_{11} \;=\; & -\frac{1}{4}e^{i\tau\psi}\widetilde{R}_{\Phi,\tau}\Big(e_1\Big(\partial_{\bar{z}}^{-1}(aq_1) - M_1(z)\Big)\Big) \\
& -\frac{1}{4}e^{-i\tau\psi}R_{\Phi,-\tau}\Big(e_1\Big(\partial_z^{-1}(\overline{a(z)})q_1 - M_3(\bar{z})\Big)\Big) \\
& -\frac{e^{i\tau\psi}}{\tau}\,\frac{e_2\Big(\partial_{\bar{z}}^{-1}(aq_1) - M_1(z)\Big)}{4\partial_z\Phi} \\
& -\frac{e^{i\tau\psi}}{\tau}\,\frac{e_2\Big(\partial_z^{-1}(\overline{a(z)}q_1) - M_3(\bar{z})\Big)}{4\overline{\partial_z\Phi}}
\end{aligned}
$$

Other remainder term

Find u_{12} such that

$$\Delta(u_{12}e^{\tau\varphi}) - q_1 u_{12} e^{\tau\varphi} - q_1 u_{11} e^{\tau\varphi} + h_1 e^{\tau\varphi} \quad \text{in } \Omega,$$

$$\begin{aligned} u_{12}\Big|_{\partial\Omega\setminus\Gamma} &= \frac{1}{4}\widetilde{R}_{\Phi,\tau}\Big(e_1\Big(\partial_{\bar{z}}^{-1}(a(z)q_1) - M_1(z)\Big)\Big) \\ &+ \frac{1}{4}R_{\Phi,-\tau}\Big(e_1\Big(\partial_z^{-1}(\overline{a(z)})q_1 - M_3(\bar{z})\Big)\Big). \end{aligned}$$

$$\|u_{12}\|_{L^2(\Omega)} = o\left(\frac{1}{\tau}\right), \quad \tau \to \infty.$$

Here

$$\boxed{\begin{aligned} h_1 \;&=\; e^{i\tau\psi}\Delta\left(\frac{e_2\Big(\partial_{\bar z}^{-1}(a(z)q_1) - M_1(z)\Big)}{4\tau\partial_z\Phi}\right) \\ &\quad + e^{-i\tau\psi}\Delta\left(\frac{e_2\Big(\partial_{z}^{-1}(\overline{a(z)}q_1) - M_3(\bar z)\Big)}{4\tau\overline{\partial_z\Phi}}\right) \\ &\quad - \frac{a_0 q_1}{\tau}e^{i\tau\psi} - \frac{\overline{a_1} q_1}{\tau}e^{-i\tau\psi}. \end{aligned}}$$

Similarly:

$$\boxed{\Delta v - q_2 v = 0 \quad \text{in } \Omega, \qquad v\Big|_{\partial\Omega\setminus\Gamma} = 0.}$$

Construct solution v of the form

$$\begin{aligned} v(x) &= e^{-\tau\Phi(z)}\Big(a(z) + b_0(z)/\tau\Big) + e^{-\tau\overline{\Phi(z)}}\Big(\overline{a(z) + b_0(z)/\tau}\Big) \\ &\quad + e^{-\tau\varphi}v_{11} + e^{-\tau\varphi}v_{12}. \end{aligned}$$

Main Term

$$R = \int_\Omega (q_1 - q_2)(a(a_0 + b_0) + \bar{a}(\overline{a_1} + \overline{b_1}))dx$$

$$+\frac{1}{4}\int_\Omega (q_1 - q_2)\left(a\,\frac{\partial_{\bar{z}}^{-1}(aq_2) - M_2(z)}{\partial_z \Phi} + \bar{a}\,\frac{\partial_{\bar{z}}^{-1}(\bar{a}q_2) - M_4(\bar{z})}{\overline{\partial_z \Phi}}\right)dx$$

$$+\frac{1}{4}\int_\Omega (q_1 - q_2)\left(a\,\frac{\partial_{\bar{z}}^{-1}(aq_1) - M_1(z)}{\partial_z \Phi} + \bar{a}\,\frac{\partial_{\bar{z}}^{-1}(\bar{a}q_1) - M_3(\bar{z})}{\overline{\partial_z \Phi}}\right)dx$$

Proof of Uniqueness for Partial Data

- Take geometiric optics solution u_1 to

$$\boxed{\Delta u_1 - q_1 u_1 = 0, \quad u_1\Big|_{\partial\Omega\setminus\Gamma} = 0.}$$

- u_2: $\boxed{\Delta u_2 - q_2 u_2 = 0, \quad u_2\Big|_{\partial\Omega} = u_1\Big|_{\partial\Omega}}$.

  DN maps are equal $\Rightarrow \nabla u_2 = \nabla u_1$ on Γ.

$$\boxed{\begin{aligned} u = u_1 - u_2 \ \Rightarrow \ \Delta u - q_2 u &= (q_1 - q_2) u_2 \\ u\Big|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\Gamma} &= 0. \end{aligned}}$$

- Take complex geometric optics solution v to

$$\boxed{\Delta v - q_2 v = 0, \quad v\Big|_{\partial\Omega\setminus\Gamma} = 0.}$$

$$\boxed{0 = \int_\Omega v(\Delta u - q_2 u)dx = -\int_\Omega (q_1 - q_2) v u_1 dx :}$$

Stationary phase + estimates for $u_{12} \Rightarrow$

$$\boxed{2\sum_{k=1}^{l} \frac{\pi((q_1 - q_2)|a|^2)(\widetilde{x_k})\mathrm{Re}\, e^{i2\tau \mathrm{Im}\,\Phi(\widetilde{x_k})}}{|(\det \mathrm{Im}\,\Phi'')(\widetilde{x_k})|^{\frac{1}{2}}} + R = o(1), \quad \text{as } \tau \to \infty.}$$

[left side] = almost perodic function in τ.

Bohr's theorem inplies [left side] $= 0$ for all τ.

Phase function

We can choose Φ such that

$$\boxed{\operatorname{Im}\Phi(\widetilde{x_k}) \neq \operatorname{Im}\Phi(\widetilde{x_j}), \quad j \neq k.}$$

Let $a(\widetilde{x_k}) \neq 0$. Then stationary phase implies

$$\boxed{q_1(\widetilde{x_k}) = q_2(\widetilde{x_k}).}$$

Partial Data for Second Order Elliptic Equations ($n = 2$)

(Imanuvilov–U–Yamamoto, 2011)

$$\boxed{\Delta_g + A(z)\frac{\partial}{\partial z} + B(z)\frac{\partial}{\partial \overline{z}} + q} \qquad z = x_1 + ix_2$$

$g = (g_{ij})$ positive definite symmetric matrix;

$$\Delta_g u = \frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det(g)}\, g^{ij} \frac{\partial u}{\partial x_j}\right) \qquad g^{ij} = (g_{ij})^{-1}$$

Includes:

- Anisotropic Calderón's Problem
- Magnetic Schrödinger Equation
- Convection terms

Anisotropic case

Cardiac muscle	6.3 mho	(longitudinal)
	2.3 mho	(transversal)

$\boxed{\gamma = (\gamma^{ij})}$ positive-definite, symmetric matrix

conductivity

$\Omega \subseteq \mathbb{R}^n, \Omega$ bounded. Under assumptions of no sources or sinks of current the potential u satisfies

$\boxed{\text{div}(\gamma \nabla u) = 0}$

$$\boxed{\begin{aligned} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_i}\right) &= 0 \text{ in } \Omega \\ u\big|_{\partial\Omega} &= f \end{aligned}} \qquad (*)$$

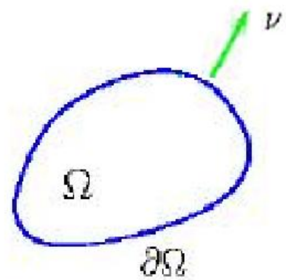
f = voltage potential at boundary

Isotropic

$$\gamma^{ij}(x) = \alpha(x)\delta^{ij}; \delta^{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) = 0 \text{ in } \Omega \qquad (*)$$
$$u\big|_{\partial\Omega} = f$$

$$\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i \gamma^{ij} \frac{\partial u}{\partial x_j}\bigg|_{\partial\Omega}$$



$\nu = \left(\nu^1, \cdots, \nu^n\right)$ is the unit outer normal to $\partial\Omega$

$\Lambda_\gamma(f)$ is the induced current flux at $\partial\Omega$.

Λ_γ is the voltage to current map or Dirichlet - to - Neumann map

$$\boxed{\begin{aligned} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) &= 0 \text{ in } \Omega \\ u\Big|_{\partial\Omega} &= f \end{aligned}} \qquad (*)$$

$$\boxed{\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i \gamma^{ij} \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}}$$

EIT: Can we recover γ in Ω from Λ_γ ?

$$\boxed{\begin{aligned} \operatorname{div}(\gamma \nabla u) &= 0 \\ u\big|_{\partial\Omega} &= f \end{aligned}}$$

$$\boxed{\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \gamma^{ij} \nu^i \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}}$$

$$\boxed{\Lambda_\gamma \Rightarrow \gamma \ ?}$$

Answer: No

$$\boxed{\Lambda_{\psi_* \gamma} = \Lambda_\gamma}$$

where $\psi : \Omega \to \Omega$ change of variables

$$\psi|_{\partial\Omega} = \text{Identity}$$

$$\boxed{\psi_* \gamma = \left(\frac{(D\psi)^T \circ \gamma \circ D\psi}{|\det D\psi|} \right) \circ \psi^{-1}}$$

$$\boxed{v = u \circ \psi^{-1}}$$

Theorem (Imanuvilov–U–Yamamoto, 2011) $\Omega \subset \mathbb{R}^2$, $\Gamma \subset \partial\Omega$, Γ open, $\gamma_k = (\gamma_k^{ij}) \in C^\infty(\bar{\Omega}), k = 1, 2$, positive definite symmetric. Assume

$$\Lambda_{\gamma_1}(f)|_\Gamma = \Lambda_{\gamma_2}(f)|_\Gamma, \quad \forall f \ \ \text{supp} f \subset \Gamma.$$

Then $\exists F : \bar{\Omega} \to \bar{\Omega}$, C^∞ diffeomorphism, $F|_\Gamma =$ Identity such that

$$F_* \gamma_1 = \gamma_2.$$

Full Data ($\Gamma = \partial\Omega$):

- $\gamma_k \in C^2(\bar{\Omega})$, Nachman (1996)
- γ_k Lipschitz, Sun–U (2001)
- $\gamma_k \in L^\infty(\Omega)$, Astala–Lassas–Pävärinta (2006)

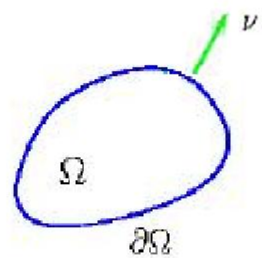
DIRICHLET-TO-NEUMANN MAP (Lee-U, 1989)

(M,g) compact Riemannian manifold with boundary.

Δ_g Laplace-Beltrami operator $g=(g_{ij})$ pos. def. symmetric matrix

$$\boxed{\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j}\right)} \quad (g^{ij}) = (g_{ij})^{-1}$$

$$\boxed{\begin{aligned} \Delta_g u &= 0 \text{ on } M \\ u\big|_{\partial M} &= f \end{aligned}}$$



Conductivity:
$\gamma^{ij} = \sqrt{\det g}\, g^{ij}$

$$\boxed{\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\Big|_{\partial M}}$$

$\nu = (\nu^1, \cdots, \nu^n)$ unit-outer normal

$$\Delta_g u = 0$$
$$u\Big|_{\partial M} = f$$

$$\boxed{\Lambda_g(f) = \frac{\partial u}{\partial \nu_g} = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\Big|_{\partial M}}$$

current flux at ∂M

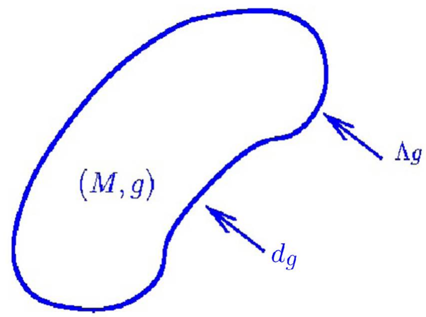
Inverse-problem (EIT)

Can we recover $\boxed{g}$ from $\boxed{\Lambda_g}$?

$\Lambda_g =$ Dirichlet-to-Neumann map or voltage to current map

ANOTHER MOTIVATION (STRING THEORY)

$\boxed{\text{HOLOGRAPHY}}$



Dirichlet-to-Neumann map is the "boundary-2pt function"

Inverse problem: Can we recover (M, g) (bulk) from boundary-2pt function ?

M. Parrati and R. Rabadan, Boundary rigidity and holography, JHEP 0401 (2004) 034

$$\boxed{\begin{aligned} \Delta_g u &= 0 \\ u\big|_{\partial M} &= f \end{aligned}}$$

$$\boxed{\Lambda_g(f) = \frac{\partial u}{\partial \nu_g}\bigg|_{\partial M}}$$

$$\boxed{\Lambda_g \Rightarrow g \quad ?}$$

Answer: No $\boxed{\Lambda_{\psi^* g} = \Lambda_g}$ where

$\psi : M \to M$ diffeomorphism, $\psi\big|_{\partial M} =$ Identity and

$$\boxed{\psi^* g = \left(D\psi \circ g \circ (D\psi)^T\right) \circ \psi}$$

Show $\Lambda_{\psi^* g} = \Lambda_g$; $\psi : M \to M$ diffeomorphism, $\psi\Big|_{\partial M} = \text{Identity}$

$$Q_g(f) = \sum_{i,j} \int_M g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \sqrt{\det g}\, dx$$

$$\boxed{Q_g(f) = -\int_{\partial M} \Lambda_g(f) f\, dS}$$

$$\boxed{Q_g \Leftrightarrow \Lambda_g}$$

$$v = u \circ \psi, \ \Delta_{\psi^* g} v = 0$$

$$\boxed{Q_{\psi^* g} = Q_g} \Rightarrow \boxed{\Lambda_{\psi^* g} = \Lambda_g}$$

Theorem ($n \geq 3$) (Lassas-U 2001, Lassas-Taylor-U 2003) $(M, g_i), i = 1, 2$, real-analytic, connected, compact, Riemannian manifolds with boundary. Let $\Gamma \subseteq \partial M$, Γ open. Assume

$$\boxed{\Lambda_{g_1}(f)|_\Gamma = \Lambda_{g_2}(f)|_\Gamma,} \quad \forall f,\ f \text{ supported in } \Gamma$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi\Big|_\Gamma =$ Identity, so that

$$\boxed{g_1 = \psi^* g_2}$$

In fact one can determine topology of M, as well (only need to know $\Lambda_g, \partial M$).

Theorem (Guillarmou-Sa Barreto, 2009) $(M, g_i), i = 1, 2$, are compact Riemannian manifolds with boundary that are Einstein. Assume

$$\boxed{\Lambda_{g_1} = \Lambda_{g_2}}$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi|_{\partial M} = Identity$ such that

$$\boxed{g_1 = \psi^* g_2}$$

Note: Einstein manifolds with boundary are real analytic in the interior.

Theorem ($n = 2$)(Lassas-U, 2001)

(M, g_i), $i = 1, 2$, connected Riemannian manifold with boundary. Let $\Gamma \subseteq \partial M$, Γ open. Assume

$$\boxed{\Lambda_{g_1}(f)|_\Gamma = \Lambda_{g_2}(f)|_\Gamma,} \quad \forall f,\ f \text{ supported in } \Gamma$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi\Big|_\Gamma =$ Identity, and $\beta > 0, \beta\Big|_\Gamma = 1$ so that

$$\boxed{g_1 = \beta \psi^* g_2}$$

In fact, one can determine topology of M as well.

$\boxed{\textbf{Moding Out the Diffeomorphism Group}}$

Some conformal class $\boxed{\Lambda_{\beta g} = \Lambda_g}, \quad \beta \in C^\infty(M)$

$$\Longrightarrow \beta = 1?$$

More general problem

$$\boxed{\begin{array}{c} (\Delta_g - q)u = 0, \ q \in C^\infty(M) \\ u|_{\partial M} = f, \\ \Lambda_g(f) = \frac{\partial u}{\partial \nu_g}|_{\partial M}. \end{array}}$$

Inverse Problem: Does $\boxed{\Lambda_g}$ determines $\boxed{q}$?

$(\Delta_g - q)u = 0, \qquad \Lambda_g(f) = \frac{\partial u}{\partial \nu_g}|_{\partial M}, \qquad \boxed{\Lambda_g \to q?}$

Theorem (n=2) (Guillarmou-Tzou, 2009)

$\boxed{\text{YES}}$

Earlier results:

- $\mathbb{R}^2$, q small (Sylvester-U, 1986)
- $\mathbb{R}^2$, q generic (Sun-U, 2001)
- $\mathbb{R}^2$, $q = \frac{\Delta\sqrt{\lambda}}{\sqrt{\lambda}}, \gamma > 0$ (Nachmann 1996)
- Riemannian surfaces, $q = \frac{\Delta\sqrt{\lambda}}{\sqrt{\lambda}}, \gamma > 0$, (Henkin-Michel, 2008)
- $q \in L^\infty$, (Bukhgeim, 2008)

MODING OUT GROUP OF DIFFEOMORPHISM

$(n \geq 3)$

$$\boxed{\begin{array}{c} (\Delta_g - q)u = 0,\ \ q \in C^\infty(M) \\ u|_{\partial M} = f, \\ \Lambda_g(f) = \frac{\partial u}{\partial \nu_g}|_{\partial M}. \end{array}}$$

$$(*) \quad g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}, \quad c > 0.$$

Theorem (Dos Santos-Kenig-Salo-U) Assume that there is a global coordinate system so that (*) is true. In addition $\boxed{g_0}$ is simple. Then $\boxed{\Lambda_g}$ determines uniquely $\boxed{q}$.

Simple: No conjugate points and strictly convex.

$$\boxed{g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}, \quad x' \in \mathbb{R}^{n-1}}$$

Examples

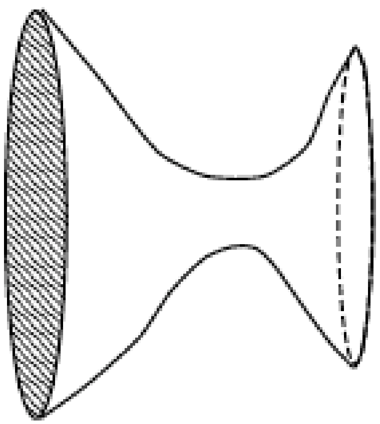
(a) $g(x)$ conformal to Euclidean metric (Sylvester-U, 1987)

(b) $g(x)$ conformal to hyperbolic metric (Isozaki, 2004)

(c) $g(x)$ conformal to metric on sphere (minus a point)

Non-uniqueness for EIT (Invisibility)

Motivation (Greenleaf-Lassas-U, MRL, 2003)



When bridge connecting the two parts of the manifold gets narrower the boundary measurements give less information about isolated area.

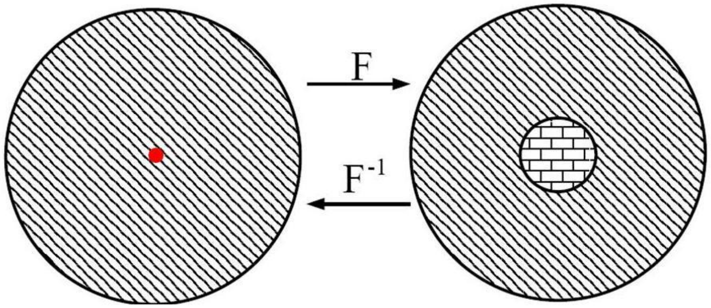
When we realize the manifold in Euclidean space we should obtain conductivities whose boundary measurements give no information about certain parts of the domain.

Greenleaf-Lassas-U (2003 MRL)

Let $\Omega = \mathcal{B}(0,2) \subset \mathbb{R}^3$, $D = \mathcal{B}(0,1)$ where $\mathcal{B}(0,r) = \{x \in \mathbb{R}^3; |x| < r\}$

$$F : \Omega \setminus \{0\} \to \Omega \setminus \overline{D}$$

$$\boxed{F(x) = (\frac{|x|}{2} + 1)\frac{x}{|x|}}$$



F diffeomorphism, $F\big|_{\partial\Omega} = \text{Identity}$

Let $\begin{array}{l} g = \text{identity metric in } \mathcal{B}(0,2) \\ \widehat{g} = (F^{-1})^* g \text{ on } \mathcal{B}(0,2) \setminus \mathcal{B}(0,1) \\ \widehat{\sigma} = \text{conductivity associated to } \widehat{g} \end{array}$

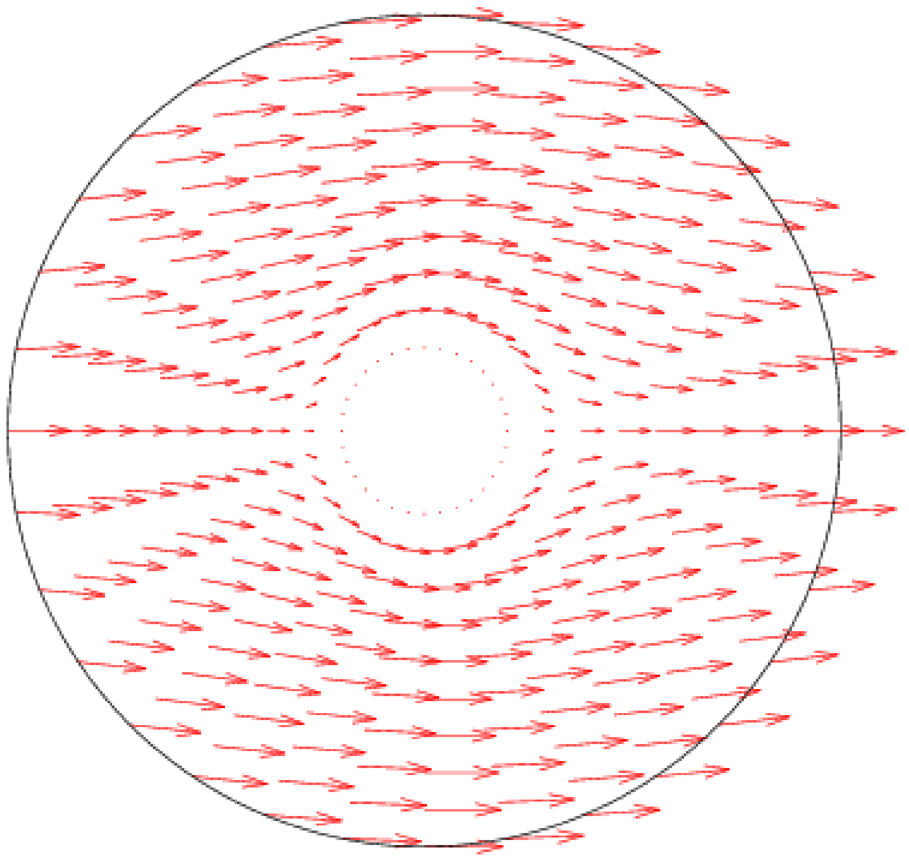
In spherical coordinates $(r, \phi, \theta) \to (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r\cos\theta)$

$$\widehat{\sigma} = \begin{pmatrix} 2(r-1)^2 \sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}$$

Let $\widetilde{g}$ be the metric in $\mathcal{B}(0,2)$ (positive definite in $\mathcal{B}(0,1)$) s.t. $\widetilde{g} = \widehat{g}$ in $\mathcal{B}(0,2) \setminus \mathcal{B}(0,1)$. Then

Theorem (Greenleaf-Lassas-U 2003)

$$\boxed{\Lambda_{\widetilde{g}} = \Lambda_g}$$



Based on work of Greenleaf-Lassas-U, MRL 2003