Summer School on Mathematical Physics

Inverse Problems: Visibility and Invisibility

Lecture II

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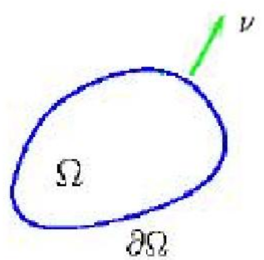
$$\boxed{\begin{aligned} \operatorname{div}(\gamma(x)\nabla u(x)) &= 0 \\ u\big|_{\partial\Omega} &= f \end{aligned}}$$

$\gamma(x)$ = conductivity,

f = voltage potential at $\partial\Omega$

Current flux at $\partial\Omega = (\nu \cdot \gamma\nabla u)\big|_{\partial\Omega}$ were ν is the unit outer normal.



Information is encoded in map

$$\boxed{\Lambda_\gamma(f) = \nu \cdot \gamma\nabla u\big|_{\partial\Omega}}$$

EIT (Calderón's inverse problem)

Does $\boxed{\Lambda_\gamma}$ determine $\boxed{\gamma}$?

Λ_γ = Dirichlet-to-Neumann map

Reduction to Schrödinger equation

$$\boxed{\mathrm{div}(\gamma \nabla w) = 0}$$

$$u = \sqrt{\gamma}\, w$$

Then the equation is transformed into:

$$\boxed{(\Delta - q)u = 0, q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}} \qquad \left(\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right)$$

$$\boxed{\begin{aligned} (\Delta - q)u &= 0 \\ u\Big|_{\partial\Omega} &= f \end{aligned}}$$

Define $\boxed{\Lambda_q(f) = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}}$

ν = unit-outer normal to $\partial\Omega$.

$\boxed{\text{COMPLEX GEOMETRICAL OPTICS}}$

(Sylvester-U) $n \geq 2$, $q \in L^\infty(\Omega)$
Let $\rho \in \mathbb{C}^n$ ($\rho = \eta + ik, \eta, k \in \mathbb{R}^n$) such that $\boxed{\rho \cdot \rho = 0}$
($|\eta| = |k|, \eta \cdot k = 0$).

Then for $|\rho|$ sufficiently large we can find solutions of

$$\boxed{(\Delta - q)w_\rho = 0 \text{ on } \Omega}$$

of the form

$$\boxed{w_\rho = e^{x\cdot\rho}(1 + \Psi_q(x,\rho))}$$

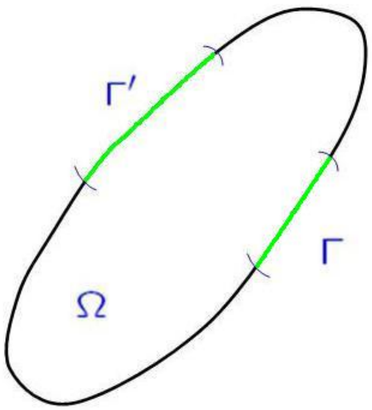
with $\Psi_q \to 0$ in Ω as $|\rho| \to \infty$.

PARTIAL DATA PROBLEM

Suppose we measure

$$\boxed{\Lambda_\gamma(f)|_\Gamma, \quad \text{supp} f \subseteq \Gamma'}$$

Γ, Γ' open subsets of $\partial\Omega$



Can one recover γ?

Important case $\Gamma = \Gamma'$.

EXTENSION OF CGO SOLUTIONS

$$\boxed{u = e^{x\cdot\rho}(1 + \Psi_q(x,\rho))}$$

$\rho \in \mathbb{C}^n, \rho \cdot \rho = 0$

(Not helpful for localizing)

Kenig-Sjöstrand-U (2007),

$$\boxed{u = e^{\tau(\varphi(x)+i\psi(x))}(a(x) + R(x,\tau))}$$

$\tau \in \mathbb{R}$, φ, ψ real-valued, $R(x,\tau) \to 0$ as $\tau \to \infty$.

φ limiting Carleman weight,

$$\nabla\varphi \cdot \nabla\psi = 0, \quad |\nabla\varphi| = |\nabla\psi|$$

Example: $\boxed{\varphi(x) = \ln|x - x_0|,}$ $\quad x_0 \notin \overline{ch(\Omega)}$

CGO SOLUTIONS

$$\boxed{\begin{aligned} u = e^{\tau(\varphi(x)+i\psi(x))}(a_0(x) + R(x,\tau)) \\ R(x,\tau) \overset{\tau\to\infty}{\longrightarrow} 0 \text{ in } \Omega \end{aligned}}$$

$$\boxed{\varphi(x) = \ln|x - x_0|}$$

Complex Spherical Waves

Theorem (Kenig-Sjöstrand-U) Ω strictly convex.

$$\Lambda_{q_1}\Big|_\Gamma = \Lambda_{q_2}\Big|_\Gamma, \quad \Gamma \subseteq \partial\Omega, \quad \Gamma \text{ arbitrary}$$

$$\Rightarrow q_1 = q_2$$

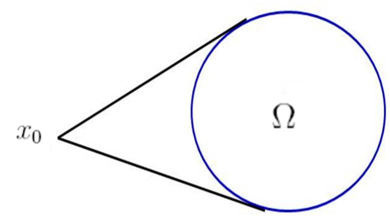
Theorem (Kenig-Sjöstrand-U) Ω strictly convex.

$$\Lambda_{q_1}\Big|_{\Gamma} = \Lambda_{q_2}\Big|_{\Gamma}, \quad \Gamma \subseteq \partial\Omega, \quad \Gamma \text{ arbitrary}$$

$$\Rightarrow q_1 = q_2$$

$\boxed{u_\tau = e^{\tau(\varphi + i\psi)} a_\tau}$ $\boxed{\varphi(x) = \ln|x - x_0|, x_0 \notin \overline{ch(\Omega)}}$

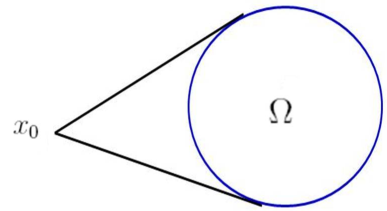
Eikonal: $\boxed{\nabla\varphi \cdot \nabla\psi = 0, |\nabla\varphi| = |\nabla\psi|}$
$\psi(x) = d(\frac{x-x_0}{|x-x_0|}, \omega), \omega \in S^{n-1}$: smooth for $x \in \bar{\Omega}$.



Transport: $\boxed{(\nabla\varphi + i\nabla\psi) \cdot \nabla a_\tau = 0}$

(Cauchy-Riemann equation in plane generated by $\nabla\varphi, \nabla\psi$)

$\varphi(x) = \ln|x - x_0|, \quad x_0 \notin \overline{ch(\Omega)}$



Carleman Estimates

$u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega_-} = 0 \qquad\qquad \partial\Omega_\pm = \{x \in \partial\Omega; \nabla\varphi \cdot \nu \gtrless 0\}$

$$\boxed{\int_{\partial\Omega_+} < \nabla\varphi, \nu > |e^{-\tau\varphi(x)}\frac{\partial u}{\partial \nu}|^2 ds \leq \frac{C}{\tau}\int_\Omega |(\Delta - q)ue^{-\tau\varphi(x)}|^2 ds}$$

This gives control of $\frac{\partial u}{\partial \nu}|_{\partial\Omega_{+,\delta}}$,

$$\boxed{\partial\Omega_{+,\delta} = \{x \in \partial\Omega, \nabla\varphi \cdot \nu \geq \delta\}}$$

$\boxed{\text{More general CGO solutions}}$

$$\boxed{u_\tau = e^{\tau(\varphi + i\psi)} a_\tau,}$$

$\tau \gg 0$, $\tau = 1/h$ (semicl.), φ, ψ real-valued

- φ is a limiting Carleman weight

$$e^{\frac{\varphi}{h}} h^2(-\Delta + q) e^{-\frac{\varphi}{h}}$$

has semiclassical principal symbol

$$P_\varphi(x, \xi) = \xi^2 - (\nabla\varphi)^2 + 2i\nabla\varphi \cdot \xi$$

Hörmander's condition:

$$\{\text{Re } P_\varphi, \text{Im } P_\varphi\} \le 0 \quad \text{on} \quad P_\varphi = 0$$

We need $\varphi, -\varphi$ to be phase of solutions.

$$\text{LCW}: \quad \boxed{\{\text{Re } P_\varphi, \text{Im } P_\varphi\} = 0}$$

$\nabla\varphi \neq 0$ in an open neighborhood of $\bar{\Omega}$.

CGO solutions $\quad u_h = e^{\frac{1}{h}(\varphi + i\psi)} a_h$

- φ LCW, φ real-valued

$$\{\mathrm{Re} P_\varphi, \mathrm{Im} P_\varphi\} = 0 \quad \text{on} \quad P_\varphi = 0$$

$\nabla\varphi \neq 0$ on an open neighborhood of $\bar{\Omega}$.

Examples (Dos Santos Ferreira-Kenig-Salo-U, 2009)

(a) $\varphi(x) = x \cdot \xi$, $\xi \in \mathbb{R}^n$, $|\xi| = 1$

(b) $\varphi(x) = a \ln|x - x_0| + b$, $(a, b$ constants), $x_0 \notin \overline{ch(\Omega)}$

(c) $\varphi(x) = \dfrac{a\langle x - x_0, \xi\rangle}{|x - x_0|^2} + b$, $\xi \in \mathbb{R}^n$

(d) $\varphi(x) = a \arctan \frac{2\langle x - x_0, \xi\rangle}{|x - x_0|^2 - |\xi|^2} + b$

(e) $\varphi(x) = a \operatorname{arctanh} \frac{2\langle x - x_0, \xi\rangle}{|x - x_0|^2 - |\xi|^2} + b$

(f) $n = 2$, φ is a harmonic function

Instead of

$$\boxed{\int_\Omega e^{2ix\cdot k} q(x)dx = 0}$$

$k \perp \xi$ ($\xi \in \mathbb{S}^{n-1}$) as in Bukhgeim-U argument we get

$$\boxed{\int_\Omega e^{i\lambda f(x)} \boxed{q(x)}\, a_1 a_2 dx = 0}$$

λ any real number, $a_1, a_2 \neq 0$, $f(x)$ real-analytic, a_1, a_2 real analytic

Analytic microlocal analysis $\Longrightarrow q = 0$ (like inversion of real-analytic Radon

$\boxed{\textbf{Linearization}}$ (Analog of Calderón)

Theorem (Dos Santos Ferreira, Kenig, Sjöstrand-U)

$$\boxed{\int_\Omega huv = 0}$$

$$\Gamma \subseteq \partial\Omega,\ \Gamma \text{ open},$$

$$\Delta u = \Delta v = 0, \quad u, v \in C^\infty(\overline{\Omega}),$$

$$\text{supp } u|_{\partial\Omega}, \text{ supp } v|_{\partial\Omega} \subseteq \Gamma,$$

$$\Rightarrow \boxed{h = 0}.$$

$\boxed{\text{Complex Spherical Waves}}$

$$\boxed{u_\tau = e^{\tau(\varphi + i\psi)} a_\tau}$$

$$\varphi(x) = \ln|x - x_0|,\ x_0 \notin \overline{\mathrm{ch}(\Omega)}$$

Also used to determine inclusions, obstacles, etc.

a) Conductivity Ide-Isozaki-Nakata-Siltanen-U

b) Helmholtz Nakamura-Yosida

c) Elasticity J.-N. Wang-U

d) 2D Systems J.-N. Wang-U

e) Maxwell T. Zhou

Complex Spherical Waves

(Loading reconperfect1.mpg)

$\boxed{\text{The Two Dimensional Case}}$

Theorem $(n = 2)$ Let $\gamma_j \in C^2(\overline{\Omega})$, $j = 1, 2$.

Assume $\boxed{\Lambda_{\gamma_1} = \Lambda_{\gamma_2}}$. Then $\boxed{\gamma_1 = \gamma_2}$.

- Nachman (1996)
- Brown-U (1997) Improved to γ_j Lipschitz
- Astala-Päivärinta (2006) Improved to $\gamma_j \in L^\infty(\Omega)$

Recall

$$\begin{gathered} \operatorname{div}(\gamma \nabla u) = 0, \quad \gamma \in L^\infty(\Omega) \\ u|_{\partial\Omega} = f \end{gathered}$$

$$Q_\gamma(f) = \int_\Omega \gamma |\nabla u|^2 dx = \langle \Lambda_\gamma f, f \rangle_{L^2(\partial\Omega)}.$$

This follows from more general result

Theorem ($n = 2$, Bukhgeim, 2008) Let $q_j \in L^\infty(\Omega)$, $j = 1, 2$.

Assume $\boxed{\Lambda_{q_1} = \Lambda_{q_2}}$. Then $\boxed{q_1 = q_2}$.

Recall

$$\begin{array}{c}(\Delta - q)u = 0,\\ u\big|_{\partial\Omega} = f.\end{array} \qquad \Lambda_q(f) = \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega}$$

with ν-unit outer normal.

$$\boxed{\Lambda_{q_1} = \Lambda_{q_2}} \;\Rightarrow\; \boxed{q_1 = q_2}$$

Sketch of proof New class of CGO solutions

$$\boxed{\begin{array}{c} u_1(z,\tau) = e^{\tau z^2}(1 + r_1(z,\tau)) \\ u_2(z,\tau) = e^{-\tau \bar{z}^2}(1 + r_2(z,\tau)) \end{array}} \qquad \tau \gg 1$$

solve $(\Delta - q_j)u_j = 0$ with $r_j(z,\tau) \to 0$ on Ω sufficiently fast.

Notation $z = x_1 + ix_2$

Remark $z^2 = x_1^2 - x_2^2 + 2ix_1x_2 = \varphi + i\psi$

$$\nabla\varphi \cdot \nabla\psi = 0, \quad |\nabla\varphi| = |\nabla\psi|$$

φ harmonic, ψ conjugate harmonic.

$$\Lambda_{q_1} = \Lambda_{q_2} \;\Rightarrow\; \boxed{\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0}$$

$$(\Delta - q_j) u_j = 0$$

$$u_1 = e^{\tau z^2} \left(1 + r_1(z,\tau)\right), \quad u_2 = e^{-\tau \bar{z}^2} \left(1 + r_2(z,\tau)\right)$$

Substituting

$$\int_\Omega (q_1 - q_2) e^{4i\tau x_1 x_2} (1 + r_1 + r_2 + r_1 r_2) dx = 0.$$

Letting $\tau \to \infty$ and using stationary phase

$$(q_1 - q_2)(0) = 0.$$

Changing z to $z - z_0$ we get

$$(q_1 - q_2)(z_0) = 0.$$

$\boxed{\text{Partial data}}$

Let $\Gamma \subseteq \partial\Omega$, Γ open.
Let $q_j \in C^{1+\varepsilon}(\Omega)$, $\varepsilon > 0$, $j = 1, 2$.

Theorem (Imanuvilov-U-Yamamoto 2010) n=2. Assume

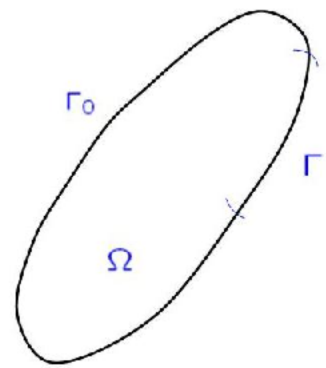
$$\boxed{\Lambda_{q_1}(f)\Big|_\Gamma = \Lambda_{q_2}(f)\Big|_\Gamma}$$

$\forall\ f$, supp$f \subseteq \Gamma$. Then

$$\boxed{q_1 = q_2}.$$

- Riemann Surfaces: Guillarmou-Tzou (2011)

$\boxed{\text{Partial data}}$ $\quad \Gamma_0 = \partial\Omega - \Gamma$



Construct CGO solutions

$$\boxed{\begin{array}{c} \Delta u_j - q_j u_j = 0 \quad \text{in } \Omega \\ u_j|_{\Gamma_0} = 0 \end{array}}$$

In this case

$$\boxed{\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0}$$

if $\Lambda_{q_1}(f)|_\Gamma = \Lambda_{q_2}(f)|_\Gamma$, supp$f \subseteq \Gamma$.

$$\int_\Omega (q_1 - q_2) u_1 u_2 = 0$$

$$u_j|_{\Gamma_0} = u_j|_{\partial\Omega - \Gamma} = 0$$

$$\boxed{u_1(x) = e^{\tau\Phi(z)}(a(z) + \frac{a_0(z)}{\tau}) + \overline{e^{\tau\Phi(z)}(a(z) + \frac{a_1(z)}{\tau})} + e^{\tau\varphi} R_\tau^{(1)}}$$

$$\boxed{u_2(x) = e^{-\tau\bar{\Phi}(z)}(\bar{a}(z) + \frac{b_0(z)}{\tau}) + \overline{e^{-\tau\bar{\Phi}(z)}(\bar{a}(z) + \frac{b_1(z)}{\tau})} + e^{\tau\varphi} R_\tau^{(2)}}$$

$\Phi = \varphi + i\psi$ holomorphic

$$u_1 = \text{Re } e^{\tau\Phi(z)}(a(z) + \cdots), \quad u_2 = \text{Re } e^{-\tau\bar{\Phi}(z)}(\bar{a}(z) + \cdots)$$

$$\boxed{\Phi(z) = \varphi + i\psi} \quad \text{holomorphic}$$

$$\boxed{u_j|_{\partial\Omega - \Gamma} = 0}$$

$p \in \Omega$, Φ has non-degenerate critical point at p (Morse function) and $Im\Phi = 0$ on Γ_0. Notice that this implies $\nabla\phi \cdot \nu = 0$ on Γ_0.

$$\bar{\partial}a = 0 \quad \text{Re } a|_{\partial\Omega - \Gamma} = 0$$

$a = 0$ at other critical points

Stationary phase in

$$\boxed{\int_\Omega (q_1 - q_2)u_1 u_2 = 0}$$

$$\boxed{\begin{array}{c} u_1 = \text{Re } e^{\tau\Phi(z)}(a(z) + \cdots) \\ u_2 = \text{Re } e^{-\tau\bar{\Phi}(z)}(\bar{a}(z) + \cdots) \\ u_j|_{\partial\Omega - \Gamma} = 0 \end{array}}$$

$\Phi(z)$ Morse function with non-degenerate critical point at p.

$$\int_\Omega (q_1 - q_2) u_1 u_2 = 0$$

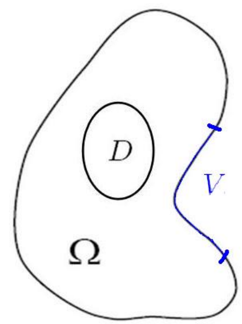
Stationary phase

$$\Longrightarrow \boxed{(q_1 - q_2)(p) = 0}$$

$\boxed{\textbf{Corollary:}}$ Obstacle Problem

$\boxed{\Omega, D \subset \mathbb{R}^2}$: smooth boundary such that $\bar{D} \subset \Omega$.

$\boxed{V \subset \partial\Omega}$: open set.



Let $\boxed{q_j \in C^{2+\alpha}(\overline{\Omega \backslash D})}$ for some $\alpha > 0$, $j = 1, 2$.

$$\widetilde{C}_{q_j} := \Big\{ (u|_V, \partial_\nu u|_V); (\Delta - q_j)u = 0 \text{ in } \Omega\backslash\bar{D} \\ \text{supp } u|_{\partial\Omega} \subset V, u \in H^1(\Omega\backslash\bar{D}) \Big\}$$

Then $\boxed{\widetilde{C}_{q_1} = \widetilde{C}_{q_2} \Longrightarrow q_1 = q_2}$.

Carleman Estimate With Degenerate Weights

Lemma

Let $\partial\Omega - \Gamma = \{x \in \partial\Omega;\ \nu \cdot \nabla\varphi = 0\}$. Then for τ sufficiently large, $\exists$ solution of

$$\boxed{\begin{array}{c} \Delta u - qu = f \quad \text{in} \quad \Omega \\ u|_{\partial\Omega-\Gamma} = g \end{array}}$$

such that

$$\|ue^{-\tau\varphi}\|_{L^2(\Omega)} \leq C\left(|\tau|^{-1/2}\|fe^{-\tau\varphi}\|_{L^2(\Omega)} + \|ge^{-\tau\varphi}\|_{L^2(\Omega-\Gamma)}\right)$$

$\boxed{\textbf{CASE OF DISJOINT SETS:}}$

Theorem (Imanuvilov-U-Yamamoto, 2011)

Let $C_q = \left\{ \left(u|_{\Gamma_+}, \left(\frac{\partial u}{\partial \nu}\right)\Big|_{\Gamma_-} \right) \; ; \; (\Delta - q)u = 0 \text{ in } \Omega, \; u|_{\Gamma_0 \cup \Gamma_-} = 0, u \in H^1(\Omega) \right\}$ Assume $q_j \in C^{2+\alpha}(\overline{\Omega})$ and

$$\boxed{\mathcal{C}_{q_1} = \mathcal{C}_{q_2}}$$

then

$$\boxed{q_1 = q_2}.$$

where

$$\Gamma_{\pm} = \bigcup_{j=1}^{2} \Gamma_{\pm j}, \;\; \Gamma_0 = \bigcup_{k=1}^{4} \Gamma_{0,k}$$

oriented clockwise.

CASE OF DISJOINT SETS:

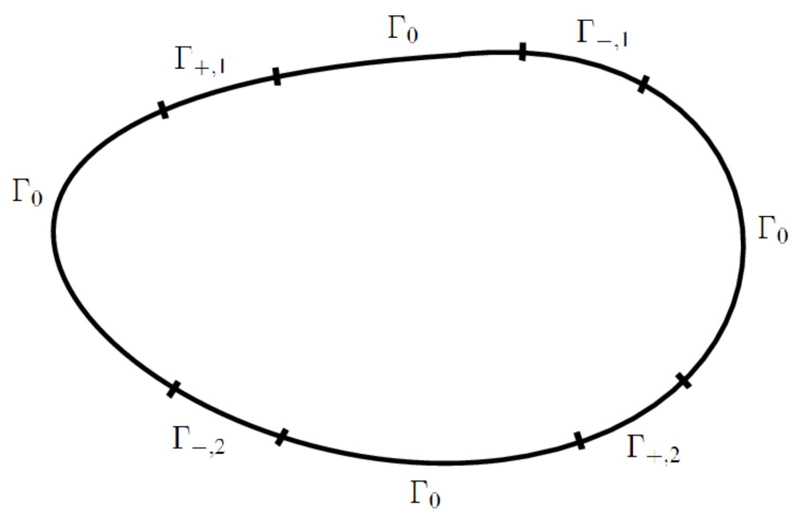


Figure 1 of Imanuvilov-U-Yamamoto (2011)

Partial Data for Second Order Elliptic Equations ($n = 2$)

(Imanuvilov–U–Yamamoto, 2011)

$$\boxed{\Delta_g + A(z)\frac{\partial}{\partial z} + B(z)\frac{\partial}{\partial \overline{z}} + q} \qquad z = x_1 + ix_2$$

$g = (g_{ij})$ positive definite symmetric matrix;

$$\Delta_g u = \frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det(g)}\, g^{ij} \frac{\partial u}{\partial x_j}\right) \qquad g^{ij} = (g_{ij})^{-1}$$

Includes:

- Anisotropic Calderón's Problem
- Magnetic Schrödinger Equation
- Convection terms

Anisotropic case

Cardiac muscle	6.3 mho	(longitudinal)
	2.3 mho	(transversal)

$\boxed{\gamma = (\gamma^{ij})}$ positive-definite, symmetric matrix

conductivity

$\Omega \subseteq \mathbb{R}^n, \Omega$ bounded. Under assumptions of no sources or sinks of current the potential u satisfies

$\boxed{\text{div}(\gamma \nabla u) = 0}$

$$\boxed{\begin{aligned} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_i}\right) &= 0 \text{ in } \Omega \\ u\big|_{\partial\Omega} &= f \end{aligned}} \qquad (*)$$

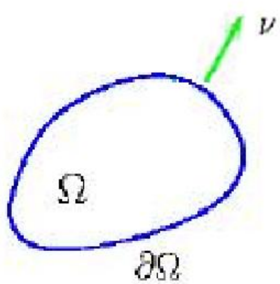
f = voltage potential at boundary

Isotropic

$$\gamma^{ij}(x) = \alpha(x)\delta^{ij}; \delta^{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\boxed{\begin{aligned}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) &= 0 \text{ in } \Omega \\ u\Big|_{\partial\Omega} &= f\end{aligned}} \qquad (*)$$

$$\boxed{\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i \gamma^{ij} \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}}$$



$\nu = \left(\nu^1, \cdots, \nu^n\right)$ is the unit outer normal to $\partial\Omega$

$\Lambda_\gamma(f)$ is the induced current flux at $\partial\Omega$.

Λ_γ is the voltage to current map or Dirichlet - to - Neumann map

$$\boxed{\begin{aligned}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) &= 0 \text{ in } \Omega\\ u\Big|_{\partial\Omega} &= f\end{aligned}} \qquad (*)$$

$$\boxed{\Lambda_\gamma(f) = \sum_{i,j=1}^{n}\nu^i\gamma^{ij}\,\frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}}$$

EIT: Can we recover γ in Ω from Λ_γ ?

$$\boxed{\begin{aligned} \operatorname{div}(\gamma \nabla u) &= 0 \\ u\big|_{\partial\Omega} &= f \end{aligned}}$$

$$\boxed{\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \gamma^{ij} \nu^i \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}}$$

$$\boxed{\Lambda_\gamma \Rightarrow \gamma\ ?}$$

Answer: No

$$\boxed{\Lambda_{\psi_* \gamma} = \Lambda_\gamma}$$

where $\psi : \Omega \to \Omega$ change of variables

$$\psi|_{\partial\Omega} = \text{Identity}$$

$$\boxed{\psi_* \gamma = \left(\frac{(D\psi)^T \circ \gamma \circ D\psi}{|\det D\psi|} \right) \circ \psi^{-1}}$$

$$\boxed{v = u \circ \psi^{-1}}$$

Theorem (Imanuvilov–U–Yamamoto, 2011) $\Omega \subset \mathbb{R}^2$, $\Gamma \subset \partial\Omega$, Γ open, $\gamma_k = (\gamma_k^{ij}) \in C^\infty(\bar{\Omega}), k = 1, 2$, positive definite symmetric. Assume

$$\Lambda_{\gamma_1}(f)|_\Gamma = \Lambda_{\gamma_2}(f)|_\Gamma, \quad \forall f \ \ \text{supp} f \subset \Gamma.$$

Then $\exists F : \bar{\Omega} \to \bar{\Omega}$, C^∞ diffeomorphism, $F|_\Gamma =$ Identity such that

$$F_*\gamma_1 = \gamma_2.$$

Full Data ($\Gamma = \partial\Omega$):

- $\gamma_k \in C^2(\bar{\Omega})$, Nachman (1996)
- γ_k Lipschitz, Sun–U (2001)
- $\gamma_k \in L^\infty(\Omega)$, Astala–Lassas–Pävärinta (2006)

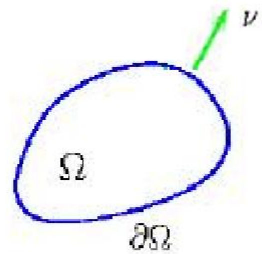
DIRICHLET-TO-NEUMANN MAP (Lee-U, 1989)

(M, g) compact Riemannian manifold with boundary.

Δ_g Laplace-Beltrami operator $g = (g_{ij})$ pos. def. symmetric matrix

$$\boxed{\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j}\right)} \quad (g^{ij}) = (g_{ij})^{-1}$$

$$\boxed{\begin{aligned} \Delta_g u &= 0 \text{ on } M \\ u\big|_{\partial M} &= f \end{aligned}}$$



Conductivity:

$\gamma^{ij} = \sqrt{\det g}\, g^{ij}$

$$\boxed{\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\,\Big|_{\partial M}}$$

$\nu = (\nu^1, \cdots, \nu^n)$ unit-outer normal

$$\Delta_g u = 0$$
$$u\Big|_{\partial M} = f$$

$$\boxed{\Lambda_g(f) = \frac{\partial u}{\partial \nu_g} = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\Big|_{\partial M}}$$

current flux at ∂M

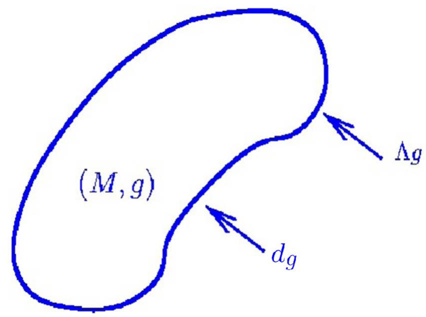
Inverse-problem (EIT)

Can we recover $\boxed{g}$ from $\boxed{\Lambda_g}$?

$\Lambda_g =$ Dirichlet-to-Neumann map or voltage to current map

ANOTHER MOTIVATION (STRING THEORY)

HOLOGRAPHY



Dirichlet-to-Neumann map is the "boundary-2pt function"

Inverse problem: Can we recover (M, g) (bulk) from boundary-2pt function ?

M. Parrati and R. Rabadan, Boundary rigidity and holography, JHEP 0401 (2004) 034

$$\boxed{\begin{aligned} \Delta_g u &= 0 \\ u\big|_{\partial M} &= f \end{aligned}}$$

$$\boxed{\Lambda_g(f) = \frac{\partial u}{\partial \nu_g}\bigg|_{\partial M}}$$

$$\boxed{\Lambda_g \Rightarrow g \quad ?}$$

Answer: No $\boxed{\Lambda_{\psi^* g} = \Lambda_g}$ where

$\psi : M \to M$ diffeomorphism, $\psi\big|_{\partial M} =$ Identity and

$$\boxed{\psi^* g = \left(D\psi \circ g \circ (D\psi)^T\right) \circ \psi}$$

Show $\Lambda_{\psi^* g} = \Lambda_g$; $\psi : M \to M$ diffeomorphism, $\psi\big|_{\partial M} =$ Identity

$$Q_g(f) = \sum_{i,j} \int_M g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \sqrt{\det g}\, dx$$

$$\boxed{Q_g(f) = -\int_{\partial M} \Lambda_g(f) f \, dS}$$

$$\boxed{Q_g \Leftrightarrow \Lambda_g}$$

$$v = u \circ \psi, \; \Delta_{\psi^* g} v = 0$$

$$\boxed{Q_{\psi^* g} = Q_g} \Rightarrow \boxed{\Lambda_{\psi^* g} = \Lambda_g}$$

Theorem ($n \geq 3$) (Lassas-U 2001, Lassas-Taylor-U 2003) $(M, g_i), i = 1, 2$, real-analytic, connected, compact, Riemannian manifolds with boundary. Let $\Gamma \subseteq \partial M$, Γ open. Assume

$$\boxed{\Lambda_{g_1}(f)|_\Gamma = \Lambda_{g_2}(f)|_\Gamma,} \quad \forall f, \ f \text{ supported in } \Gamma$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi\Big|_\Gamma =$ Identity, so that

$$\boxed{g_1 = \psi^* g_2}$$

In fact one can determine topology of M, as well (only need to know $\Lambda_g, \partial M$).

Theorem (Guillarmou-Sa Barreto, 2009) $(M, g_i), i = 1, 2$, are compact Riemannian manifolds with boundary that are Einstein. Assume

$$\boxed{\Lambda_{g_1} = \Lambda_{g_2}}$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi|_{\partial M} = Identity$ such that

$$\boxed{g_1 = \psi^* g_2}$$

Note: Einstein manifolds with boundary are real analytic in the interior.

Theorem ($n = 2$)(Lassas-U, 2001)

(M, g_i), $i = 1, 2$, connected Riemannian manifold with boundary. Let $\Gamma \subseteq \partial M$, Γ open. Assume

$$\boxed{\Lambda_{g_1}(f)|_\Gamma = \Lambda_{g_2}(f)|_\Gamma,} \quad \forall f,\ f \text{ supported in } \Gamma$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi\Big|_\Gamma = \text{Identity}$, and $\beta > 0, \beta\Big|_\Gamma = 1$ so that

$$\boxed{g_1 = \beta \psi^* g_2}$$

In fact, one can determine topology of M as well.

$\boxed{\textbf{Moding Out the Diffeomorphism Group}}$

Some conformal class $\boxed{\Lambda_{\beta g} = \Lambda_g}, \quad \beta \in C^\infty(M)$

$$\Longrightarrow \beta = 1?$$

More general problem

$$\boxed{\begin{array}{c} (\Delta_g - q)u = 0, \ q \in C^\infty(M) \\ u|_{\partial M} = f, \\ \Lambda_g(f) = \frac{\partial u}{\partial \nu_g}|_{\partial M}. \end{array}}$$

Inverse Problem: Does $\boxed{\Lambda_g}$ determines $\boxed{q}$?

$(\Delta_g - q)u = 0, \qquad \Lambda_g(f) = \frac{\partial u}{\partial \nu_g}|_{\partial M}, \qquad \boxed{\Lambda_g \to q?}$

Theorem (n=2) (Guillarmou-Tzou, 2009)

YES

Earlier results:

- $\mathbb{R}^2$, q small (Sylvester-U, 1986)
- $\mathbb{R}^2$, q generic (Sun-U, 2001)
- $\mathbb{R}^2$, $q = \frac{\Delta\sqrt{\lambda}}{\sqrt{\lambda}}, \gamma > 0$ (Nachmann 1996)
- Riemannian surfaces, $q = \frac{\Delta\sqrt{\lambda}}{\sqrt{\lambda}}, \gamma > 0$, (Henkin-Michel, 2008)
- $q \in L^\infty$, (Bukhgeim, 2008)

$\boxed{\text{MODING OUT GROUP OF DIFFEOMORPHISM}}$

$(n \geq 3)$

$$\boxed{\begin{array}{c} (\Delta_g - q)u = 0,\ q \in C^\infty(M) \\ u|_{\partial M} = f, \\ \Lambda_g(f) = \frac{\partial u}{\partial \nu_g}|_{\partial M}. \end{array}}$$

$$(*) \quad g(x_1, x') = c(x)\begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}, \quad c > 0.$$

Theorem (Dos Santos-Kenig-Salo-U) Assume that there is a global coordinate system so that (*) is true. In addition $\boxed{g_0}$ is simple. Then $\boxed{\Lambda_g}$ determines uniquely $\boxed{q}$.

Simple: No conjugate points and strictly convex.

$$\boxed{g(x_1, x') = c(x) \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix}, \quad x' \in \mathbb{R}^{n-1}}$$

Examples

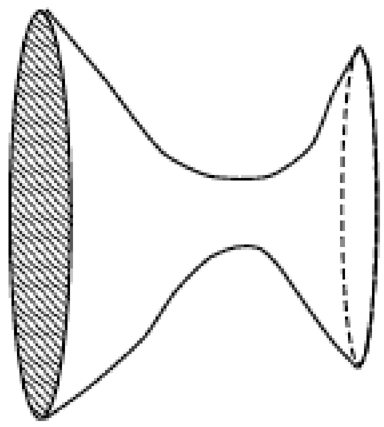
(a) $g(x)$ conformal to Euclidean metric (Sylvester-U, 1987)

(b) $g(x)$ conformal to hyperbolic metric (Isozaki, 2004)

(c) $g(x)$ conformal to metric on sphere (minus a point)

Non-uniqueness for EIT (Invisibility)

Motivation (Greenleaf-Lassas-U, MRL, 2003)



When bridge connecting the two parts of the manifold gets narrower the boundary measurements give less information about isolated area.

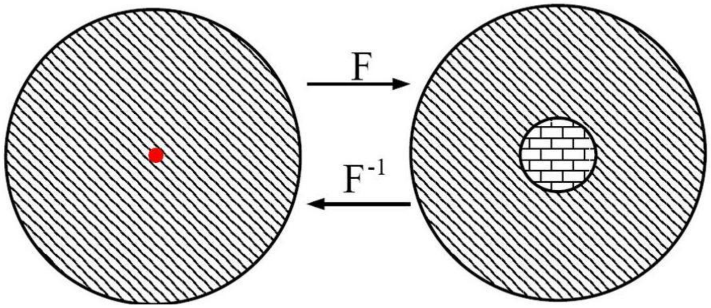
When we realize the manifold in Euclidean space we should obtain conductivities whose boundary measurements give no information about certain parts of the domain.

Greenleaf-Lassas-U (2003 MRL)

Let $\Omega = \mathcal{B}(0,2) \subset \mathbb{R}^3,$ $D = \mathcal{B}(0,1)$ where $\mathcal{B}(0,r) = \{x \in \mathbb{R}^3; |x| < r\}$

$$F : \Omega \setminus \{0\} \to \Omega \setminus \overline{D}$$

$$\boxed{F(x) = (\frac{|x|}{2} + 1)\frac{x}{|x|}}$$



F diffeomorphism, $F\Big|_{\partial\Omega} =$ Identity

Let $\begin{aligned} g &= \text{identity metric in } \mathcal{B}(0,2) \\ \widehat{g} &= (F^{-1})^* g \text{ on } \mathcal{B}(0,2) \setminus \mathcal{B}(0,1) \\ \widehat{\sigma} &= \text{conductivity associated to } \widehat{g} \end{aligned}$

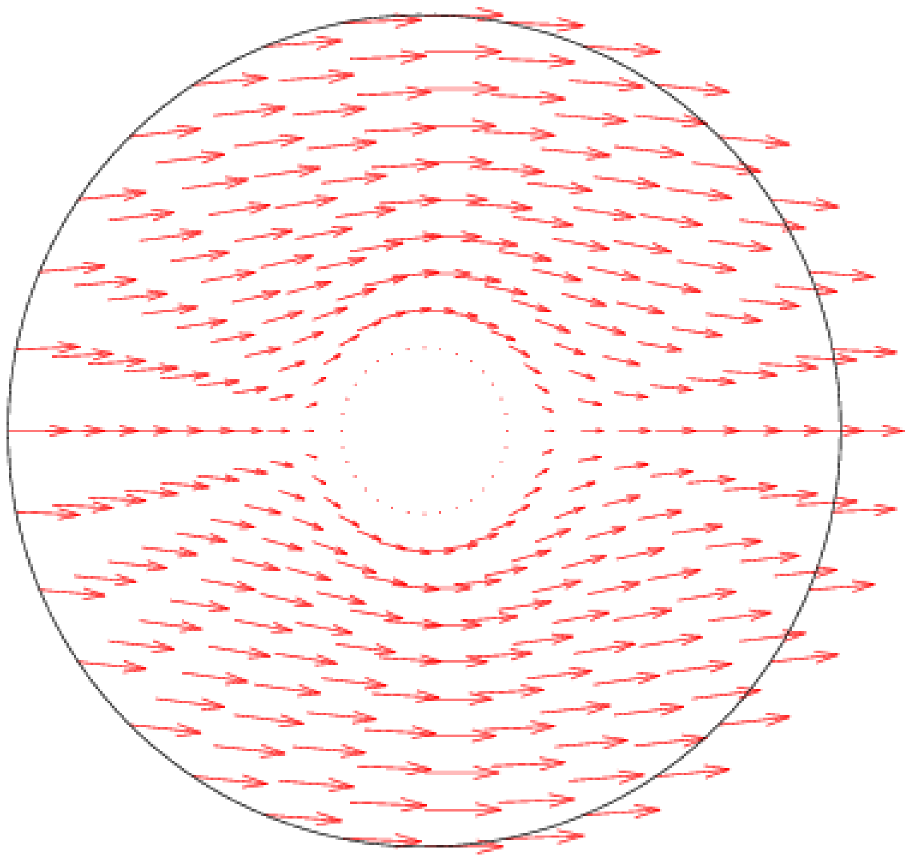
In spherical coordinates $(r, \phi, \theta) \to (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$

$$\widehat{\sigma} = \begin{pmatrix} 2(r-1)^2 \sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}$$

Let $\widetilde{g}$ be the metric in $\mathcal{B}(0,2)$ (positive definite in $\mathcal{B}(0,1)$) s.t. $\widetilde{g} = \widehat{g}$ in $\mathcal{B}(0,2) \setminus \mathcal{B}(0,1)$. Then

Theorem (Greenleaf-Lassas-U 2003)

$$\boxed{\Lambda_{\widetilde{g}} = \Lambda_g}$$



Based on work of Greenleaf-Lassas-U, MRL 2003