Summer School on Mathematical Physics

Inverse Problems: Visibility and Invisibility
Lecture III

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A difficult problem for radiologists: breast cancer detection

Ultrasound images of different breast lesions

- **benign**
- **Malign**
- **benign**

- Fibrotic Lesion
- Carcinoma Grade II
- Viscous cyst

Good sensitivity but bad specificity
How to improve specificity?

Hybrid Methods

Superposition of 2 images each obtained with a single wave

One single wave in sensitive only to a given contrast

Ultrasound to bulk compressibility

Photoacoustic Imaging

Optical wave to dielectric permittivity

Thermoacoustic Imaging

LF Electromagnetic wave to electrical impedance, conductivity.
Photoacoustic Tomography

Photoacoustic Effect: The sound of light

Graham Bell: When rapid pulses of light are incident on a sample of matter they can be absorbed and the resulting energy will then be radiated as heat. This heat causes detectable sound waves due to pressure variation in the surrounding medium.

Picture from Economist (The sound of light)
Lihong Wang (Washington U.)
First Step: in PAT and TAT is to reconstruct $H(x)$ from $u(x, t)|_{\partial \Omega \times (0, T)}$, where $u$ solves

$$(\partial_t^2 - c^2(x) \Delta) u = 0 \quad \text{on } \mathbb{R}^n \times \mathbb{R}^+$$

$u|_{t=0} = \beta H(x)$

$\partial_t u|_{t=0} = 0$

Second Step: in PAT and TAT is to reconstruct the optical or electrical properties from $H(x)$ (internal measurements).
First Step:

**IP for Wave Equation**

\[ c(x) > 0 \]: acoustic speed

\[
\begin{cases}
(\partial_t^2 - c^2 \Delta) u &= 0 \quad \text{in } (0, T) \times \mathbb{R}^n, \\
u|_{t=0} &= f, \\
\partial_t u|_{t=0} &= 0.
\end{cases}
\]

\( f \): supported in \( \bar{\Omega} \).

**Measurements**:

\[ \Lambda f := u|_{[0, T] \times \partial \Omega}. \]

The problem is to reconstruct the unknown \( f \) from \( \Lambda f \).
Prior results

Constant Speed

Kruger; Agranovsky, Ambartsoumian, Finch, Georgieva-Hristova, Jin, Haltmeier, Kuchment, Nguyen, Patch, Quinto, Rakesh, Wang, Xu . . .

Variable Speed (Numerical Results)

Anastasio et. al., Burgholzer, Cox et. al., Georgieva-Hristova, Grun, Haltmeir, Hofer, Kuchment, Nguyen, Paltauff, Wang, Xu... (Time reversal)

Partial Data

Problem is uniqueness, stability and reconstruction with measurements on a part of the boundary. There were no results so far for the variable coefficient case, and there is a uniqueness result in the constant coefficients one by Finch, Patch and Rakesh (2004).
\[ \Omega = \text{ball, constant speed} \]

\[ c = 1, \quad \Omega : \text{unit ball, } n = 3. \text{ Explicit Reconstruction Formulas (Finch, Haltmeier, Kunyansky, Nguyen, Patch, Rakesh, Xu, Wang).} \]

\[ g(x, t) = \Lambda f, \quad x \in S^{n-1}. \text{ In 3D,} \]

\[
f(x) = -\frac{1}{8\pi^2} \Delta_x \int_{|y|=1} \frac{g(y, |x-y|)}{|x-y|} dS_y.
\]

\[
f(x) = -\frac{1}{8\pi^2} \int_{|y|=1} \left( \frac{1}{t} \frac{d^2}{dt^2} g(y, t) \right) \bigg|_{t=|y-x|} dS_y.
\]

\[
f(x) = \frac{1}{8\pi^2} \nabla_x \cdot \int_{|y|=1} \left( \nu(y) \frac{1}{t} \frac{d}{dt} \frac{g(y, t)}{t} \right) \bigg|_{t=|y-x|} dS_y.
\]

The latter is a partial case of an explicit formula in any dimension (Kunyansky).
$T = \infty$: a backward Cauchy problem with zero initial data.
$T < \infty$: time reversal

\[
\begin{cases}
(\partial^2_t - c^2 \Delta)v_0 = 0 & \text{in } (0, T) \times \Omega, \\
v_0|_{[0, T] \times \partial \Omega} = \chi h, \\
v_0|_{t=T} = 0, \\
\partial_t v_0|_{t=T} = 0,
\end{cases}
\]

where $h = \Lambda f$; $\chi$: cuts off smoothly near $t = T$.

**Time Reversal**

\[ f \approx A_0 h := v_0(0, \cdot) \quad \text{in } \tilde{\Omega}, \text{ where } h = \Lambda f. \]
Underlying metric: $c^{-2}dx^2$. Set

$$T_0 = \max_{x \in \Omega} \text{dist}(x, \partial \Omega).$$

**Theorem (Stefanov–U)**

$T \geq T_0 \implies$ uniqueness. $T < T_0 \implies$ no uniqueness. We can recover $f(x)$ for $\text{dist}(x, \partial \Omega) \leq T$ and nothing else.

The proof is based on the unique continuation theorem by Tataru.
$T_1 \leq \infty$: length of the longest (maximal) geodesic through $\tilde{\Omega}$.

The "stability time": $T_1/2$. If $T_1 = \infty$, we say that the speed is **trapping** in $\Omega$.

**Theorem (Stefanov–U)**

$T > T_1/2 \implies \text{stability}$.

$T < T_1/2 \implies \text{no stability, in any Sobolev norms}$.

The second part follows from the fact that $\Lambda$ is a smoothing FIO on an open conic subset of $T^*\Omega$. In particular, if the speed is **trapping**, there is no stability, whatever $T$. 
Reconstruction. Modified time reversal

A modified time reversal, harmonic extension

Given $h$ (that eventually will be replaced by $\Lambda f$), solve

$$
\begin{aligned}
\left\{
\begin{array}{ll}
(\partial_t^2 - c^2 \Delta)v &= 0 \quad \text{in } (0, T) \times \Omega, \\
v|_{[0, T] \times \partial \Omega} &= h, \\
v|_{t=T} &= \phi, \\
\partial_t v|_{t=T} &= 0,
\end{array}
\right.
\end{aligned}
$$

where $\phi$ is the harmonic extension of $h(T, \cdot)$:

$$
\Delta \phi = 0, \quad \phi|_{\partial \Omega} = h(T, \cdot).
$$

Note that the initial data at $t = T$ satisfies compatibility conditions of first order (no jump at $\{T\} \times \partial \Omega$). Then we define the following pseudo-inverse

$$
Ah := v(0, \cdot) \quad \text{in } \tilde{\Omega}.
$$
We are missing the Cauchy data at $t = T$; the only thing we know there is its value on $\partial \Omega$. The time reversal methods just replace it by zero. We replace it by that data (namely, by $(\phi, 0)$), having the same trace on the boundary, that minimizes the energy.

Given $U \subset \mathbb{R}^n$, the energy in $U$ is given by

$$E_U(t, u) = \int_U (|\nabla u|^2 + c^{-2}|u_t|^2) \, dx.$$ 

We define the space $H_D(U)$ to be the completion of $C_0^\infty(U)$ under the Dirichlet norm

$$\|f\|_{H_D}^2 = \int_U |\nabla u|^2 \, dx.$$ 

The norms in $H_D(\Omega)$ and $H^1(\Omega)$ are equivalent, so

$$H_D(\Omega) \cong H^1_0(\Omega).$$ 

The energy norm of a pair $[f, g]$ is given by

$$\| [f, g] \|_{\mathcal{H}(\Omega)}^2 = \|f\|_{H_D(\Omega)}^2 + \|g\|_{L^2(\Omega, c^{-2}dx)}^2.$$
\[ A\Lambda f = f - Kf \]

where \( Kf = w(0, \cdot) \)

where \( w \) solves

\[
\begin{cases}
(\partial_t^2 - c^2 (x) \Delta) w = 0 & \text{in } (0, T) \times \Omega, \\
|w|_{[0, T] \times \partial\Omega} = 0, \\
|w|_{t=T} = |u|_{t=T} - \phi, \\
|w_t|_{t=T} = |u_t|_{t=T},
\end{cases}
\]

where \( u \) solves

\[
\begin{cases}
(\partial_t^2 - c^2 \Delta) u = 0 & \text{in } (0, T) \times \mathbb{R}^n, \\
|u|_{t=0} = f, \\
\partial_t u|_{t=0} = 0.
\end{cases}
\]
Consider the “error operator” \( K \),

\[ A \Lambda f = f - Kf \]

\( Kf = \) first component of: \( U_{\Omega,D}(-T)\Pi_{\Omega}U_{\mathbb{R}^n}(T)[f,0] \),

where

- \( U_{\mathbb{R}^n}(t) \) is the dynamics in the whole \( \mathbb{R}^n \),
- \( U_{\Omega,D}(t) \) is the dynamics in \( \Omega \) with Dirichlet BC,
- \( \Pi_{\Omega} : \mathcal{H}(\mathbb{R}^n) \rightarrow \mathcal{H}(\Omega) \) is the orthogonal projection.

That projection is given by \( \Pi_{\Omega}[f, g] = [f|_\Omega - \phi, g|_\Omega] \), where \( \phi \) is the harmonic extension of \( f|_{\partial\Omega} \).

Obviously,

\[ \|Kf\|_{H_D} \leq \|f\|_{H_D}. \]
Reconstruction (whole boundary)

**Theorem (Stefanov–U, IP 2009)**

Let $T > T_1/2$. Then $A\Lambda = I - K$, where $\|K\|_{\mathcal{L}(H_D(\Omega))} < 1$. In particular, $I - K$ is invertible on $H_D(\Omega)$, and the inverse thermoacoustic problem has an explicit solution of the form

$$f = \sum_{m=0}^{\infty} K^m Ah, \quad h := \Lambda f.$$

If $T > T_1$, then $K$ is compact.
We have the following estimate on $\|K\|:

**Theorem (Stefanov–U, IP 2009)**

$$\|Kf\|_{H_D(\Omega)} \leq \left( \frac{E_\Omega(u, T)}{E_\Omega(u, 0)} \right)^{\frac{1}{2}} \|f\|_{H_D(\Omega)}, \quad \forall f \in H_D(\Omega), \; f \neq 0,$$

where $u$ is the solution with Cauchy data $(f, 0)$. 
Summary: Dependence on $T$

(i) $T < T_0 \implies$ no uniqueness

$\Lambda f$ does not recover uniquely $f$. $\|K\| = 1$.

(ii) $T_0 < T < T_1/2 \implies$ uniqueness, no stability

We have uniqueness but not stability (there are invisible singularities). We do not know if the Neumann series converges. $\|Kf\| < \|f\|$ but $\|K\| = 1$.

(iii) $T_1/2 < T < T_1 \implies$ stability and explicit reconstruction

This assumes that $c$ is non-trapping. The Neumann series converges exponentially but maybe not as fast as in the next case ($K$ contraction but not compact). There is stability (we detect all singularities but some with $1/2$ amplitude). $\|K\| < 1$

(iv) $T_1 < T \implies$ stability and explicit reconstruction

The Neumann series converges exponentially, $K$ is contraction and compact (all singularities have left $\Omega$ by time $t = T$). There is stability. $\|K\| < 1$

If $c$ is trapping ($T_1 = \infty$), then (iii) and (iv) cannot happen.
Numerical Experiments (Qian-Stefanov-U-Zhao, SIAM J. Imaging, 2011)

Sound speed models

non-trapping speed $c_1$  \hspace{1cm}  radial trapping speed $c_2$  \hspace{1cm}  trapping speed $c_3$

**Figure:** Sound speed models
Shepp-Logan phantom: non-trapping $c_1$ (1)

![Boundary distance map](image1)

**The boundary distance**

![The exact initial condition](image2)

**Exact**

![The Neumann series solution](image3)

![The time reversal solution](image4)

**NS**

**TR**

**Figure:** Example 1, non-trapping $c_1$, $T = 2T_0$. 
Shepp-Logan phantom: non-trapping $c_1$ (2)

The boundary distance

Exact

NS

TR

Figure: Example 1, non-trapping $c_1$, $T = 4T_0$. 
Shepp-Logan phantom: non-trapping $c_1$ (3)

The boundary distance  

Exact

NS  

TR

**Figure:** Example 1, non-trapping $c_1$, $T = 4 T_0$, with 10% noise.
Shepp-Logan phantom: trapping $c_3$ (4)

The exact initial condition

The Neumann series solution

The boundary distance

NS

TR

Figure: Example 1, trapping $c_3$, $T = 4T_0$. 
**Zebras: non-trapping** \( c_1 \) (1)

**Figure:** Example 2, non-trapping \( c_1 \), \( T = 4T_0 \).
Zebras: trapping $c_3$ (2)

The boundary distance

Exact

NS

TR

Figure: Example 2, trapping $c_3$, $T = 4T_0$. 
Zebras: radial trapping $c_2$ (3)

The boundary distance

The exact initial condition

The Neumann series solution

The time reversal solution

NS

TR

**Figure:** Example 2, radial trapping $c_2$, $T = 4T_0$. 
Let $c$ be piecewise smooth with a jump across a smooth closed surface $\Gamma$. The direct problem is a transmission problem, and there are reflected and refracted rays.

In brain imaging, the interface is the skull. The sound speed jumps by about a factor of 2 there. Experiments show that the ray that arrives first carries about 20% of the energy.
Propagation of singularities is the key again. (Completely) trapped singularities are a problem, as before. Let $\mathcal{K} \subset \Omega$ be a compact set such that all rays originating from it are never tangent to $\Gamma$ and non-trapping. For $f$ satisfying 

$$\text{supp } f \subset \mathcal{K}$$

the Neumann series above still converges (uniformly to $f$).

We need a small modification to keep the support in $\mathcal{K}$ all the time. We use the projection

$$\Pi_{\mathcal{K}} : H_D(\Omega) \rightarrow H_D(\mathcal{K})$$

for that purpose.
Reconstruction

Theorem (Stefanov–U, IP 2011)

Let all rays from $K$ have a path never tangent to $\Gamma$ that reaches $\partial \Omega$ at time $|t| < T$. Then

$$\Pi_K A \Lambda = I - K \text{ in } H_D(K), \text{ with } \|K\|_{H_D(K)} < 1.$$ 

In particular, $I - K$ is invertible on $H_D(K)$, and $\Lambda$ restricted to $H_D(K)$ has an explicit left inverse of the form

$$f = \sum_{m=0}^{\infty} K^m \Pi_K A h, \quad h = \Lambda f.$$ 

The assumption $\text{supp } f \subset K$ means that we need to know $f$ outside $K$; then we can subtract the known part.

In the numerical experiments below, we do not restrict the support of $f$, and still get good reconstruction images but the invisible singularities remain invisible.
Numerical experiments
discontinuous sound speed models

Figure: **Left**: a discontinuous piecewise sound speed $c_4$; **Right**: a non-piecewise constant discontinuous sound speed $c_5$. 
Shepp-Logan phantom: discontinuous speed $c_4$ (1)

The boundary distance

Exact

NS

TR

**Figure:** Example 3, discontinuous sound speed $c_4$, $T = 4 T_0$. 
Shepp-Logan phantom: discontinuous speed $c_5$ (2)

The boundary distance

The exact initial condition

The Neumann series solution

The time reversal solution

NS

TR

**Figure:** Example 3, discontinuous sound speed $c_5$, $T = 4T_0$. 
Zebras: discontinuous speed $c_5$

**Figure:** Example 2, discontinuous sound speed $c_5$, $T = 4T_0$. 
The speed jumps by a factor of 2 in average from the exterior of the "skull". The region \( \Omega \), as before, is smaller: \( \Omega = [-1.28, 1.28]^2 \).
A “skull” speed, Neumann series

\[
T = 2 T_0, \text{ error } = 15% \\
T = 4 T_0, \text{ error } = 9.75% \\
T = 8 T_0, \text{ error } = 7.55%
\]

Neumann Series, 15 steps
A “skull” speed, Time Reversal

original

$T = 2T_0$, error = 68%

$T = 4T_0$, error = 23.7%

$T = 8T_0$, error = 78.5%

Time Reversal. There is a lot of “white clipping” in the last image, many values in $[1, 1.6]$. 
A “skull” speed, Time Reversal

Time Reversal. The values in last image are compressed from $[0,1]$ to $[-0.05, 1.6]$
Original vs. Neumann Series vs. Time Reversal

\[ T = 8T_0. \text{ Original vs. Neumann Series vs. Time Reversal} \]

\( \text{original} \quad \text{NS, error} = 7.55\% \quad \text{TR, error} = 78.5\% \)

(the latter compressed from \([0, 1]\) to \([-0.05, 1.6]\))
Assume that $c = 1$ outside $\Omega$. Let $\Gamma \subset \partial \Omega$ be a relatively open subset of $\partial \Omega$. Assume now that the observations are made on $[0, T] \times \Gamma$ only, i.e., we assume we are given

$$\Lambda f |_{[0, T] \times \Gamma}.$$ 

We consider $f$’s with

$$\text{supp } f \subset \mathcal{K},$$

where $\mathcal{K} \subset \Omega$ is a fixed compact.
Heuristic arguments for uniqueness: To recover $f$ from $\Lambda f$ on $[0, T] \times \Gamma$, we must at least be able to get a signal from any point, i.e., we want for any $x \in K$, at least one “signal” from $x$ to reach some $\Gamma$ for $t < T$. Set

$$T_0(K) = \max_{x \in K} \text{dist}(x, \Gamma).$$

The uniqueness condition then should be

$$T \geq T_0(K). \quad (\ast)$$

Theorem (Stefanov–U, IP 2011)

Let $c = 1$ outside $\Omega$, and let $\partial \Omega$ be strictly convex. Then if $T \geq T_0(K)$, if $\Lambda f = 0$ on $[0, T] \times \Gamma$ and $\text{supp} \, f \subset K$, then $f = 0$.

Proof based on Tataru’s uniqueness continuation results. Generalizes a similar result for constant speed by Finch, Patch and Rakesh.

As before, without $(\ast)$, one can recover $f$ on the reachable part of $K$. Of course, one cannot recover anything outside it, by finite speed of propagation. Therefore, $(\ast)$ is an “if and only if” condition for uniqueness with partial data.
Heuristic arguments for stability: To be able to recover $f$ from $\Lambda f$ on $[0, T] \times \Gamma$ in a stable way, we need to recover all singularities. In other words, we should require that

$$\forall (x, \xi) \in K \times S^{n-1}, \text{ the ray (geodesic) through it reaches } \Gamma \text{ at time } |t| < T.$$ 

We show next that this is an “if and only if” condition (up to replacing an open set by a closed one) for stability. Actually, we show a bit more.

Proposition (Stefanov–U)

If the stability condition is not satisfied on $[0, T] \times \overline{\Gamma}$, then there is no stability, in any Sobolev norms.
A reformulation of the stability condition

- Every geodesic through $\mathcal{K}$ intersects $\Gamma$.
- $\forall (x, \xi) \in \mathcal{K} \times S^{n-1}$, the travel time along the geodesic through it satisfies $|t| < T$.

Let us call the least such time $T_{1/2}$, then $T > T_{1/2}$ as before. In contrast, any small open $\Gamma$ suffices for uniqueness.
Let \( A \) be the “modified time reversal” operator as before. Actually, \( \phi \) will be 0 because of \( \chi \) below. Let \( \chi \in C_0^\infty([0, T] \times \partial \Omega) \) be a cutoff (supported where we have data).

**Theorem**

\[ A \chi \Lambda \text{ is a zero order classical } \Psi DO \text{ in some neighborhood of } K \text{ with principal symbol} \]

\[ \frac{1}{2} \chi(\gamma_x, \xi(\tau_+(x, \xi))) + \frac{1}{2} \chi(\gamma_x, \xi(\tau_-(x, \xi))). \]

If \([0, T] \times \Gamma\) satisfies the stability condition, and \(|\chi| > 1/C > 0\) there, then

(a) \( A \chi \Lambda \) is elliptic,
(b) \( A \chi \Lambda \) is a Fredholm operator on \( H_D(K) \),
(c) there exists a constant \( C > 0 \) so that

\[ \|f\|_{H_D(K)} \leq C\|\Lambda f\|_{H^1([0, T] \times \Gamma)}. \]

(b) follows by building a parametrix, and (c) follows from (b) and from the uniqueness result.

In particular, we get that for a fixed \( T > T_1 \), the classical Time Reversal is a parametrix (of infinite order, actually).
Reconstruction

One can constructively write the problem in the form

Reducing the problem to a Fredholm one

\[(I - K)f = BA\chi\Lambda f \quad \text{with the r.h.s. given,}\]

i.e., \(B\) is an explicit operator (a parametrix), where \(K\) is compact with 1 not an eigenvalue.

Constructing a parametrix without the ΨDO calculus.

Assume that the stability condition is satisfied in the interior of \(\text{supp} \chi\). Then

\[A\chi\Lambda f = (I - K)f,\]

where \(I - K\) is an elliptic ΨDO with \(0 \leq \sigma_p(K) < 1\). Apply the formal Neumann series of \(I - K\) (in Borel sense) to the l.h.s. to get

\[f = (I + K + K^2 + \ldots)A\chi\Lambda f \mod C^\infty.\]
Numerical Experiments: partial data

Zebras: non-trapping speed $c_1$, one-side missing

The boundary distance map

The exact initial condition

The Neumann series solution

**Figure:** non-trapping sound speed $c_1$, one-side missing, $T = 4.7$. 
Zebras: non-trapping speed $c_1$, two-side missing

The boundary distance map

**Figure:** non-trapping sound speed $c_1$, two-side missing, $T = 4.7$.  

The exact initial condition

The Neumann series solution
Modified zebras: non-trapping speed \( c_1 \), two-side missing

The exact initial condition

The Neumann series solution

The boundary distance map

**Figure:** non-trapping sound speed \( c_1 \), two-side missing, \( T = 4.7 \).
Zebras: trapping speed $c_3$, one-side missing

The boundary distance map

Exact NS

Figure: trapping sound speed $c_3$, one-side missing, $T = 4.92$. 
Zebras: trapping speed $c_3$, two-side missing

The boundary distance map

**Figure:** trapping sound speed $c_3$, two-side missing, $T = 4.92$. 
Examples: Non-trapping speed, 1 and 2 sides missing

original    NS, 3 sides, error = 7.99%  NS, 2 sides, error = 12.2%

Partial data reconstruction, non-trapping speed, $T = 4T_0$. 
In the diffusive regime, **optical radiation** is modeled by:

\[-\nabla \cdot \gamma(x) \nabla u + \sigma(x) u = 0 \text{ in } X \quad u = g \text{ on } \partial X \quad \text{Illumination},\]

\[H(x) = \Gamma(x)\sigma(x)u(x) \text{ in } X \quad \text{Internal Functional}.\]

The **objectives** of quantitative PAT are to understand:

- What we can reconstruct of \((\gamma(x), \sigma(x), \Gamma(x))\) from knowledge of \(H_j(x), \quad 1 \leq j \leq J\) obtained for illuminations \(g = g_j, \quad 1 \leq j \leq J\).

- How **stable** the reconstructions are.

- How to choose \(J\) and the illuminations \(g_j\).
In Thermo-Acoustic Tomography, low-frequency radiation is used.

Using a (scalar) Helmholtz model for radiation, quantitative TAT is

\[ \Delta u + n(x)k^2 u + ik\sigma(x)u = 0 \quad \text{in } X, \quad u = g \quad \text{on } \partial X \quad \text{Illumination,} \]

\[ H(x) = \sigma(x)|u|^{2}(x) \quad \text{in } X \quad \text{Internal Functional.} \]

QTAT consists of uniquely and stably reconstructing \( \sigma(x) \) from knowledge of \( H(x) \) for appropriate illuminations \( g \).
QPAT with two/more measurements

\[-\nabla \cdot \gamma(x)\nabla u + \sigma(x)u = 0 \text{ in } X, \quad u = g \text{ on } \partial X, \quad H(x) = \Gamma(x)\sigma(x)u(x).\]

Let \((g_1, g_2)\) providing \((H_1, H_2)\). Define \(\beta = H_1^2 \nabla \frac{H_2}{H_1}\). \textbf{IF:} \(|\beta| \geq c_0 > 0\), then

Theorem (Bal-U’10, Bal-Ren’11)

(i) \((H_1, H_2)\) uniquely determine the whole measurement operator \(g \in H^{\frac{1}{2}}(\partial X) \mapsto \mathcal{H}(g) = H \in H^1(X)\).

(ii) The measurement operator \(\mathcal{H}\) uniquely determines

\[
\chi(x) := \frac{\sqrt{\gamma}}{\Gamma \sigma}(x), \quad q(x) := -\left(\frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}} + \frac{\sigma}{\gamma}\right)(x).
\]

(iii) \((\chi, q)\) uniquely determine \((H_1, H_2)\).

Two well-chosen measurements suffice to reconstruct \((\chi, q)\) and thus \((\gamma, \sigma, \Gamma)\) up to transformations leaving \((\chi, q)\) invariant.
The proof of (i) & (ii) is based on the elimination of $\sigma$ to get

$$-\nabla \cdot \chi^2 \left[ H_1^2 \nabla \frac{H}{H_1} \right] = 0 \text{ in } X \quad (\chi, H) \text{ known on } \partial X.$$  

Then we verify that $q := -\left( \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}} + \frac{\sigma}{\gamma} \right)(x) = -\frac{\Delta(\chi H_1)}{\chi H_1}$.

(iii) Finally, define $(\Delta + q)v_j = 0$ to get $H_j = \frac{v_j}{\chi}$. The IF implies that vector field $H_1^2 \nabla \frac{\nu_2}{\nu_1} \neq 0$. This is a qualitative statement on the absence of critical points of elliptic solutions.
Assuming **IF** satisfied, then the reconstruction of (e.g.) $\chi$ is **stable**.

**CGO** method. Analyzing the transport equation by the method of characteristics and using CGO solutions, we show that for appropriate illuminations (and for $k \geq 3$):

$$\| \chi - \tilde{\chi} \|_{C^{k-1}(\mathcal{X})} \leq C \| H - \tilde{H} \|_{(C^k(\mathcal{X}))^2}.$$
Reconstruction of two discontinuous parameters
Stability result for QTAT

$$\Delta u + k^2 u + i\sigma(x)u = 0 \text{ in } X, \quad u = g \text{ on } \partial X, \quad H(x) = \sigma(x)|u|^2.$$ 

**Theorem (Bal, Ren, U, Zhou’11)**

Let $\sigma$ and $\tilde{\sigma}$ be uniformly bounded functions in $Y = H^p(X)$ for $p > n$ with $X$ the bounded support of the unknown conductivity.

Then there is an open set of illuminations $g$ such that

$$H(x) = \tilde{H}(x) \text{ in } Y \quad \text{implies that} \quad \sigma(x) = \tilde{\sigma}(x) \text{ in } Y.$$ 

Moreover, there exists $C$ such that

$$\|\sigma - \tilde{\sigma}\|_Y \leq C\|H - \tilde{H}\|_Y.$$ 

The **inverse scattering problem with internal data** is well posed. We apply a Banach fixed point IF appropriate functional is a contraction.
Discontinuous conductivity in TAT
Vector fields and complex geometrical optics

- Take $\rho \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$. Then $\Delta e^{\rho \cdot x} = 0$. For $u_j = e^{\rho_j \cdot x}$, $j = 1, 2$:

$$\Im \left( e^{-(\rho_1 + \rho_2) \cdot x} u_1^2 \nabla u_2 \right) = \Im (\rho_2 - \rho_1),$$

is a constant vector field $2k$ for $\rho_1 = k + ik^\perp$ and $\rho_2 = \bar{\rho}_1$.

- Let $u_\rho(x) = e^{\rho \cdot x} (1 + \psi_\rho(x))$ solution of $\Delta u_\rho + qu_\rho = 0$.

**Theorem (Bal-U’10)**

For $q$ sufficiently smooth and $k \geq 0$, we have

$$|\rho| \|\psi_\rho\|_{H^{n/2 + k + \varepsilon}(X)} + \|\psi_\rho\|_{H^{n/2 + k + 1 + \varepsilon}(X)} \leq C \|q\|_{H^{n/2 + k + \varepsilon}(X)}.$$

- For illuminations $g$ on $\partial X$ close to traces of CGO solutions constructed in $\mathbb{R}^d$, we obtain “nice” vector fields $|\beta| \geq c_0 > 0$ and thus an open set of illuminations $g$ for which stable reconstructions are guaranteed.
Several HIPs require to verify qualitative properties of elliptic solutions:

- the absence of critical points in QPAT
- the contraction of appropriate functionals in QTAT

The existence of open sets of illuminations $g_j$ such that these properties hold is obtained by means of CGO solutions.
Consider a general scalar elliptic equation

\[ \nabla \cdot a \nabla u + b \cdot \nabla u + cu = 0 \quad \text{in } X, \quad u = f \quad \text{on } \partial X \]

with \(a, b, c, \nabla \cdot a\) of class \(C^{0,\alpha}(\bar{X})\) for \(\alpha > 0\), complex-valued, and \(\alpha_0|\xi|^2 \leq \xi \cdot (\Re a) \xi \leq \alpha_0^{-1}|\xi|^2\). For \(\tau > 0\) a function on \(X\), define

\[ a_\tau = \tau a, \quad b_\tau = \tau b - a \nabla \tau, \quad c_\tau = \tau c \]

and the equivalence class \(c := (a, b, c) \sim (a_\tau, b_\tau, c_\tau)\).

Let \(l \in \mathbb{N}^*\) and \(f_i \in H^{\frac{1}{2}}(\partial X)\) for \(1 \leq i \leq l\) be \(l\) boundary conditions. Define \(\tilde{f} = (f_1, \ldots, f_l)\). The measurement operator \(M_{\tilde{f}}\) is

\[ M_{\tilde{f}} : c \mapsto M_{\tilde{f}}(c) = (u_1, \ldots, u_l), \]

with \(u_j\) solution of the above elliptic problem with \(f = f_j\).
Unique reconstruction up to gauge transformation

\[ \nabla \cdot a \nabla u_j + b \cdot \nabla u_j + cu_j = 0 \quad \text{in} \ X, \quad u_j = f_j \quad \text{on} \ \partial X, \quad 1 \leq j \leq I. \]

**Theorem** [Bal-U 2012]. Let \( c \) and \( \tilde{c} \) be two classes of coefficients with \((a, b, c)\) and \( \nabla \cdot a \) of class \( C^{m,\alpha}(\overline{X}) \) for \( \alpha > 0 \) and \( m = 0 \) or \( m = 1 \). We assume that the above elliptic equation is well posed for \( c = (a, b, c) \).

Then for \( I \) sufficiently large and for an open set of boundary conditions \( f = (f_j)_{1 \leq j \leq I} \), then \( M_f(c) \) uniquely determines \( c \). Moreover, we have the following stability estimate

\[
\| (a, b + \nabla \cdot a, c) - (\tilde{a}, \tilde{b} + \nabla \cdot \tilde{a}, \tilde{c}) \|_{W^{m,\infty}(X)} \leq C \| M_f(c) - M_f(\tilde{c}) \|_{W^{m+2,\infty}(X)},
\]

\[
\| b - \tilde{b} \|_{L^\infty(X)} \leq C \| M_f(c) - M_f(\tilde{c}) \|_{W^{3,\infty}(X)},
\]

for \( m = 0, 1 \) and for an appropriate \((\tilde{a}, \tilde{b}, \tilde{c})\) of \( \tilde{c} \).

In several settings (typically when CGO solutions are available), we can choose \( I = I_n = \frac{1}{2} n(n+3) \) with \( n \) spatial dimension and with \( I_n \) the dimension of \( c \).
In some situations (such as the practical settings of QPAT and TE/MRE), the gauge in $c$ can be uniquely and stably determined:

**Corollary** [Bal-U 2012] Under the hypotheses of the preceding theorem, and in the setting where $b = 0$, then $M_f(c)$ uniquely determines $(\gamma, 0, c)$. Let us define $\gamma = \tau M^0$ where $M^0$ has a determinant equal to 1. Then we have the following stability result:

$$\|\tau - \tilde{\tau}\|_{W^{1,\infty}(X)} + \|(M^0, c) - (\tilde{M}^0, \tilde{c})\|_{L^{\infty}(X)} \leq C\|M_f(c) - \tilde{M}_f(\tilde{c})\|_{W^{2,\infty}(X)}.$$ 

From the practical point of view, the reconstruction of the determinant of $\gamma$ is *more stable* than the reconstruction of the anisotropy of the (possibly complex valued) tensor $\gamma$. This has been observed numerically in a different setting.