Summer School on Mathematical Physics

Inverse Problems: Visibility and Invisibility

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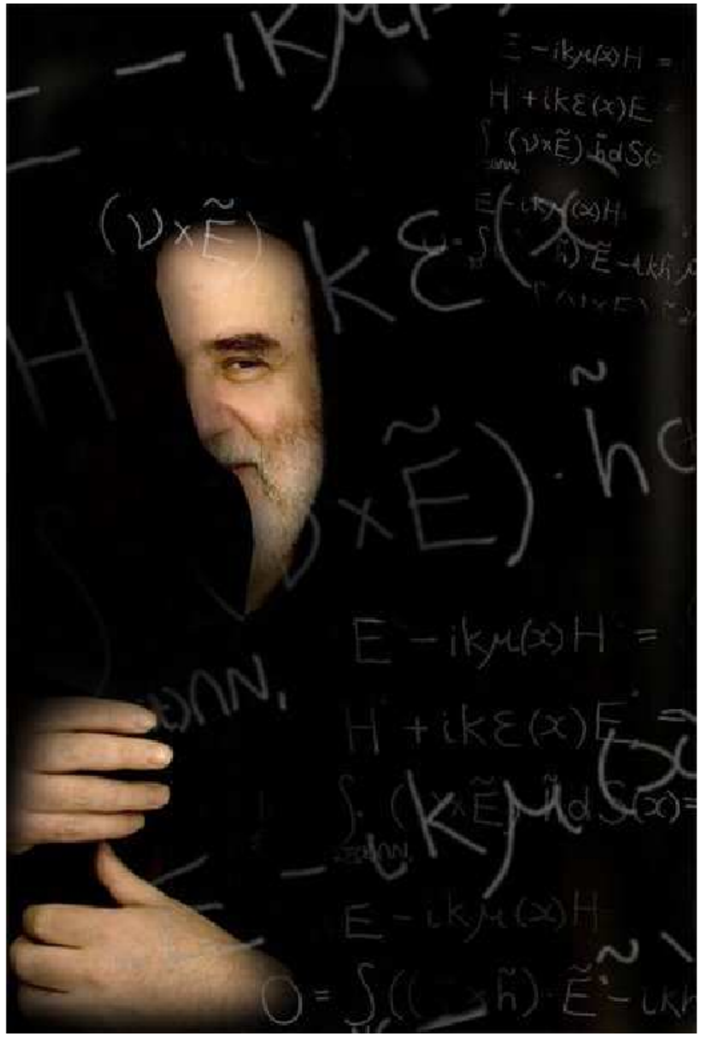
H. G. Wells: The Invisible Man (1897)

Susan Storm Richards: The Invisible Woman (1961)

Star Trek: Cloaked Romulan Bird of Prey

Harry Potter's Cloak



Photographer: Mary Levin

Photographer: Steve Zylius and Hang X. Pham

Photographer: Steve Zylius and Hang X. Pham

Transformation Optics

Two Articles in 2006 in Science on Invisibility "Controlling Electromagnetic Fields", **J.B. Pendry, D. Schurig, D.R. Smith**, Science **312**, pp. 1780-1782, (June 2006).

Related article by **Ulf Leonhard** "Optical Conformal Mapping" in same issue.

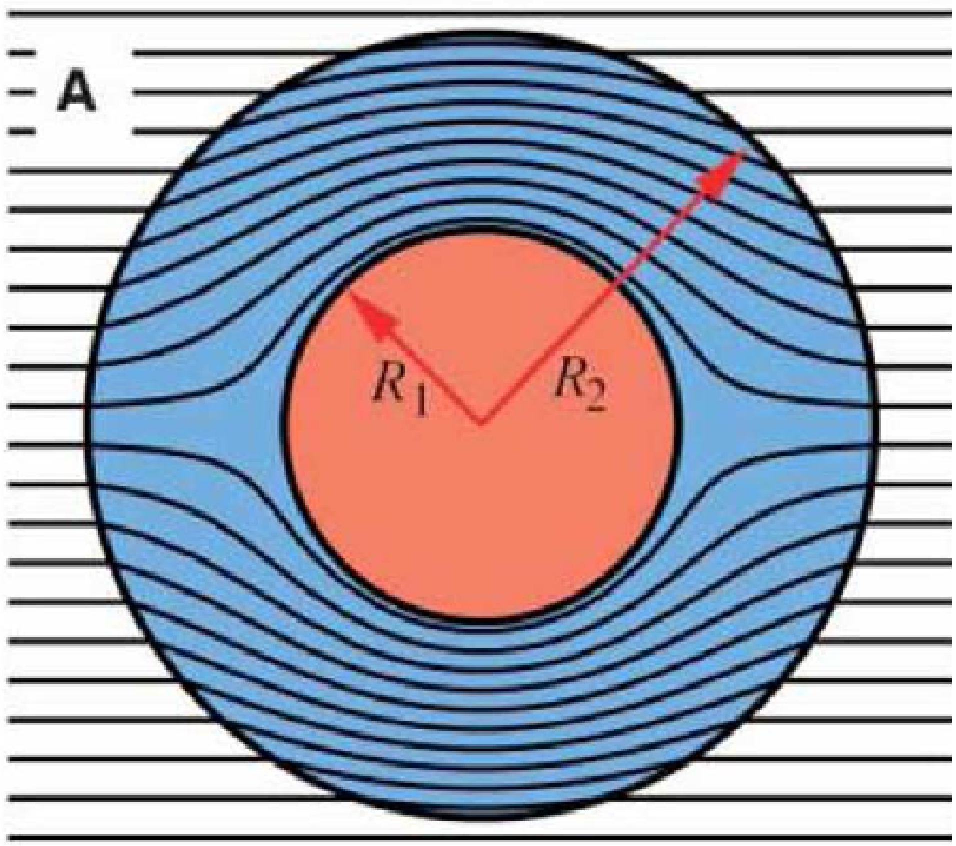
Earlier article of **A. Greenleaf, M. Lassas and G-U**, "Non-uniqueness for Calderón's problem", Math. Research Letters, 2003.

Transformation Optics

Science Magazine

No. 5, Breakthrough of 2006: THE ULTIMATE CAMOUFLAGE

"The real breakthrough may lie in the theoretical tools used to make the cloak. In such "transformation optics," researchers imagine–à la Einstein–warping empty space to bend the path of electromagnetic waves. A mathematical transformation then tells them how to mimic the bending by filling unwarped space with a material whose optical properties vary from point to point. The technique could be used to design antennas, shields, and myriad other devices. Any way you look at it, the ideas behind invisibility are likely to cast a long shadow."



From Pendry et al's paper

All Boundary measurements for the homogeneous conductivity $\gamma = 1$ and the degenerate conductivity $\tilde{\sigma}$ are the same

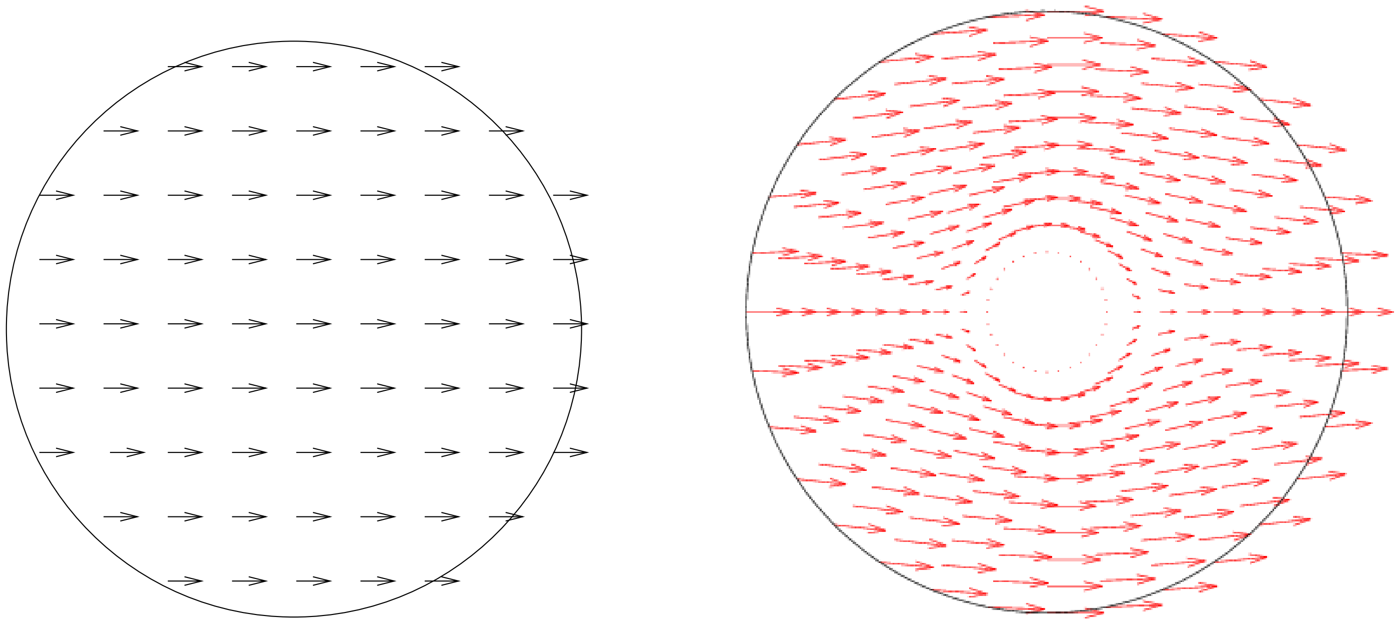
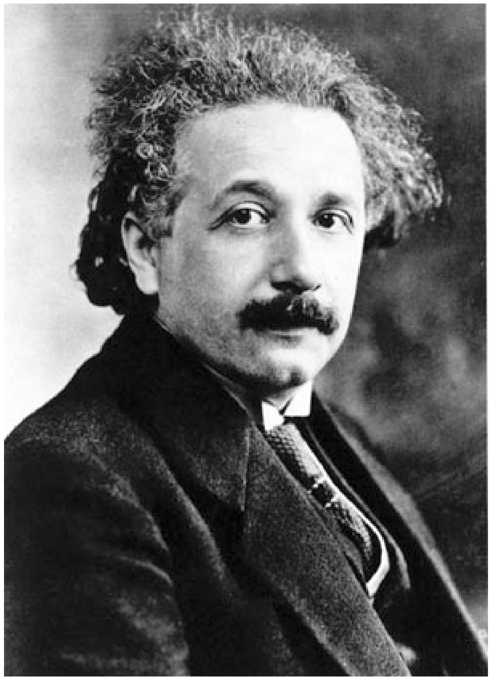
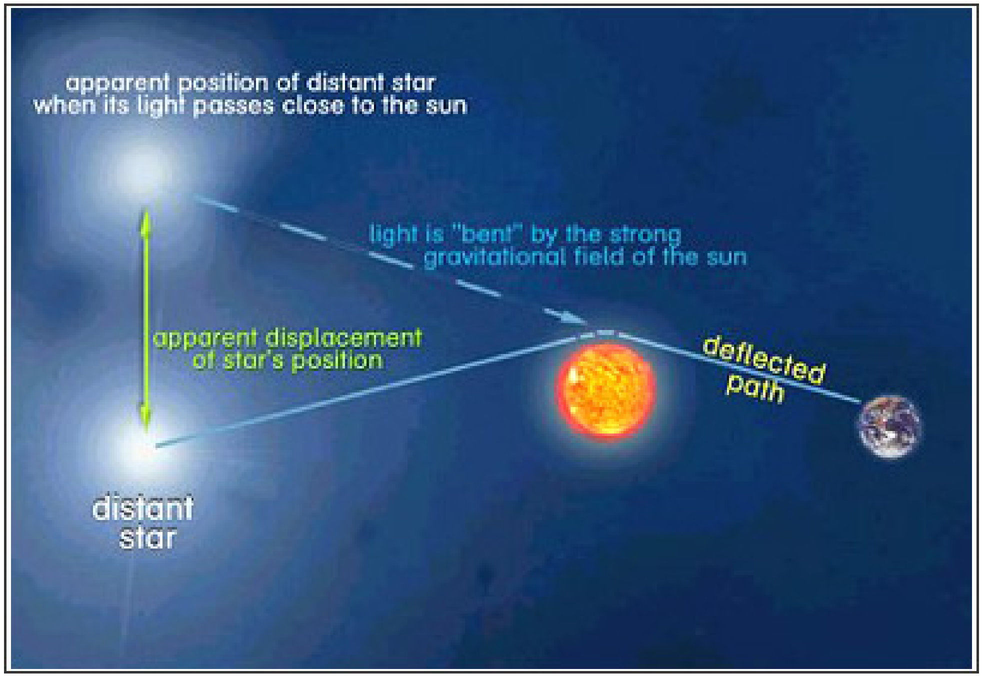


Figure: Analytic solutions for the currents

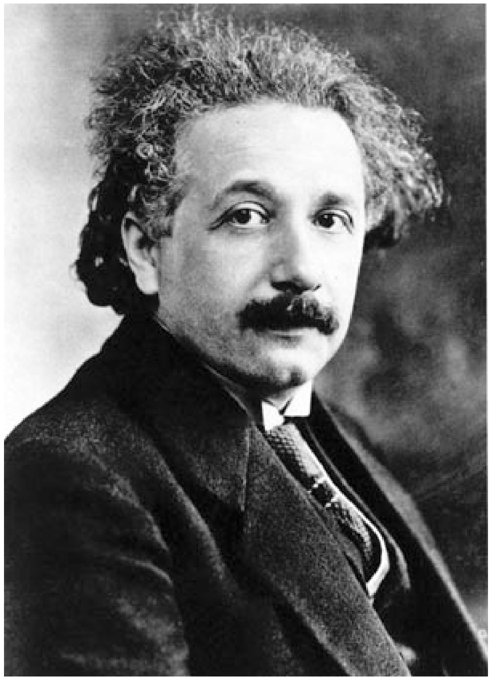
Based on work of Greenleaf-Lassas-U, MRL 2003

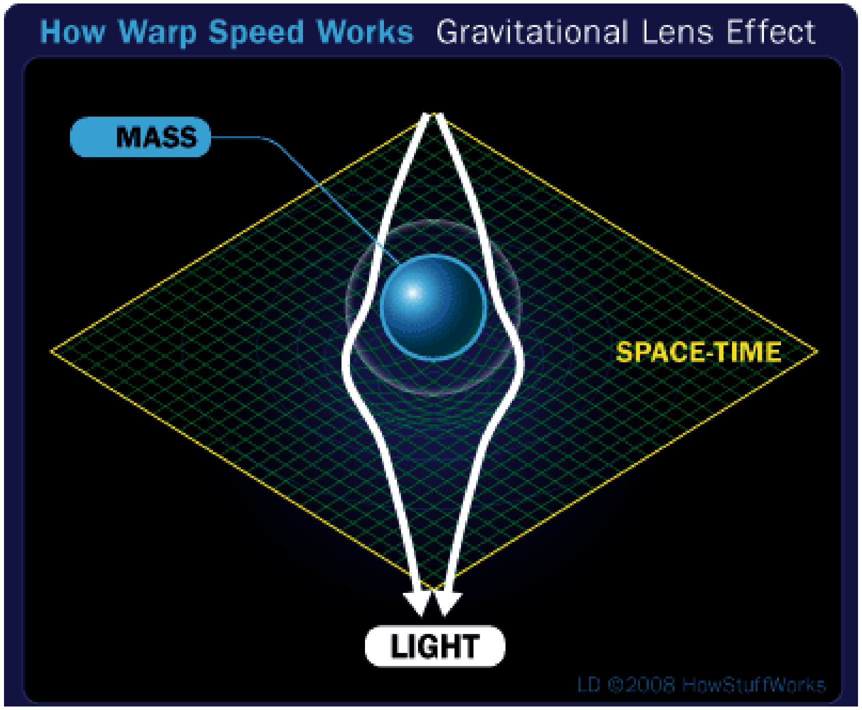
Bending Light



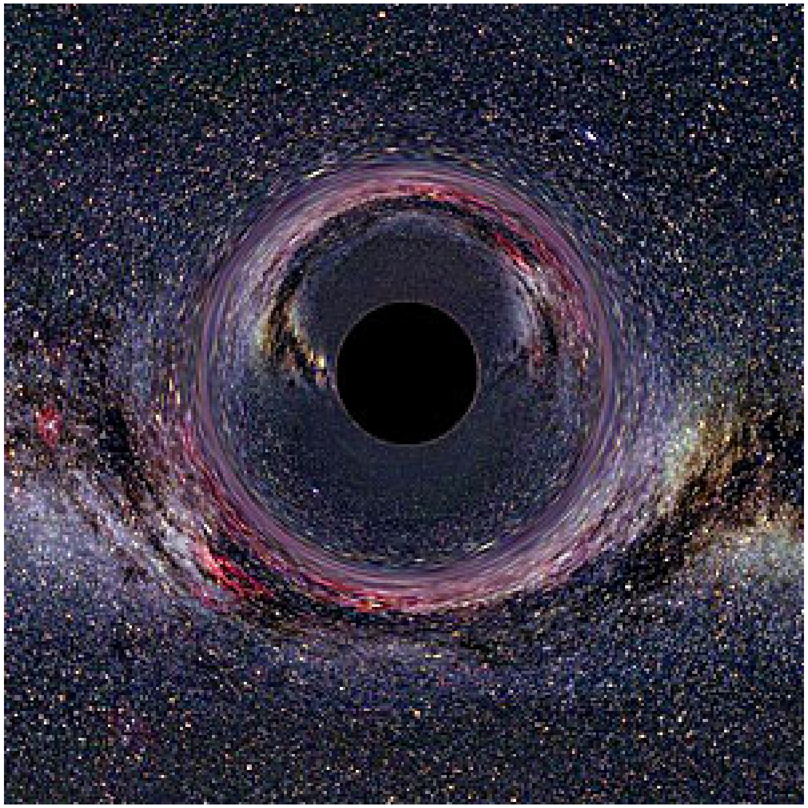


How to Deflect Light

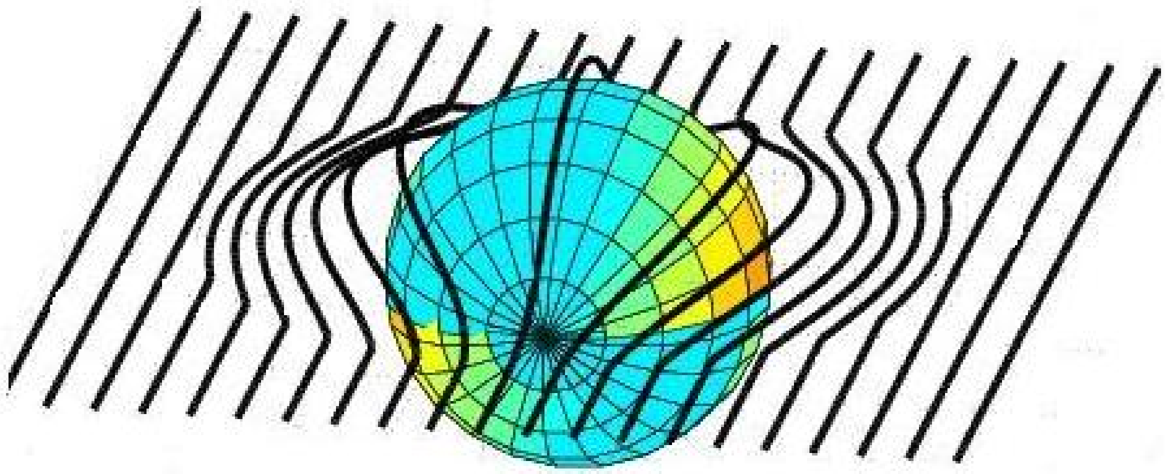




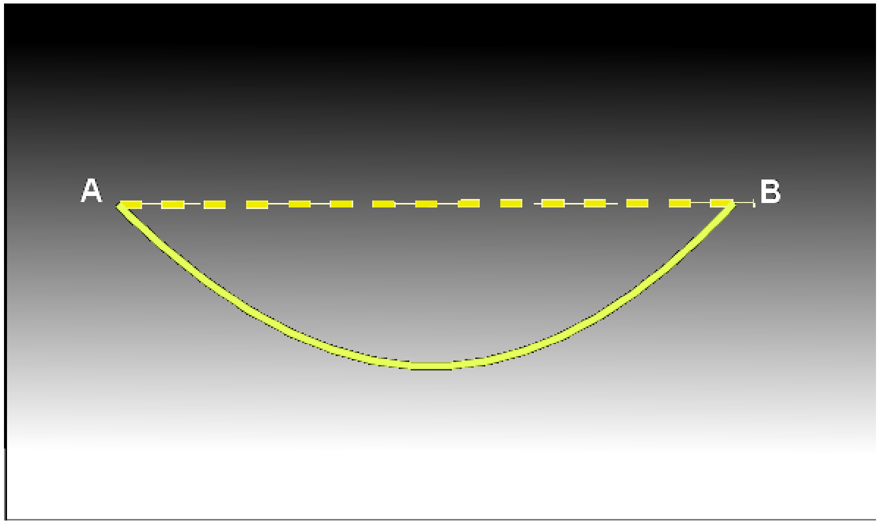
Black Hole



Optical White Hole



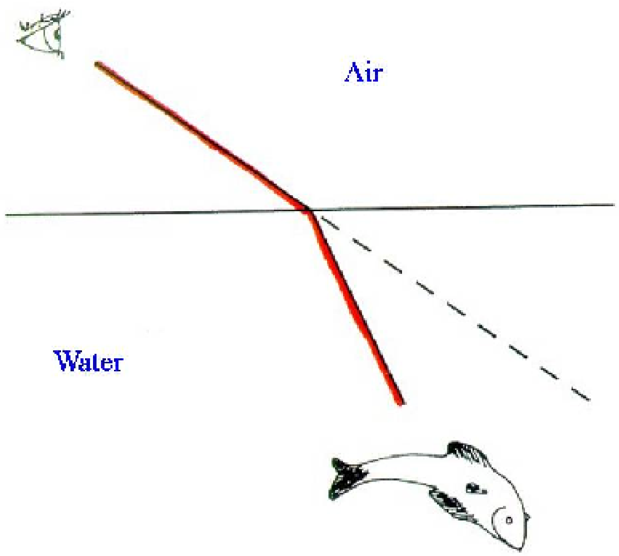
Ray Theory of Light: Fermat's principle



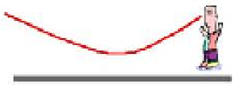
Fermat's principle. Light takes the shortest optical path from A to B (solid line) which is not a straight line (dotted line) in general. The optical path length is measured in terms of the refractive index n integrated along the trajectory. The greylevel of the background indicates the refractive index; darker tones correspond to higher refractive indices.

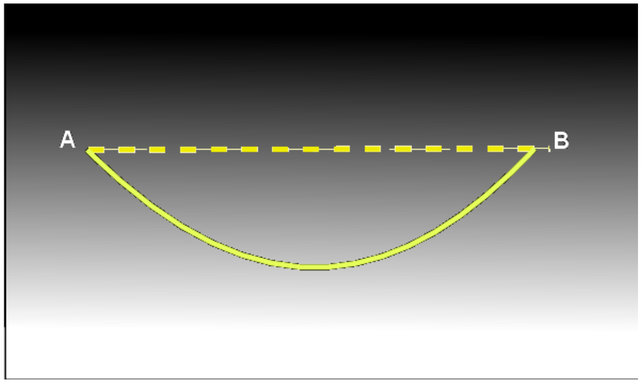
Index of Refraction

Fermat's Principle: Minimize Optical Length



Mirage





S. Cummer, B. Pope, D. Schurig, D. Smith and J. Pendry, "Full-wave simulations of electromagnetic cloaking structures", Phys. Rev. E **74** 036621 (2006)

"It is open problem whether full-wave cloaking is possible, even in theory"

Answer: It is possible for all frequencies for electromagnetic waves. This is joint work with A. Greenleaf, Y. Kurylev and M. Lassas. (Comm. Math. Physics, 2007).

Based on work of A. Greenleaf, Y. Kurylev, M. Lassas–U

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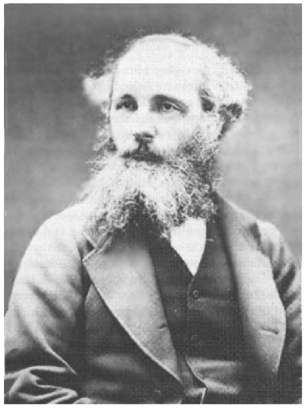
Wave Theory of Light: Maxwell's Equations

$$\boxed{\begin{aligned}&\nabla \times E - ik\mu(x)H = 0\\ &\nabla \times H + ik\epsilon(x)E = 0\\ &D = \epsilon E, B = \mu H\\ &\text{div}D = 0, \text{div}B = 0\end{aligned}}$$



J. Maxwell

(E, H) Electromagnetic Field

D=electric displacement B=magnetic displacement	$\epsilon(x)$=electric permittivity $\mu(x)$=magnetic permeability

Index of refraction: $\sqrt{\epsilon\mu}$

We will consider a special case of anisotropic materials:

$\boxed{\epsilon(x) = \mu(x) = g^{-1}\sqrt{\det g}}$ were $g = (g_{ij})$ is a semipositive definite symmetric matrix

Similar (Helmholtz, Acoustic waves, Quantum waves)

$$\boxed{\begin{aligned} \Delta_g E + k^2 E = 0 \\ \Delta_g H + k^2 H = 0 \end{aligned}}$$

$$\begin{aligned} \Delta_g \quad &= \text{Laplace-Beltrami operator} \\ &= \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}(\sqrt{\det g}\ g^{ij}\frac{\partial}{\partial x_j}) \end{aligned}$$

$(g^{ij}) = (g_{ij})^{-1}$.

(Helmholtz)

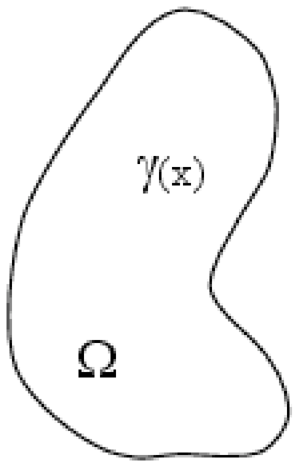
$$\boxed{\begin{array}{l} \Delta_g E + k^2 E = 0 \\ \Delta_g H + k^2 H = 0 \end{array}}$$

Consider first static case ($k = 0$)

$$\Delta_g E = \Delta_g H = 0$$

This problem in dimension $n \geq 3$ is equivalent to the Electrical Impedance Tomography (Calderón's Problem)

CALDERÓN'S PROBLEM



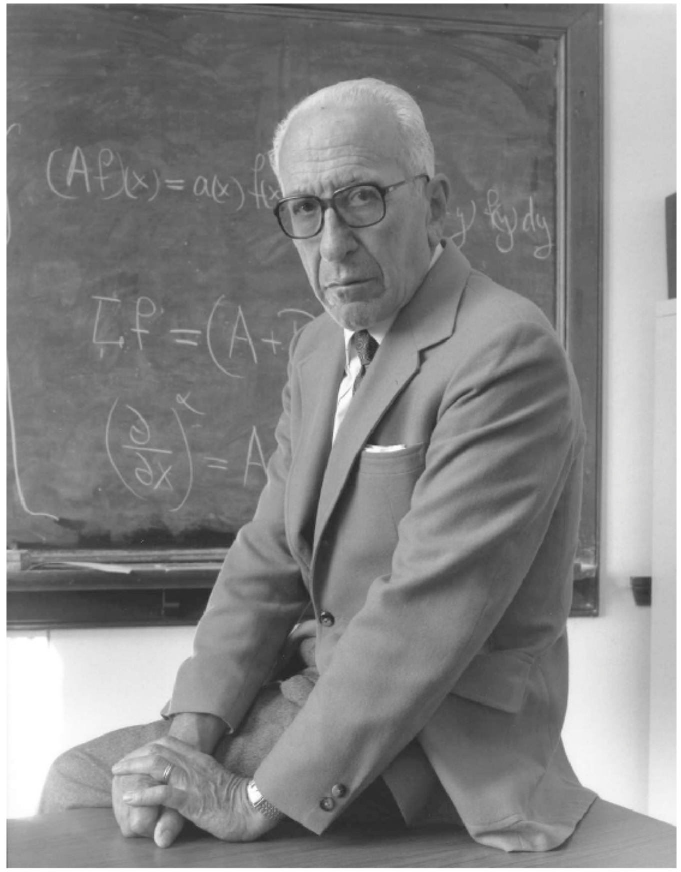
$\Omega \subset \mathbb{R}^n$

$(n = 2, 3)$

Can one determine the electrical conductivity of $\Omega, \gamma(x)$, by making voltage and current measurements at the boundary?
(Calderón; Geophysical prospection)

Early breast cancer detection

Normal breast tissue	0.3 mho
Cancerous breast tumor	2.0 mho



REMINISCENCIA DE MI VIDA MATEMATICA

Speech at Universidad Autónoma de Madrid accepting the 'Doctor Honoris Causa':

> *My work at "Yacimientos Petroliferos Fiscales" (YPF) was very interesting, but I was not well treated, otherwise I would have stayed there.*

$\boxed{\text{CALDERÓN'S PROBLEM (EIT)}}$

Consider a body $\Omega \subset \mathbb{R}^n$. An electrical potential $u(x)$ causes the current

$$\boxed{I(x) = \gamma(x)\nabla u}$$

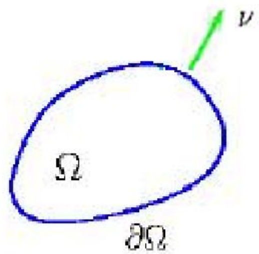
The conductivity $\gamma(x)$ can be isotropic, that is, scalar, or anisotropic, that is, a matrix valued function. If the current has no sources or sinks, we have

$$\boxed{\operatorname{div}(\gamma(x)\nabla u) = 0} \text{ in } \Omega$$

$$\boxed{\begin{aligned} \operatorname{div}(\gamma(x)\nabla u(x)) &= 0 \\ u\big|_{\partial\Omega} &= f \end{aligned}}$$

$\gamma(x)$ = conductivity,
f = voltage potential at $\partial\Omega$

Current flux at $\partial\Omega$ = $(\nu \cdot \gamma \nabla u)\big|_{\partial\Omega}$ were ν is the unit outer normal.



Information is encoded in map

$$\boxed{\Lambda_\gamma(f) = \nu \cdot \gamma \nabla u\big|_{\partial\Omega}}$$

EIT (Calderón's inverse problem)

Does $\boxed{\Lambda_\gamma}$ determine $\boxed{\gamma}$?

Λ_γ = Dirichlet-to-Neumann map

$$\boxed{\begin{aligned}\mathrm{div}(\gamma\nabla u) &= 0\\ u\big|_{\partial\Omega} &= f\end{aligned}}$$

$$\boxed{\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \gamma^{ij}\nu^i \frac{\partial u}{\partial x_j}\Big|_{\partial\Omega}}$$

$$\boxed{\Lambda_\gamma \Rightarrow \gamma\ ?}$$

Answer: No

$$\boxed{\Lambda_{\psi_*\gamma} = \Lambda_\gamma}$$

where $\psi : \Omega \to \Omega$ change of variables

$$\psi|_{\partial\Omega} = \text{Identity}$$

$$\boxed{\psi_*\gamma = \left(\frac{(D\psi)^T \circ \gamma \circ D\psi}{|\det D\psi|}\right) \circ \psi^{-1}}$$

$$\boxed{v = u \circ \psi^{-1}}$$

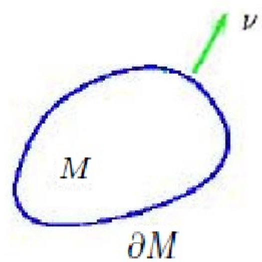
DIRICHLET-TO-NEUMANN MAP (Lee-U, 1989)

(M, g) compact Riemannian manifold with boundary.

Δ_g Laplace-Beltrami operator $g = (g_{ij})$ pos. def. symmetric matrix

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j} \right) \quad (g^{ij}) = (g_{ij})^{-1}$$

$$\begin{aligned} \Delta_g u &= 0 \text{ on } M \\ u\big|_{\partial M} &= f \end{aligned}$$



Conductivity:
$\gamma^{ij} = \sqrt{\det g}\, g^{ij}$

$$\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\Big|_{\partial M}$$

$\nu = (\nu^1, \cdots, \nu^n)$ unit-outer normal

$$\Delta_g u = 0$$
$$u\Big|_{\partial M} = f$$

$$\boxed{\Lambda_g(f) = \frac{\partial u}{\partial \nu_g} = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\Big|_{\partial M}}$$

current flux at ∂M

Inverse-problem (EIT)

Can we recover $\boxed{g}$ from $\boxed{\Lambda_g}$?

$\Lambda_g =$ Dirichlet-to-Neumann map or voltage to current map

$$\boxed{\begin{aligned} \Delta_g u &= 0 \\ u\Big|_{\partial M} &= f \end{aligned}}$$

$$\boxed{\Lambda_g(f) = \frac{\partial u}{\partial \nu_g}\Big|_{\partial M}}$$

$$\boxed{\Lambda_g \Rightarrow g \quad ?}$$

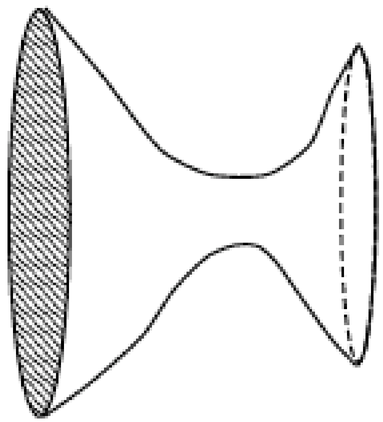
Answer: No $\boxed{\Lambda_{\psi^* g} = \Lambda_g}$ where

$\psi : M \to M$ diffeomorphism, $\psi\Big|_{\partial M} =$ Identity and

$$\boxed{\psi^* g = \left(D\psi \circ g \circ (D\psi)^T \right) \circ \psi}$$

Non-uniqueness for EIT and Cloaking

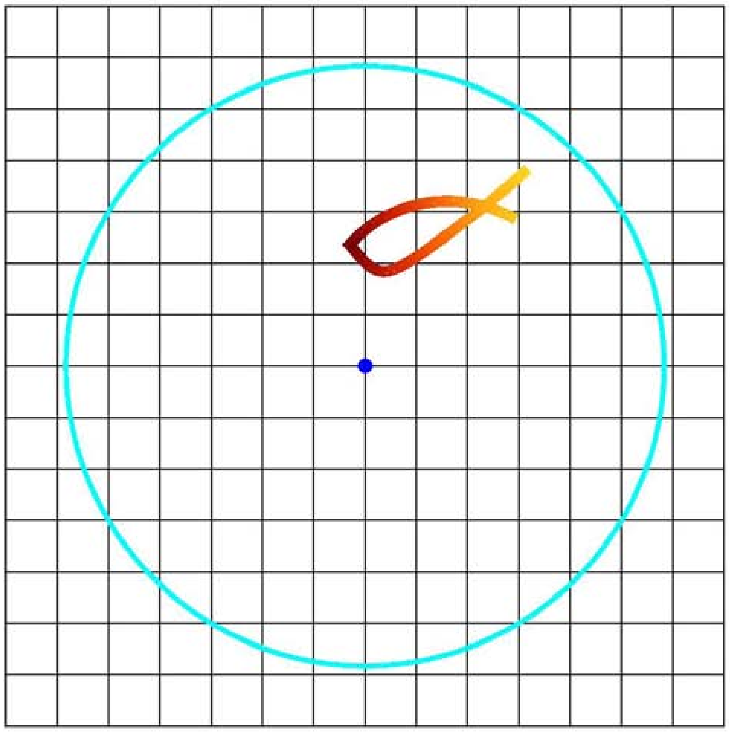
Motivation (Greenleaf-Lassas-U, MRL, 2003)



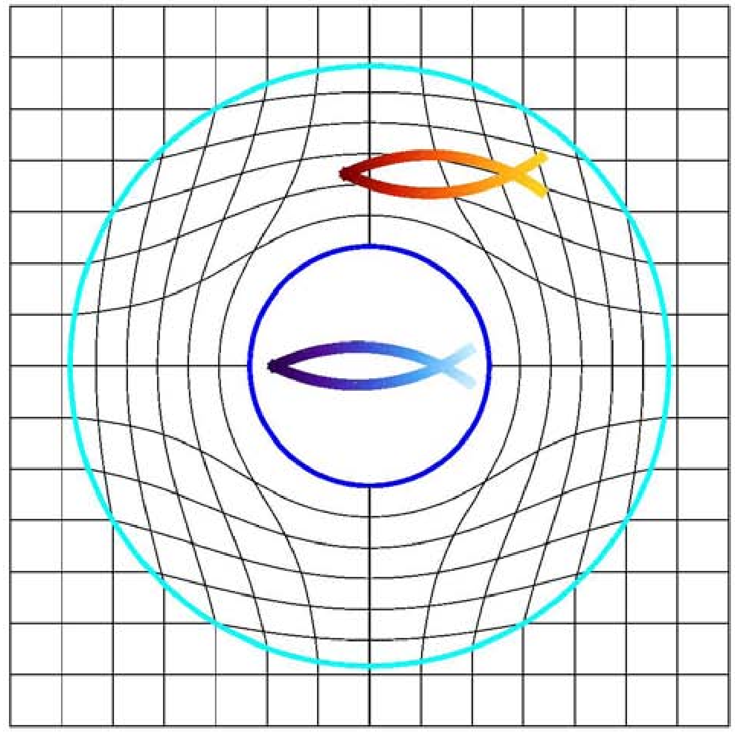
When bridge connecting the two parts of the manifold gets narrower the boundary measurements give less information about isolated area.

When we realize the manifold in Euclidean space we should obtain conductivities whose boundary measurements give no information about certain parts of the domain.

Transformation Optics



virtual space



physical space

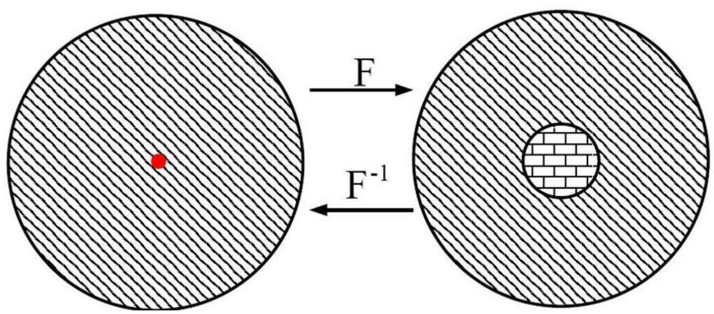
Greenleaf-Lassas-U (2003 MRL)

Let $\begin{array}{l}\Omega = \mathcal{B}(0,2) \subset \mathbb{R}^3, \\ D = \mathcal{B}(0,1),\end{array}$ where $\mathcal{B}(0,r) = \{x \in \mathbb{R}^3;\ |x| < r\}$

$$F:\ \Omega \setminus \{0\} \to \Omega \setminus \overline{D}$$

$$\boxed{F(x) = \left(\frac{|x|}{2} + 1\right)\frac{x}{|x|}}$$

F - diffeomorphism, $\quad F|_{\partial\Omega} = \text{Identity}$



$$\text{Let}\qquad \begin{array}{l} \gamma = g = \text{identity on } \mathcal{B}(0,2), \\ \widehat{\gamma} = F_*\gamma \text{ on } \mathcal{B}(0,2)\backslash\mathcal{B}(0,1), \\ \widehat{g} = \text{metric associated to } \widehat{\gamma}. \end{array}$$

In spherical coordinates

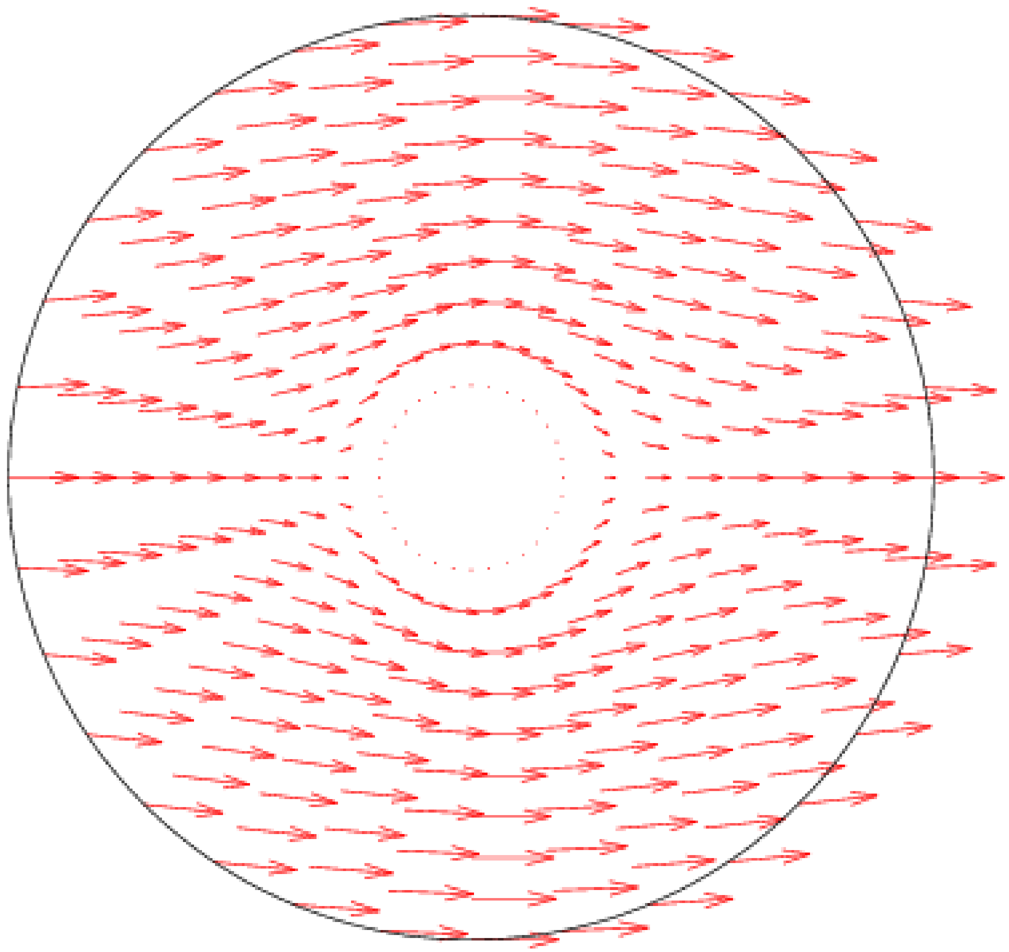
$$(r,\phi,\theta) \to (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta),$$

$$\widehat{\gamma} = \begin{pmatrix} 2(r-1)^2\sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}$$

Let $\widetilde{\gamma}$ (*resp.* $\widetilde{g}$) be the conductivity (*resp.* metric) in $\mathcal{B}(0,2)$ such that $\widetilde{\gamma} = \widehat{\gamma}$ (*resp.* $\widetilde{g} = \widehat{g}$) on $\mathcal{B}(0,2)\backslash\mathcal{B}(0,1)$ and arbitrarily positive definite on $B(0,1)$. Then

Theorem (Greenleaf-Lassas-U 2003)

$$\boxed{\Lambda_{\widetilde{\gamma}} = \Lambda_\gamma} \quad \left(\textit{resp. } \boxed{\Lambda_{\widetilde{g}} = \Lambda_g}\right)$$



Based on work of Greenleaf-Lassas-U, MRL 2003

Helmholtz Equation (Acoustic Cloaking)

$$\boxed{\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j}\right) + k^2 u = 0} \quad (g^{ij}) = (g_{ij})^{-1}.$$

Acoustic equation with density $\rho = \sqrt{\det g} g^{ij}$ and bulk modulus $\lambda^{-1} = \sqrt{\det g}$

$$g^{ij} = \begin{pmatrix} 2(r-1)^2 \sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}$$

Theorem (Greenleaf-Kurylev-Lassas-U, CMP 2007)

$$\Lambda_g = \Lambda_{\{I=(\delta_{ij})\}}$$

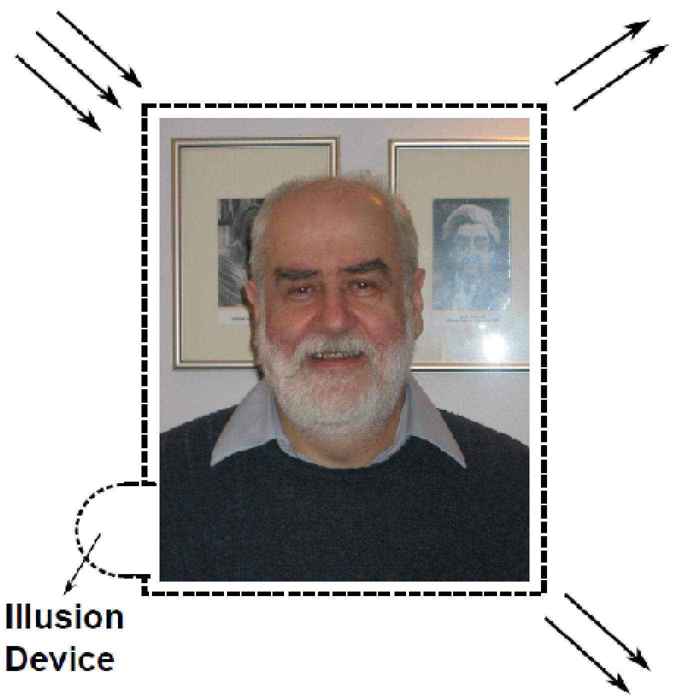
Acoustic Cloaking:

H. Chen and C. J. Chan, Appl. Phy. Lett., (2007)

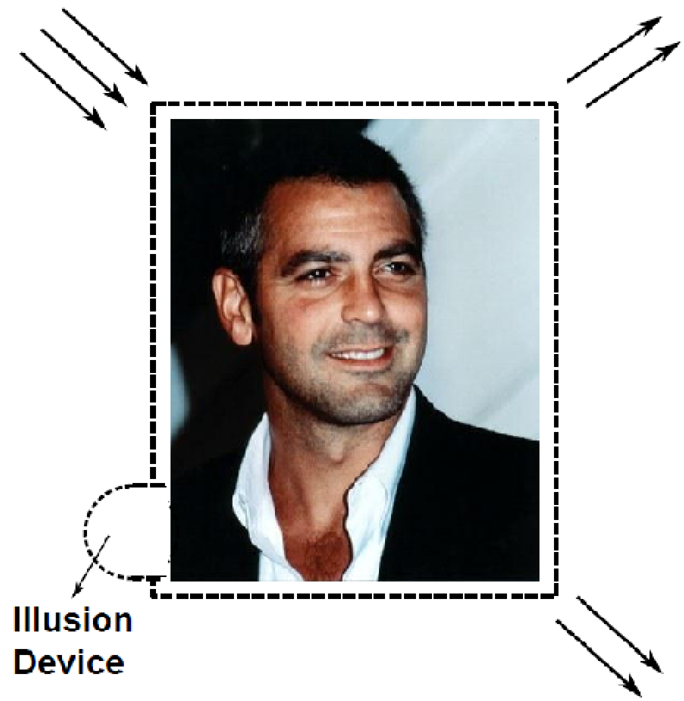
S. Cummer et al, PRL (2008);

Transformation of an object into another object

(Lai et al, PRL 2008, H. Liu IP 2009)



Real space



Illusion space

Cloaking for Maxwell's equations (Passive Devices)

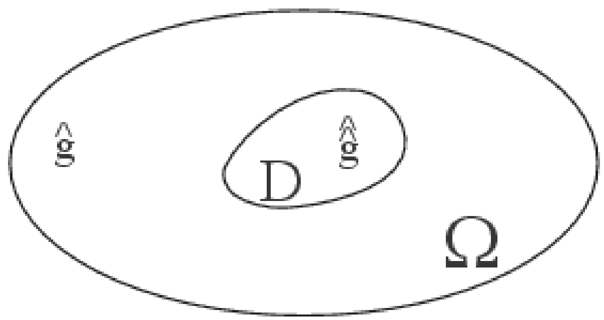
Model: Maxwell's Equations for Time Harmonic Waves

$$\boxed{\begin{aligned}&\nabla\times E - ik\mu(x)H = 0\\ &\nabla\times H + ik\epsilon(x)E = 0\\ &D = \epsilon E, B = \mu H\\ &\mathrm{div}D = 0, \mathrm{div}B = 0\end{aligned}}$$

(E, H) Electromagnetic Field

D=electric displacement B=magnetic displacement	$\epsilon(x)$=electric permittivity $\mu(x)$=magnetic permeability

Take $\boxed{\epsilon(x) = \mu(x) = g^{-1}\sqrt{\det g}}$ where $g = (g_{ij})$ is a semipositive definite symmetric matrix



$$\tilde{g} = \begin{cases} \hat{g} \text{ on } \Omega \setminus D, & \hat{g} = (F^{-1})^* e \\ \hat{\hat{g}} \text{ on } \overline{D}, & \hat{\hat{g}} \text{ Riemannian metric on } D \end{cases}$$

Theorem (Greenleaf-Kurylev-Lassas-U, CMP, 2007)

$$\Lambda_I = \Lambda_{\tilde{g}}$$

Here

$$\Lambda_g(E \times \nu) = H \times \nu$$

where ν is the inner-unit normal.
More generally we define the Cauchy data $(E \times \nu, H \times \nu)$.

$\boxed{\text{Approximate Cloaking}}$

$$\boxed{\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(\gamma^{ij}\frac{\partial u}{\partial x_j}) + \frac{k^2}{\sqrt{\det \gamma}} u = 0}.$$

$$F(x) = \begin{cases} x, & |x| > 2 \\ (1 + \frac{|x|}{2})\frac{x}{|x|}, & 0 < |x| < 2 \end{cases}, \quad F_*\gamma = \frac{(DF) \circ \gamma \circ (DF)^T}{|\det F|} \circ F^{-1}$$

$$\tilde{\gamma} = \begin{cases} F_*(\delta^{jk}), & x \in B(0,3) \setminus B(0,1) \\ 2(\delta^{jk}), & x \in B(0,1) \end{cases},$$

$$\boxed{(\tilde{\gamma}_R^{jk})(x) = \begin{cases} (\tilde{\gamma}^{jk})(x), & |x| > R \\ 2(\delta^{jk}), & |x| \leq R \end{cases}}$$

$1 < R < 2$, non-singular anisotropic.

Approximate Quantum Cloaking

(Greenleaf-Kurylev-Lassas-U, PRL 2008, NJP 2008)

Quantum Waves:

$$\boxed{(-\Delta + V)u = Eu}$$

Isotropic Conductivity γ:

$$\boxed{\mathrm{div}(\gamma \nabla v) + k^2 v = 0}$$

$\boxed{u = \gamma^{1/2} v :}$

$$\boxed{(-\Delta + V)u = k^2 u, \quad V = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}} - \frac{k^2}{\gamma} + k^2.}$$

$$\boxed{\Lambda_{\widetilde{\gamma}_R} \to \Lambda_1} \quad \text{as} \quad R \to 1.$$

Approximate Cloaking

Truncation:

- Greenleaf-Kurylev-Lassas-U Opt. Exp. 2007; NJP 2008; PRL 2008
- Yan-Ruan-Qiu Opt. Exp. 2007

Other approaches and 2D acoustic cloaking

- Liu-J. Li-Rondi-U (CMP 2015, general case)
- Bao-Liu (Maxwell, JMPA 2013)
- Lassas-Zhou MRL 2011 (2D)
- Liu-Zhou CDDS-A 2011 (2D)
- Liu-Zhou SIAP 2011 (Maxwell)
- Kohn-Onofrei-Vogelius-Weinstein CPAM 2010 (Helmholtz)
- Liu (virtual reshaping, 2009)
- Kohn-Shen-Vogelius-Weinstein IP 2008 (Conductivity)

$\boxed{\text{Approximate Cloaking with Isotropic Materials}}$

Homogenization (Allaire, Cherkaev, Milton)

Construct isotropic conductivities $\widetilde{\gamma}_{R,\varepsilon}$ that give almost cloaking for acoustic and conductivity equation

$$\boxed{\Lambda_{\widetilde{\gamma}_{R,\varepsilon}} \xrightarrow[\varepsilon\to 0, R\to 1]{} \Lambda_I,} \qquad I = (\delta^{jk})$$

$$\boxed{\mathrm{div}(\widetilde{\gamma}_{R,\varepsilon}\nabla u) + \frac{k^2}{\sqrt{\det \widetilde{\gamma}_{R,\varepsilon}}} u = 0.}$$

QUANTUM CLOAKING

"What I want to talk about is the problem of manipulating and controlling things on a small scale."

Richard Feynman, There is Plenty of Room at the Bottom (1959)

QUANTUM CLOAKING

(Zhang et al., PRL 2008)

$$\boxed{-\frac{h^2}{2}\sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}(m^{ij}\frac{\partial}{\partial x_j}\psi)+V\psi=E\psi}$$

(m^{ij}) = effective mass

Cloaking $m^{ij}=\sqrt{\det(g)}g^{ij}$

$$(g^{ij})=\begin{pmatrix} 2(r-1)^2\sin\theta & 0 & 0 \\ 0 & 2\sin\theta & 0 \\ 0 & 0 & 2(\sin\theta)^{-1} \end{pmatrix}$$

Approximate quantum cloaks (Greenleaf-Kurylev-Lassas-U, PRL 2008, NJP 2008)

E is not a Neumann eigenvalue in cloaked region,

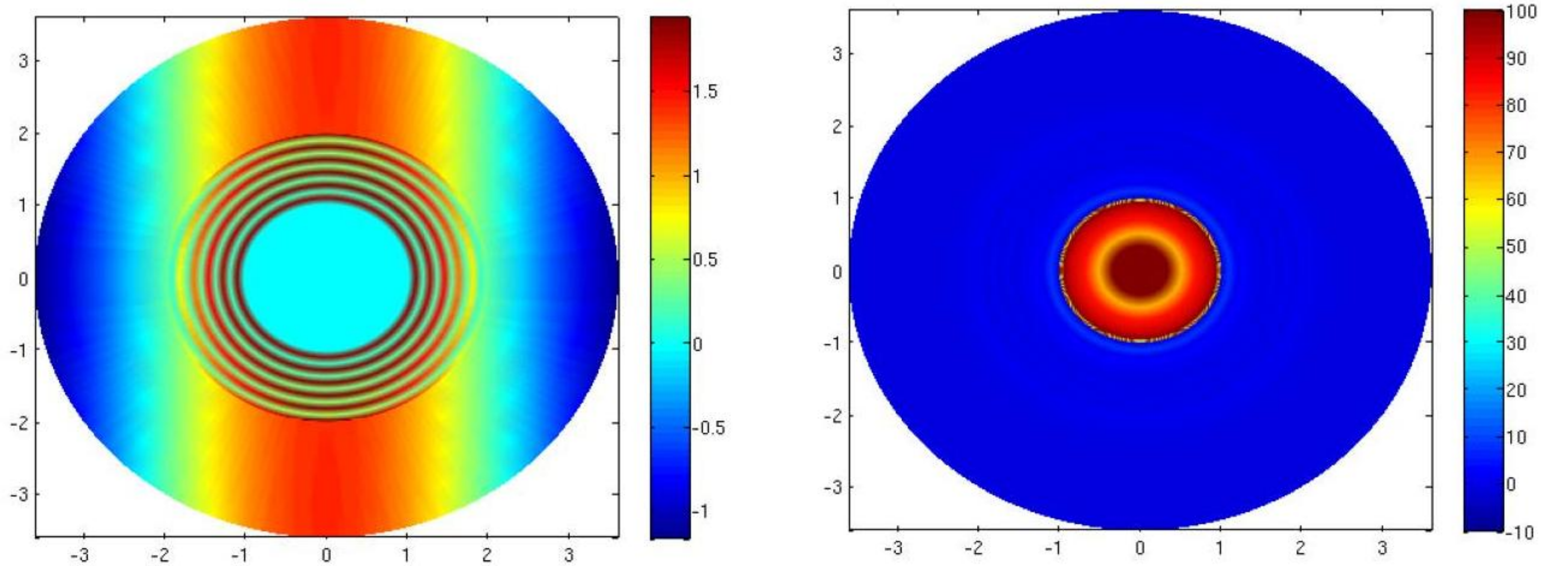
$$\varphi = \sqrt{\widetilde{\gamma}_{R,\varepsilon}}\, u :$$

$$\boxed{\left(-\Delta + V_{E,R,\varepsilon}\right)\varphi = E\varphi,}$$

$$E = k, \quad V_{E,R,\varepsilon} = \frac{\Delta\sqrt{\widetilde{\gamma}_{R,\varepsilon}}}{\sqrt{\widetilde{\gamma}_{R,\varepsilon}}} + E\left(1 - \frac{1}{\det \widetilde{\gamma}_{R,\varepsilon}}\right)$$

W bounded potential on $B(0,1)$. Construct a sequence $V_n(E)$ s.t.

$$\boxed{\Lambda_{V_n(E)+W} \underset{n\to\infty}{\longrightarrow} \Lambda_0}$$

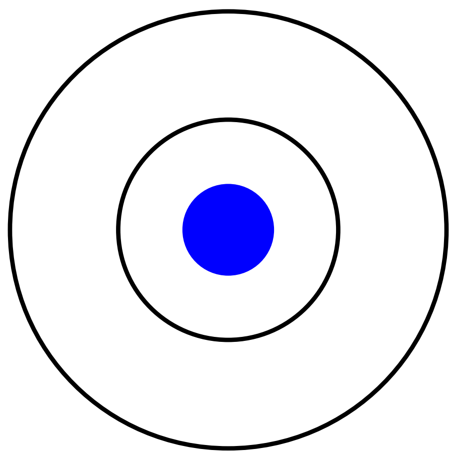


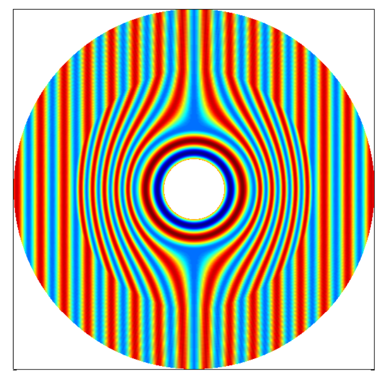
Left: Approximate cloak with E not a Neumann eigenvalue; matter wave passes unaltered.

Right: E a Neumann eigenvalue: cloak supports almost bound state.

Invisible sensors (Alu-Engheta, PRL 2009),(Greenleaf-Kurylev-Lassas-U PRE 2011, Chen-U Optics Express 2011) Let us put an obstacle $B(1/2)$ with a Robin boundary condition an approximate acoustic cloak with $R > 1$,

$$\nabla \cdot \gamma_R \nabla u + \frac{k^2}{\gamma_R} u = 0 \quad \text{in } B(2) \setminus B(1/2),$$
$$\partial_\nu u + \alpha u|_{\partial B(1/2)} = 0, \quad u|_{\partial B(2)} = f.$$





Depending on the Robin coefficient α, we have three different modes.

(Loading generic mode)

Generic Mode

(Loading sensor mode)

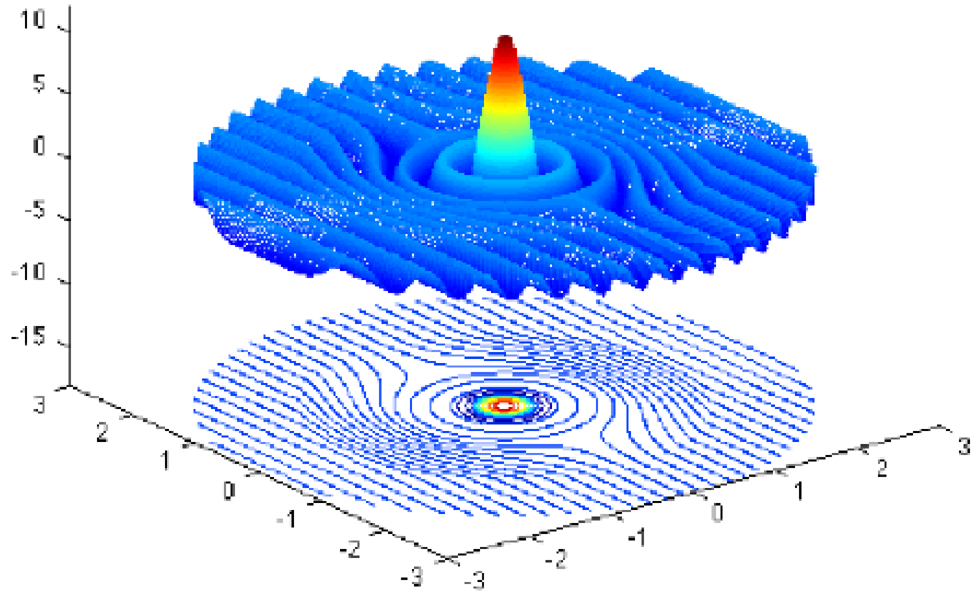
Sensor Mode

(Loading trapping mode)

Trapping Mode

SCHRÖDINGER'S HAT

(A. Greenleaf, Y. Kurylev, M. Lassas, U. Leonhardt, G–U, PNAS 2012)

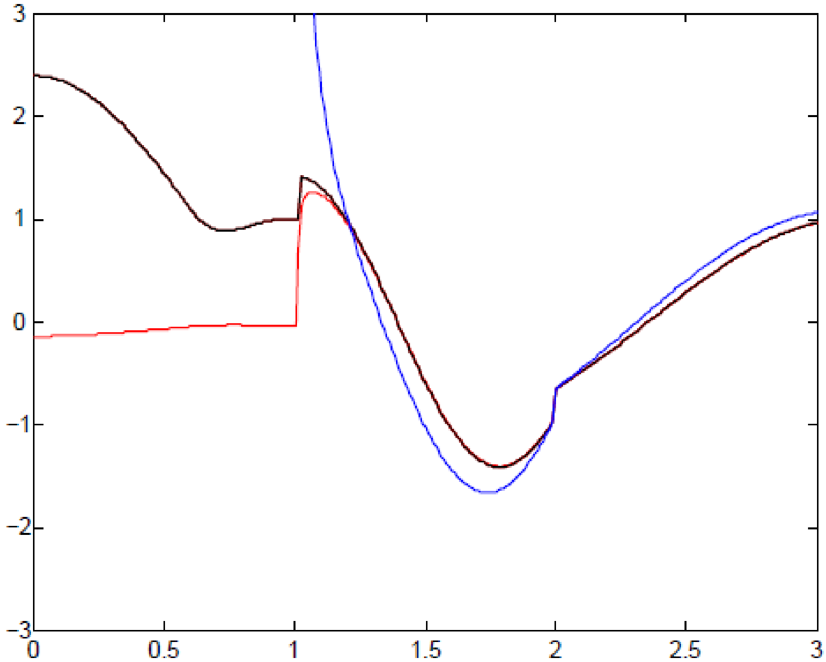


A Quasmon Inside a Schrödinger Hat

SCHRÖDINGER'S HAT

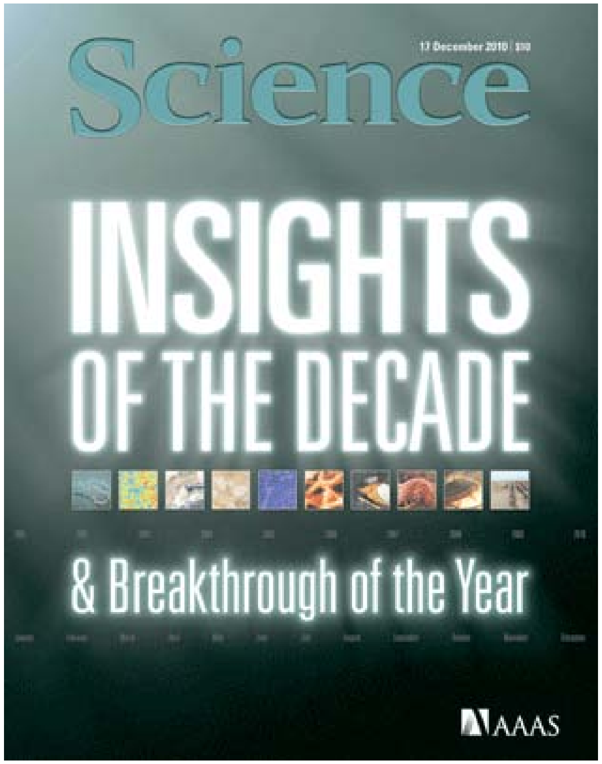
(Loading SchrodingerHat)

THREE CLOAKING REGIMES



The cloak - Schrödinger hat. The real part of the wave function on the positive x-axis for three sets of potential parameters. (Red) Quantum cloak, for which incident wave does not penetrate the cloaked region. (Black) Schrödinger hat; probability mass is almost entirely captured by the cloaked region, yet scattering is negligible. (Blue) Almost trapped wave. The cloaking effect is destroyed due to the strong resonance inside the cloak, and values within cloaked region are off-scale.

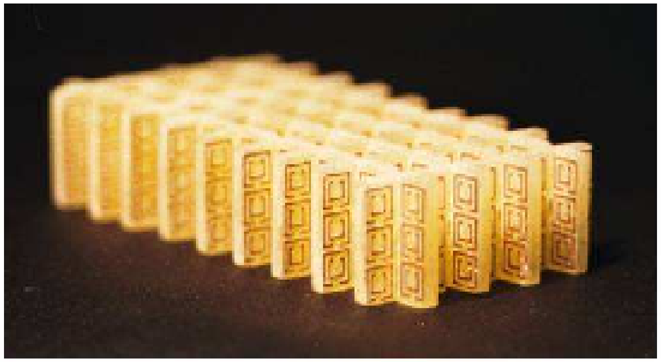
Metamaterials

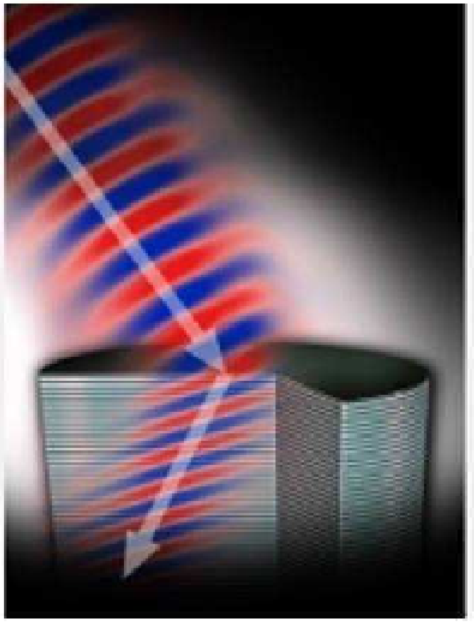


In the past decade, physicists and engineers pioneered new ways to guide and manipulate light, creating lenses that defy the fundamental limit on the resolution of an ordinary lens and even constructing "cloaks" that make an object invisible-sort of.

Metamaterials

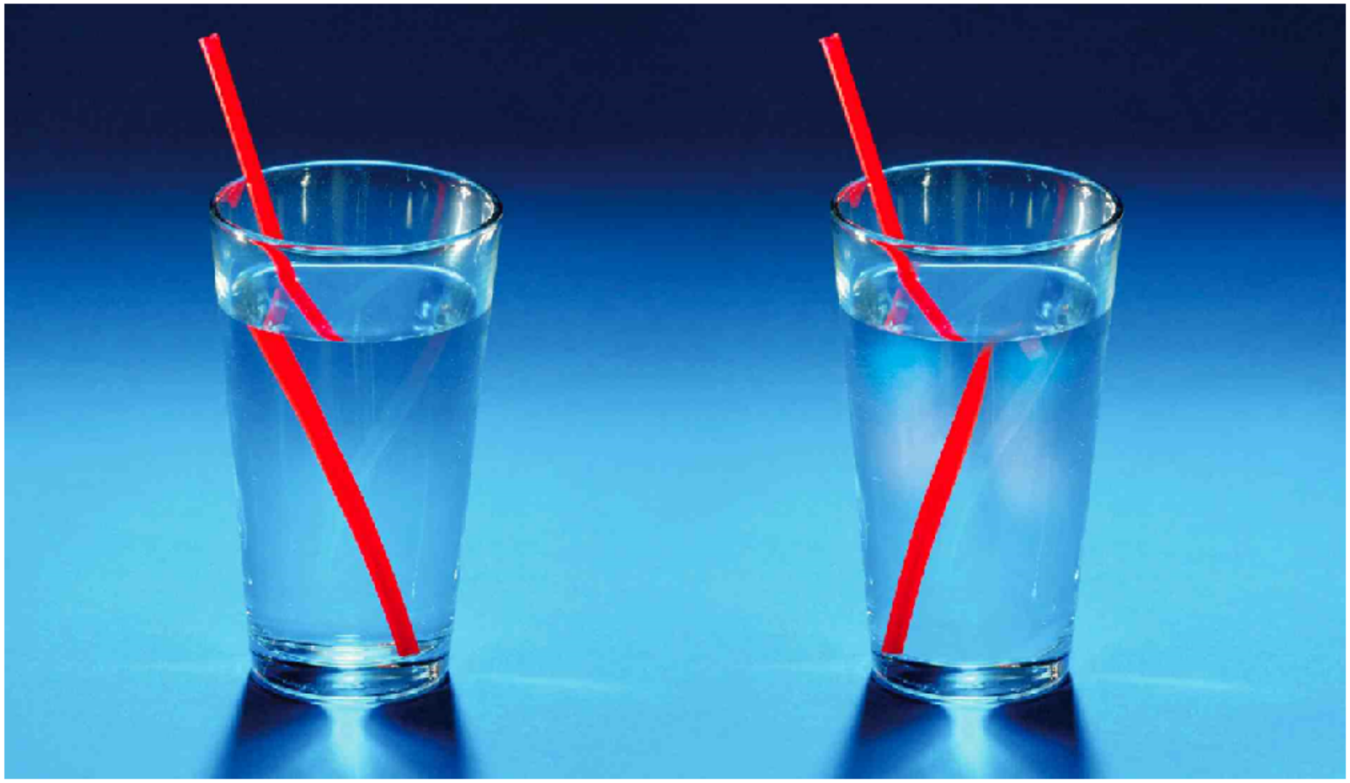
$\boxed{\textit{Negative Refractive Index: } -\sqrt{\epsilon\mu}}$



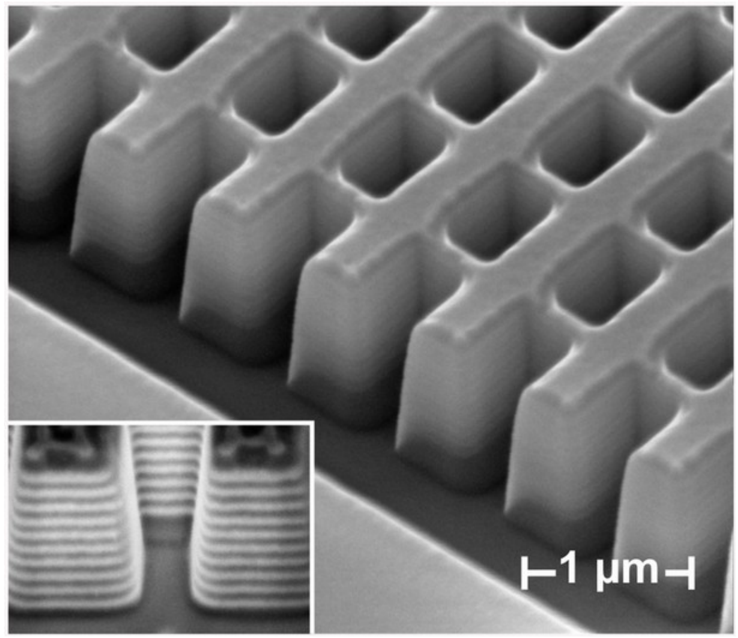


Credit: Keith Drake

Negative Refraction Simulation: Hess 2008



Negative index of refraction for visible light

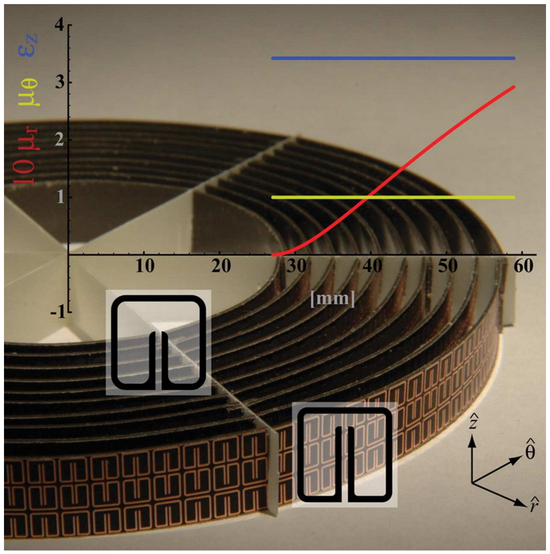


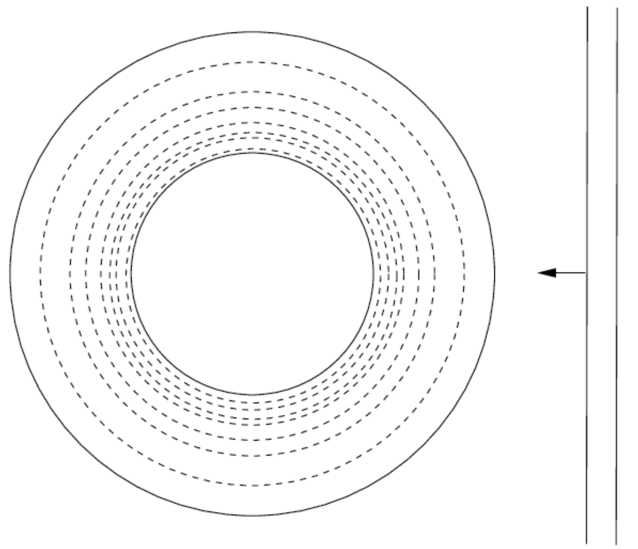
Nature 2008, Valentine et al, X. Zhang's group at UC Berkeley

Metamaterials

Cloaking

At microwave frequencies (D. Smith et al):

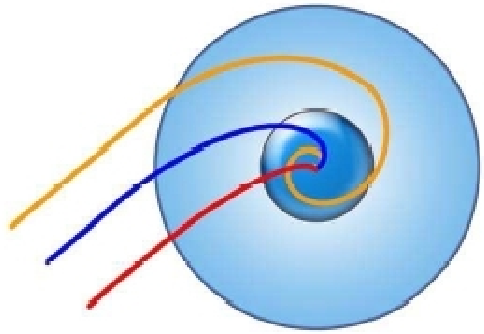




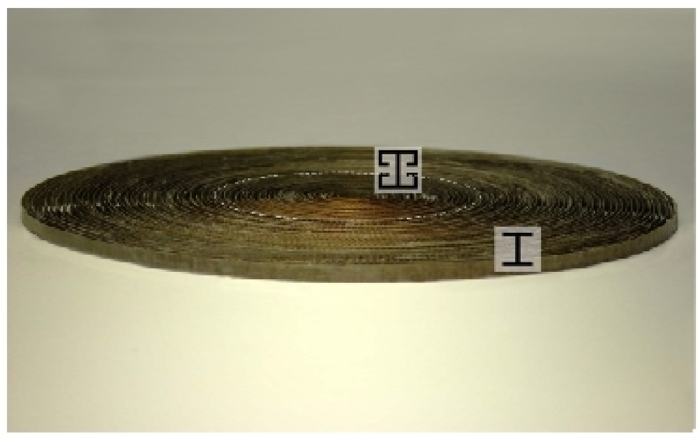
"Two-dimensional metamaterial structure exhibiting reduced visibility at 500nm", I.I. Smolyaninov et al, (Opt. Lett. 2008).

Artificial Black Holes

a



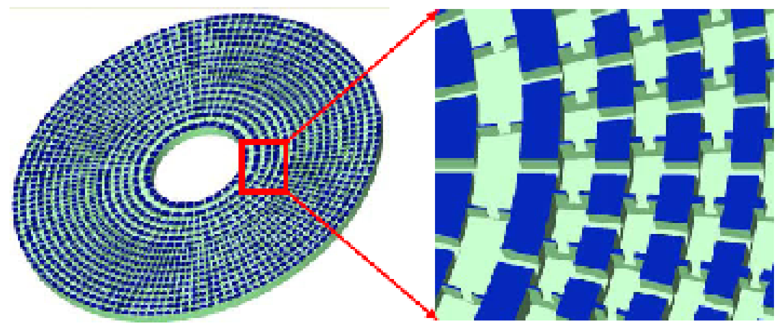
b



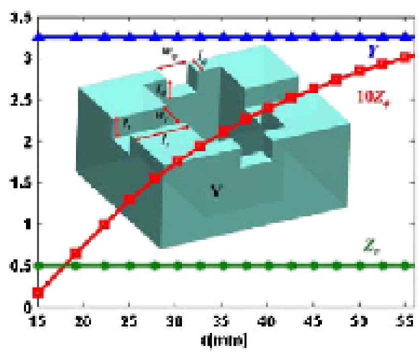
Q. Chen et al (N. J. Physics 2010)

Metamaterials

Acoustic Cloaking: The Sound of Silence



(a)



(b)

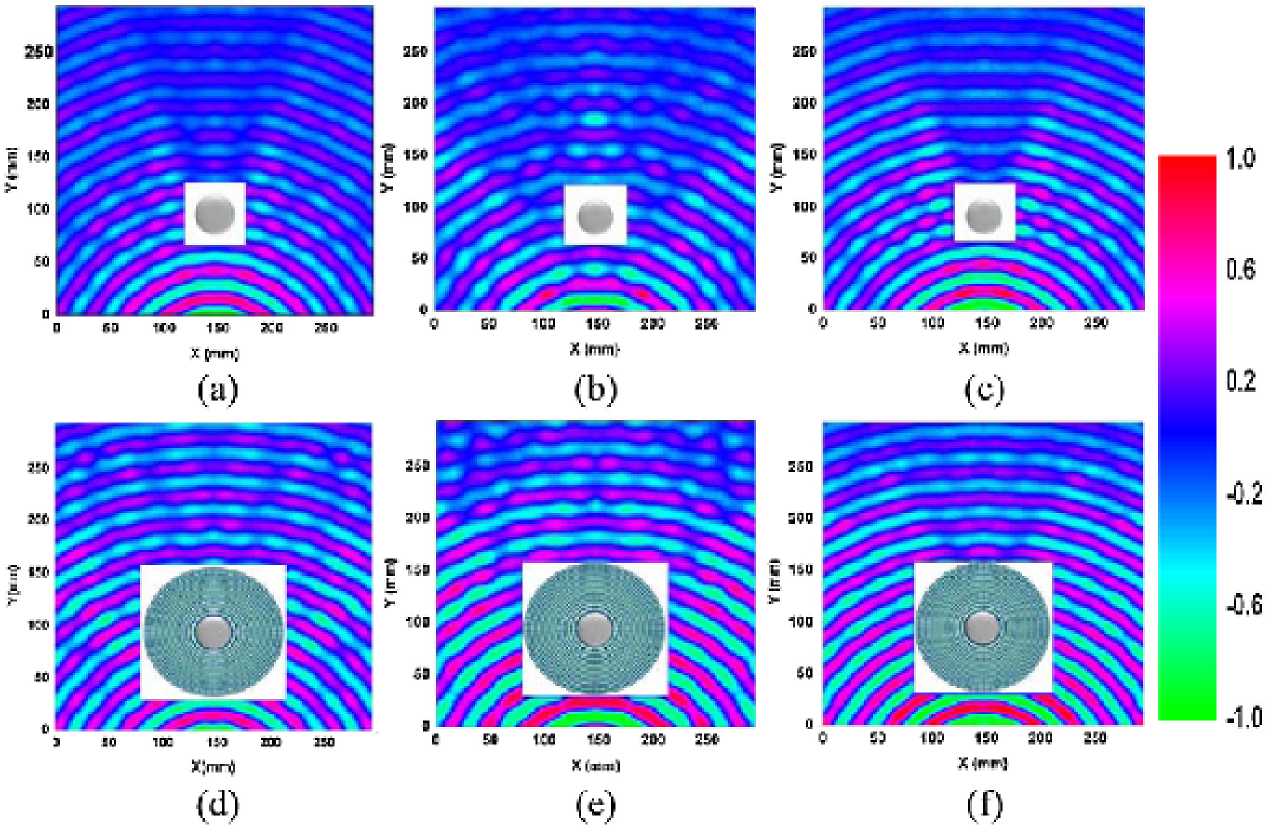
Layer	l_r(mm)	l_φ(mm)	V(mm^3)
1	2.05	0.10	3.00
3	1.37	0.22	2.29
5	1.24	0.41	2.06
7	1.24	0.30	2.06
9	1.24	0.41	2.06
11	1.24	0.52	2.06
13	1.24	0.63	2.06
15	1.24	0.74	2.06

(c)

(Zhang et al. PRL 2011)

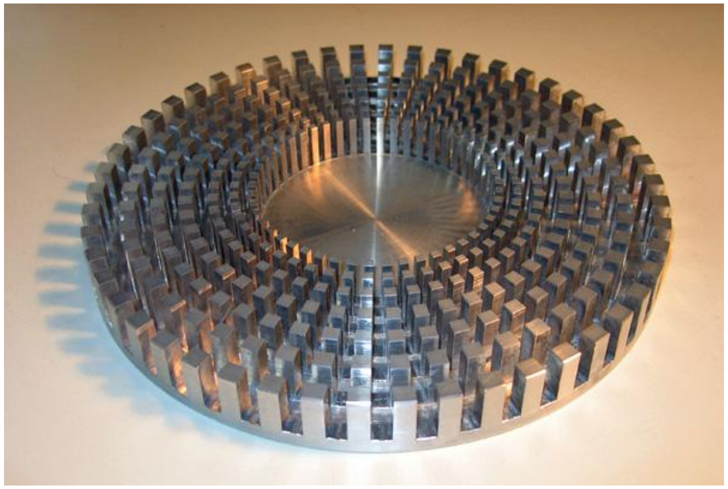
Metamaterials

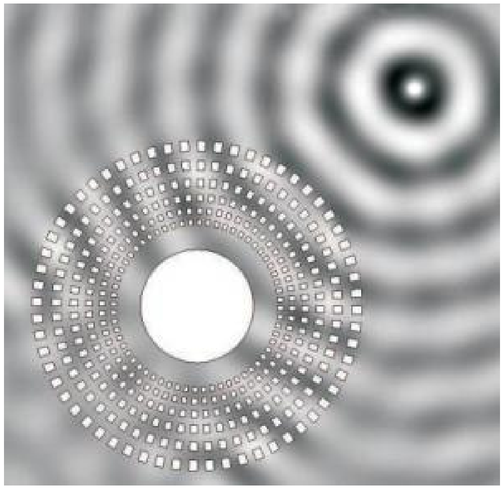
Acoustic Cloaking



(Zhang et al. PRL 2011)

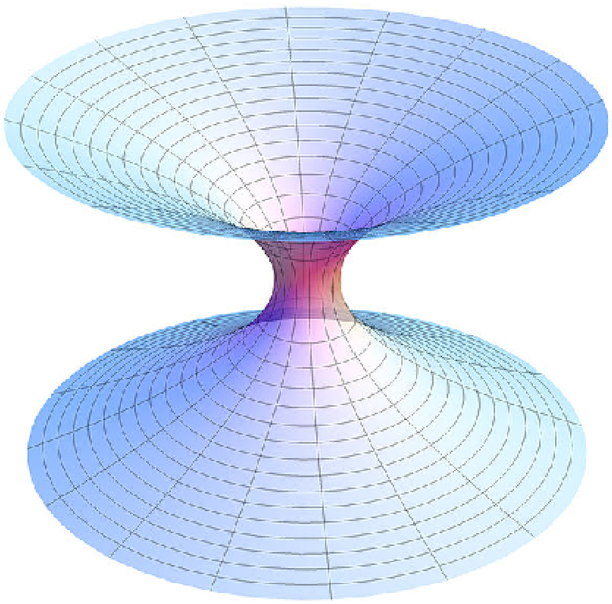
Tsunami Cloaking

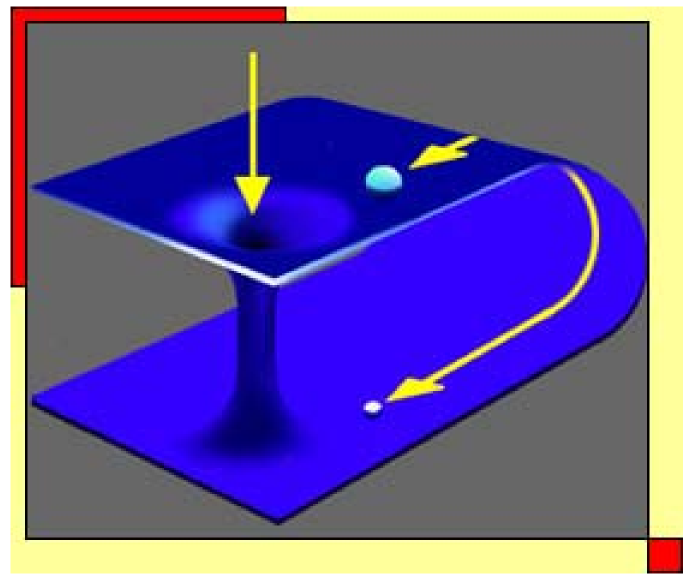




"Broadband cylindrical acoustic cloak for linear surface waves in a fluid", M. Farhat et al, PRL (2008).

Einstein-Rose Wormholes: A shortcut in space-time





(Loading Warmhole from Contact)

Electromagnetic Wormholes

Harry Potter's invisibility sleeve (Scientific American)

(A. Greenleaf, Y. Kurylev, M. Lassas–U, PRL, 2007)

How to construct a device that functions like a wormhole for electromagnetic fields?

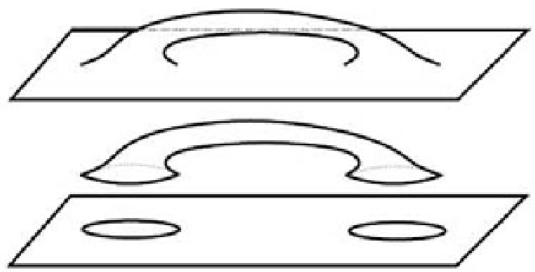
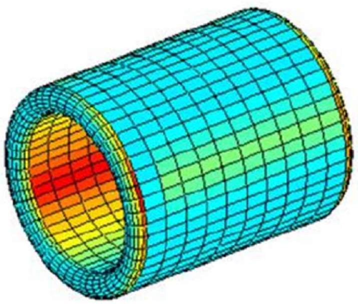


Figure: Schematic two-dimensional figure and an artistic interpretation of the wormhole device by Scientific American.

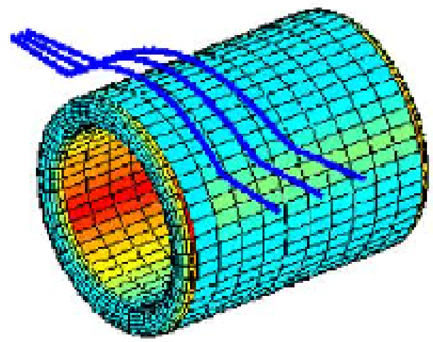
Blueprints of a Wormhole for Electromagnetic Waves

Take a cylinder

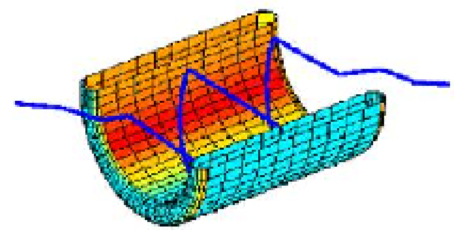


make parallel corrugation on its surface and coat it with suitable metamaterials. Such material has already been constructed for microwaves. Studies on optical frequencies are going on.

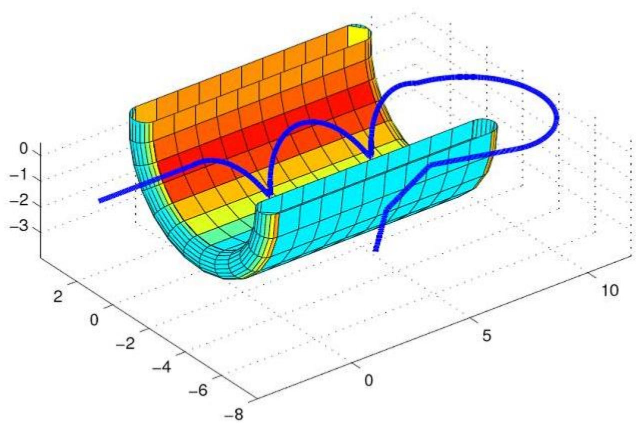
Ray tracing simulations:



Rays travelling outside.

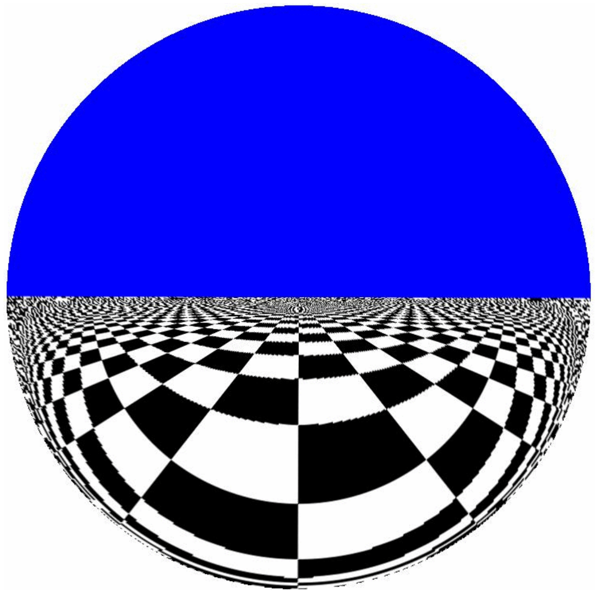


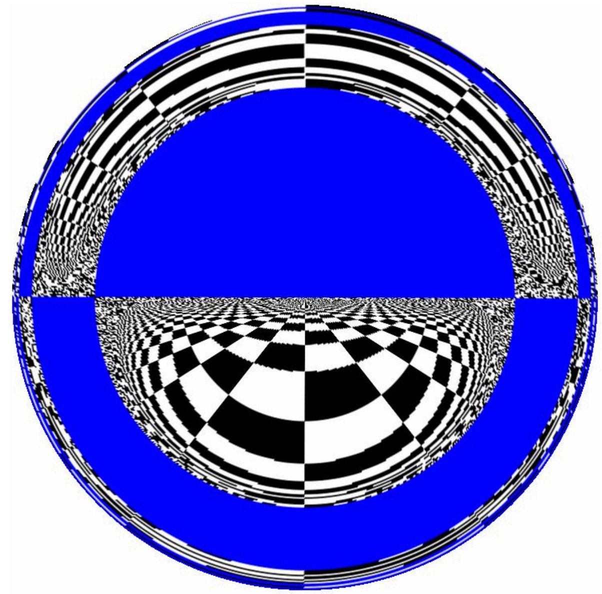
A ray traveling inside.



A ray coming back

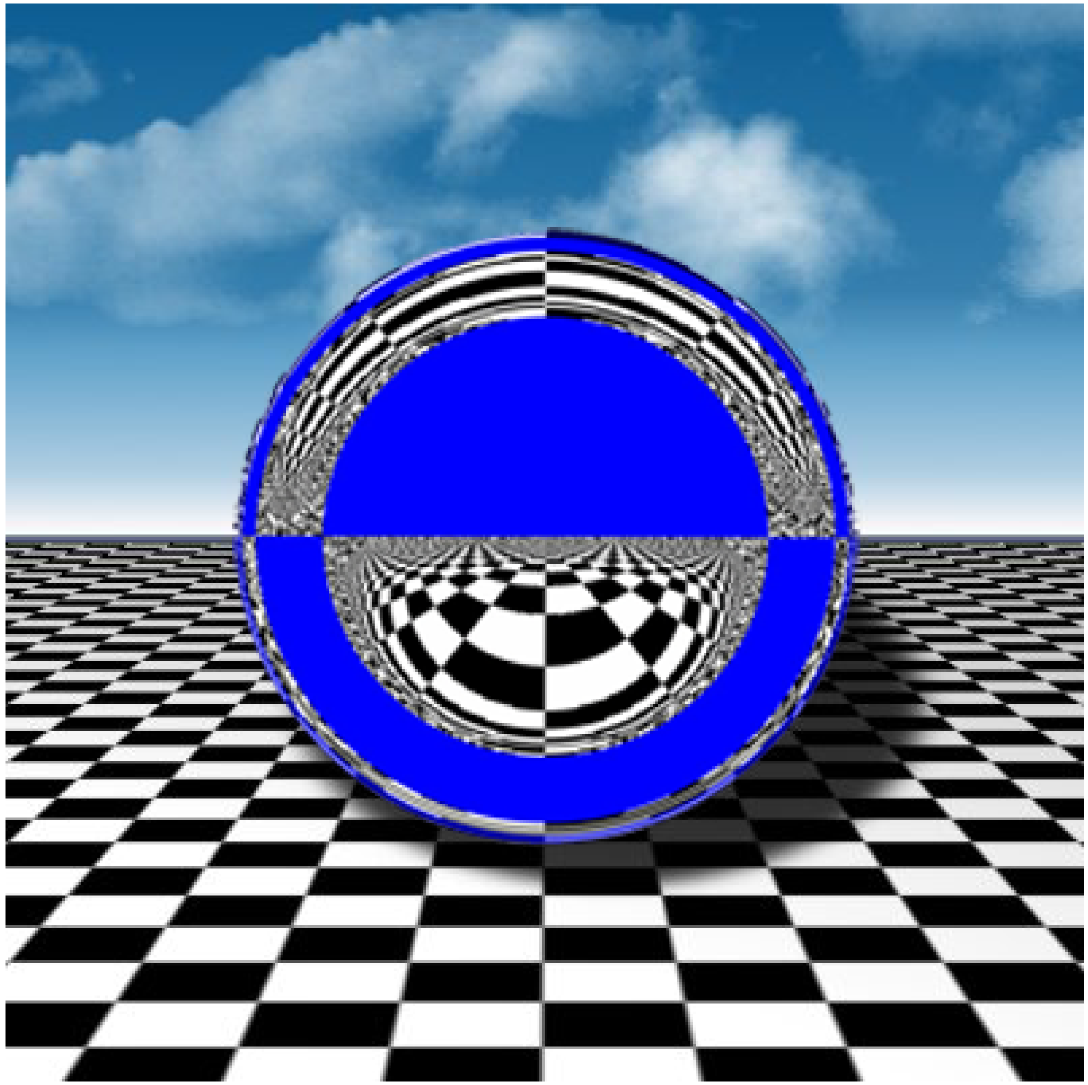
Ray tracing simulations: the end of a wormhole

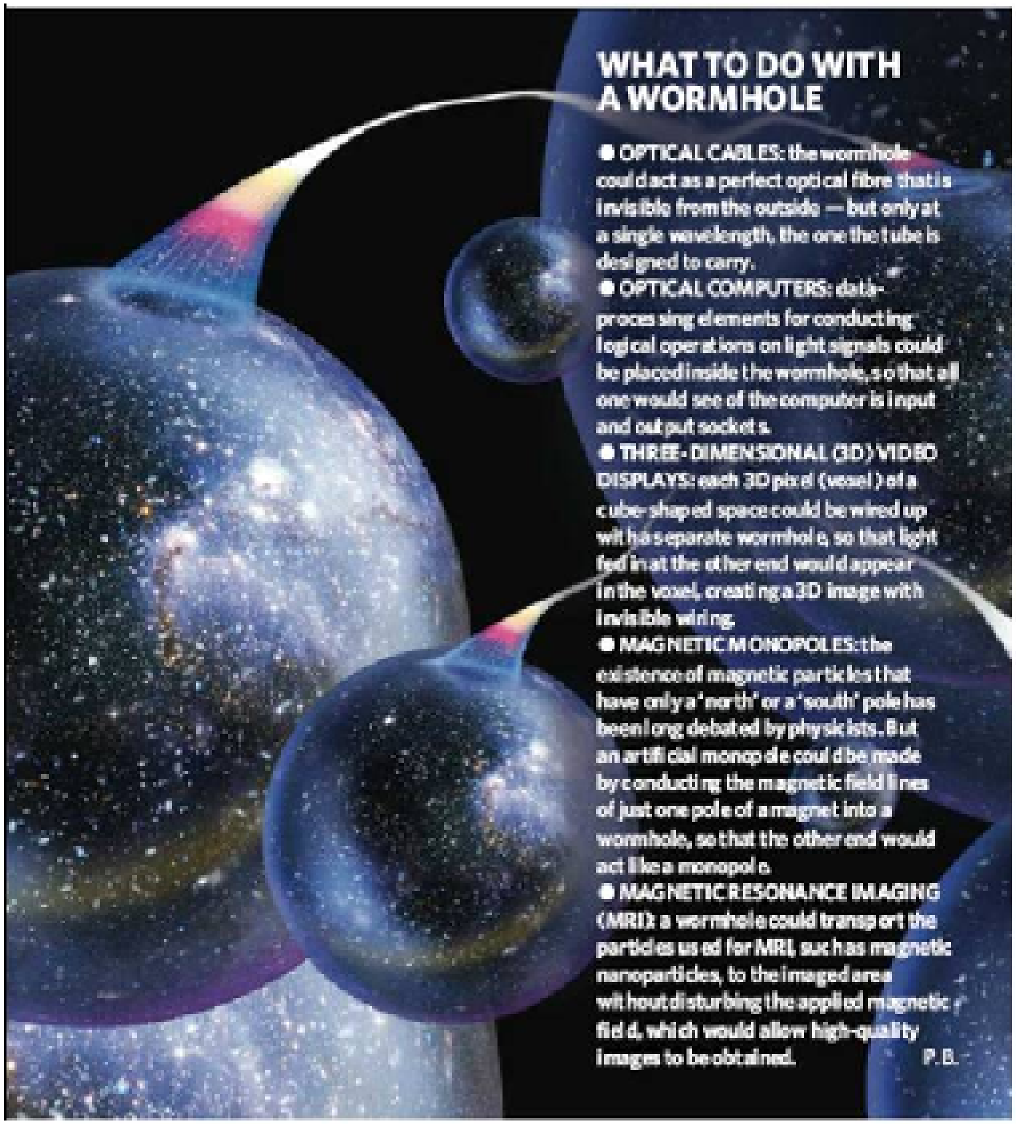




Length of handle $<< 1$. Length of handle ≈ 1.

An end of the wormhole is sphere. The other end is over an infinite chess board and under the blue sky.

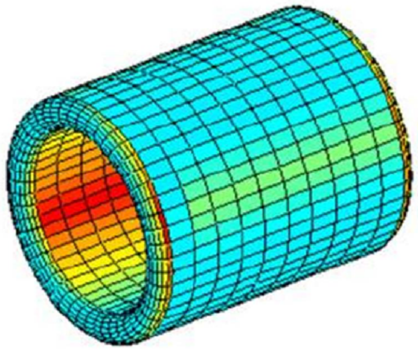




Nature News (Nov. 15, 2007)

Possible applications in future:

- Invisible optical cables.
- Components for optical computers.

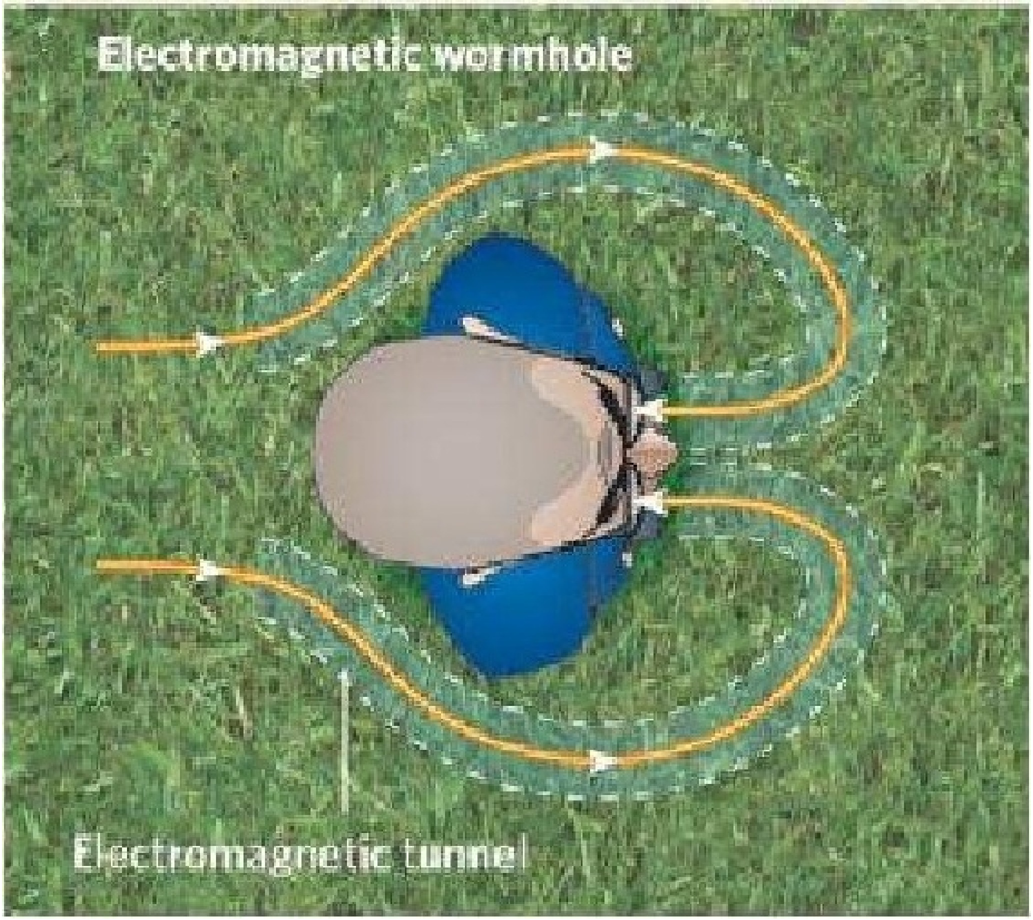


- 3D video displays: ends of invisible tunnels work as light source in 3D voxels.
- Light beam collimation.
- Virtual magnetic monopoles.
- Scopes for Magnetic Resonance Imaging devices.

Optics: Watch your back

(Kosmas L. Tsakmakidis and Ortwin Hess)

Nature 451, 27(January 3, 2008)



"Any sufficiently advanced technology is indistinguishable from magic." (Arthur C. Clarke)

"Imagination is more important than knowledge. For knowledge is limited to all we now know and understand, while imagination embraces the entire world, and all there ever will be to know and understand."
(Albert Einstein)

(Loading einwell-maxstein)