Quantum Vortices

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These lectures are conceived as a survey of selected theoretical results about *quantized vortices* in dilute, ultra-cold Bose gases under rotation, including a discussion of the *giant vortex phase transition* at sufficiently rapid rotation in an anharmonic trap. Most of the results presented have been obtained in collaboration with Michele Correggi and Nicolas Rougerie.

Vortices in rotating quantum gases are a fascinating manifestation of *superfluidity*. Their mathematical study involves methods from variational calculus and the theory of nonlinear, elliptic PDE's.

On the experimental side, sophisticated cooling and trapping techniques have led to the possibility of producing ultra-cold gases of alkali atoms exhibiting Bose-Einstein Condensation (BEC) and vortices in laboratories since the mid 1990's. The experimental and theoretical study of quantum gases is one of the most active areas in condensed matter physics and a separate sub-topic on the ArXiv.

OVERVIEW

Vortices in Fluid Dynamics

Consider a fluid with velocity field $\mathbf{v}(\mathbf{x})$. The circulation around a closed loop $\mathcal{C}$ enclosing a domain $\mathcal{D}$ is, by Stokes,

$$\oint_{\mathcal{C}} \mathbf{v} \cdot d\ell = \int_{\mathcal{D}} (\nabla \times \mathbf{v}) \cdot \mathbf{n} \, \mathrm{d}S.$$

Hence nonzero circulation requires that the vorticity

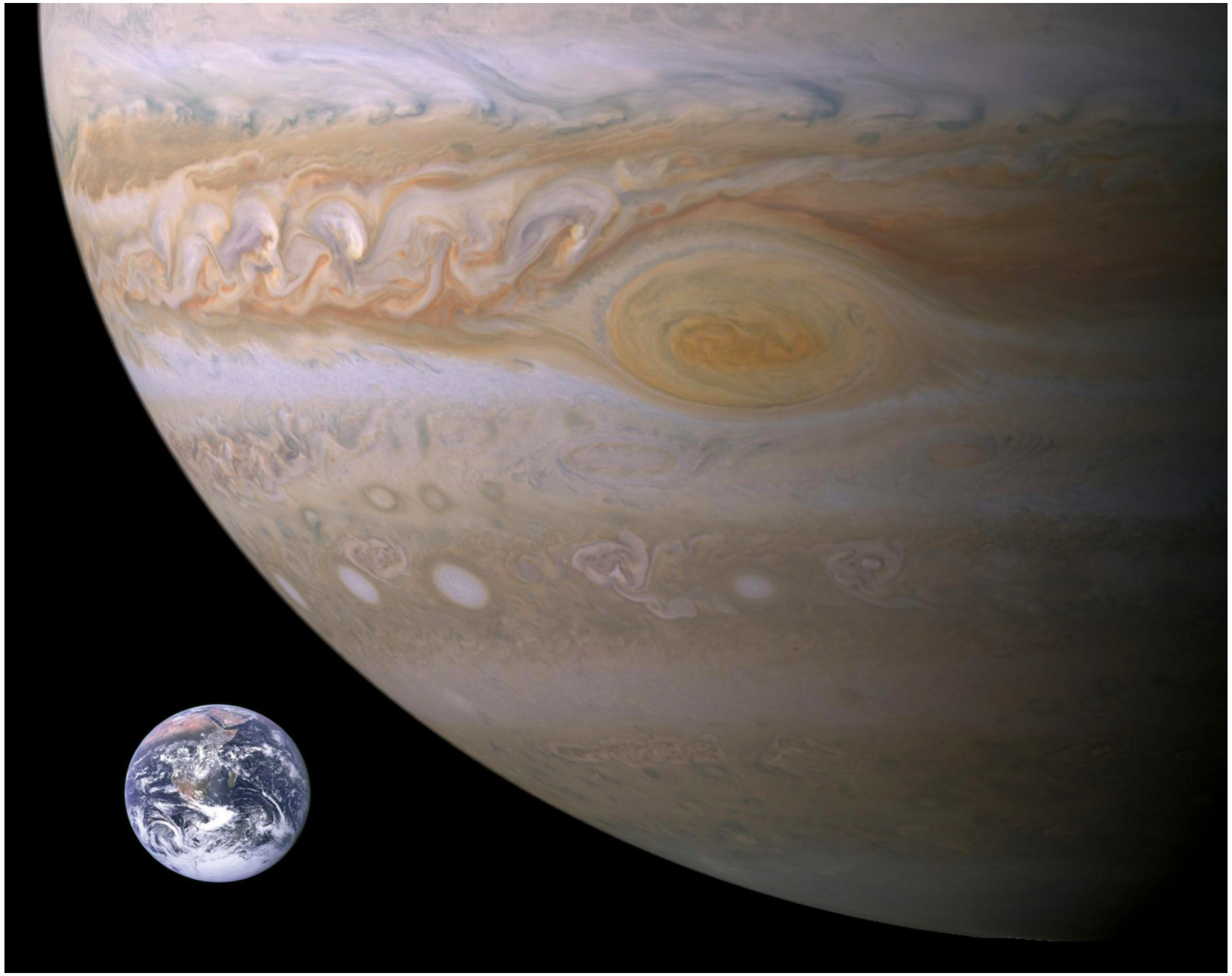
$$\nabla \times \mathbf{v}$$

is nonzero somewhere in $\mathcal{D}$.

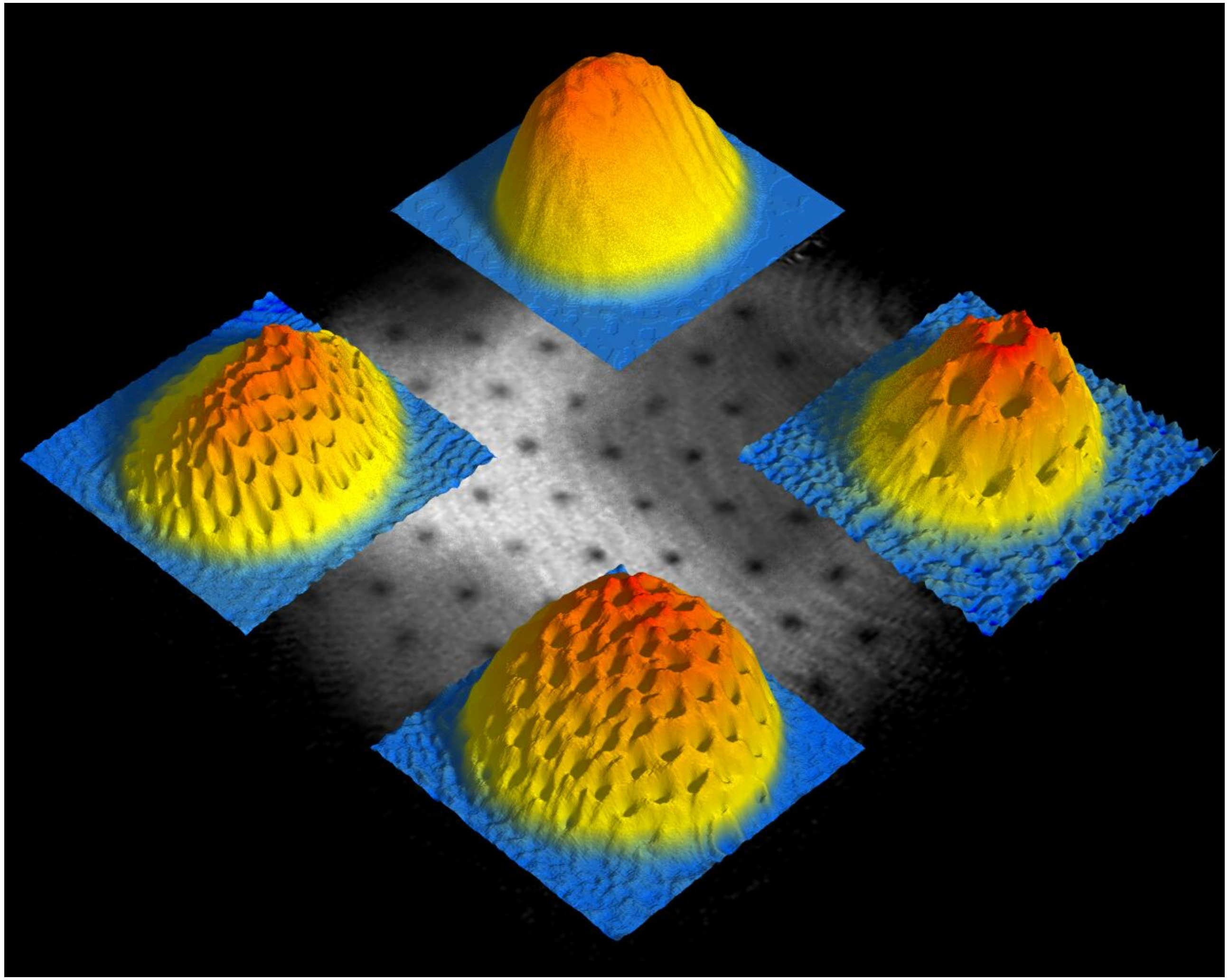
A region where $\nabla \times \mathbf{v} \neq 0$ is called a vortex.

The circulation around a vortex divided by 2π is called the degree of the vortex.

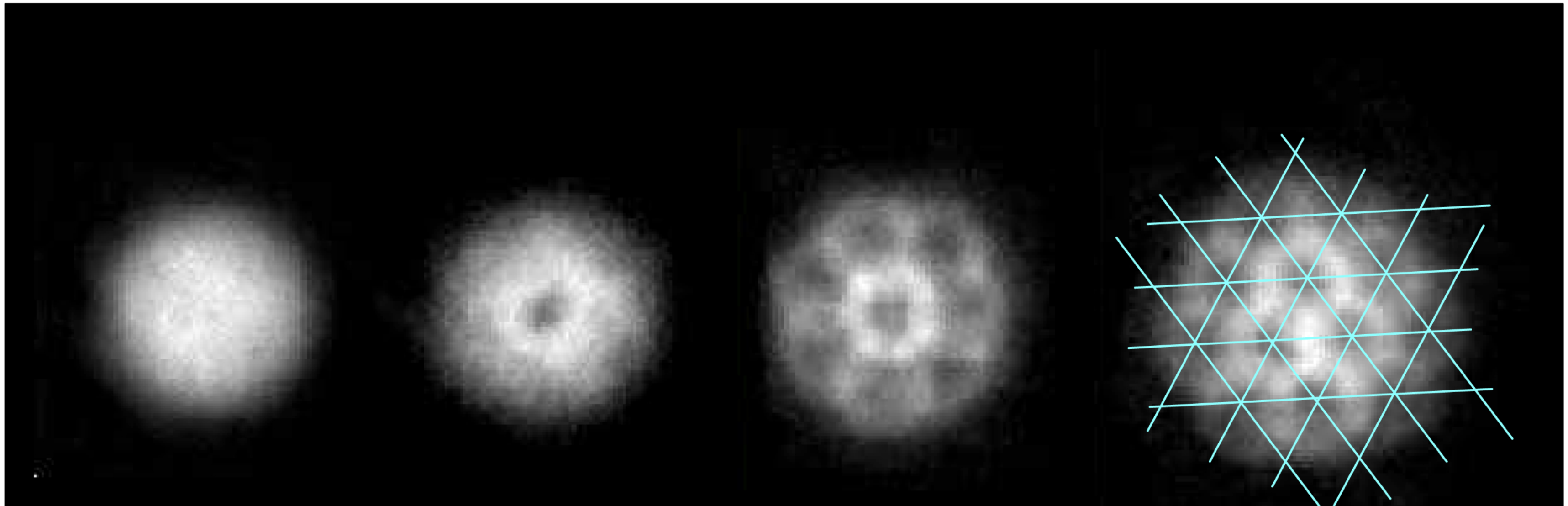
A bathtub vortex



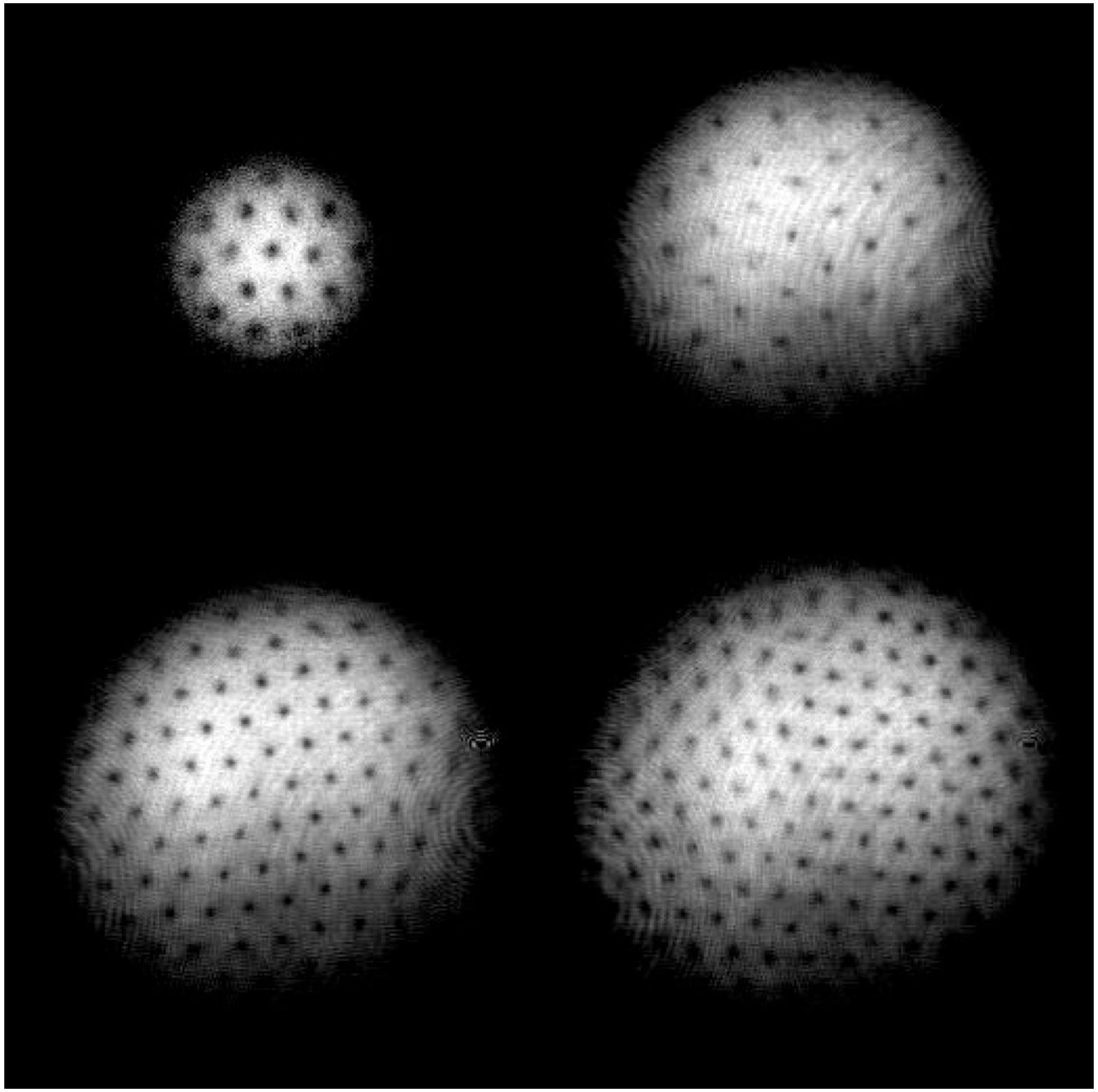
Quantum vortices



Creation of quantized vortices in a rotating container



Just below the critical frequency · *Just above the critical frequency* · *Notably above the critical frequency* · *Even larger freqs.: Abrikosov lattice*



Rotational and irrotational vortices

Two autumn leaves in a counter-clockwise vortex (reference position).

Two autumn leaves in a rotational vortex rotate with the counter-clockwise flow.

Two autumn leaves in an irrotational vortex preserve their original orientation while moving counter-clockwise.

Basic facts about quantum vortices

- Quantum vortices are irrotational, the vorticity is concentrated in vortex lines (vortex points in 2d).

- Vorticity is quantized in units of *h/m*.

  This was probably first realized by Lars Onsager (1903-1976) in 1949.

Quantization of Vorticity in a Superfluid

Describe the superfluid by a complex valued function ("order Parameter") ψ satisfying a nonlinear Schrödinger Equation (Gross-Pitaevskii equation). The phase of ψ determines the velocity: If $\psi = e^{\mathrm{i}\varphi}|\psi|$ then

$$\mathbf{v} = \frac{\hbar}{m}\nabla\varphi.$$

Since ψ is single valued we have $\oint_{\mathcal{C}} \nabla\varphi \cdot d\ell = n\,2\pi$ with $n \in \mathbb{Z}$, so

$$\oint_{\mathcal{C}} \mathbf{v} \cdot d\ell = n\frac{h}{m}\,.$$

On the other hand, where the phase is nonsingular, i.e., where $|\psi(\mathbf{r})| \neq 0$, we have

$$\nabla \times \mathbf{v} = 0.$$

The Superfluid Current Density

To understand why $\mathbf{v} = (\hbar/m)\nabla\varphi$ is the superfluid velocity note that a time-dependent non-linear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + F(\psi)\psi$$

implies the continuity equation

$$\partial\rho/\partial t + \nabla\cdot\mathbf{J} = 0$$

with the density $\rho(\mathbf{x}) = |\psi(\mathbf{x})|^2$ and the superfluid current density

$$\mathbf{J} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*).$$

The interpretation of $\mathbf{v} = (\hbar/m)\nabla\varphi$ as velocity follows from

$$\mathbf{J}(\mathbf{x}) = \rho(\mathbf{x})\mathbf{v}(\mathbf{x}).$$

The Basic Many-Body Hamiltonian

The quantum mechanical Hamiltonian for N spinless bosons with a pair interaction potential v and external potential, V, in a rotating frame with angular velocity $\mathbf{\Omega}$ is

$$H = \sum_{j=1}^{N} \left(-\tfrac{1}{2}\nabla_j^2 + V(\mathbf{x}_j) - \mathbf{L}_j \cdot \mathbf{\Omega}\right) + \sum_{1\le i<j\le N} v(|\mathbf{x}_i - \mathbf{x}_j|).$$

Here $\mathbf{x}_j \in \mathbb{R}^3$, $j = 1, \dots, N$ are the positions and $\mathbf{L}_j = -\mathrm{i}\,\mathbf{x}_j \times \nabla_j$ the angular momentum operators of the particles. Units have been chosen so that $\hbar = m = 1$ and thus $h/m = 2\pi$.

The pair interaction potential v is assumed to be radially symmetric, of short range, and repulsive.

H operates on *symmetric* functions in $L^2(\mathbb{R}^{3N})$.

Hamiltonian, Magnetic Version

The Hamiltonian can alternatively be written in the form

$$H = \sum_{j=1}^{N} \left(\tfrac{1}{2}[\mathrm{i}\nabla_j + \mathbf{A}(\mathbf{x}_j)]^2 + V(\mathbf{x}_j) - \tfrac{1}{2}\Omega^2 r_j^2\right) + \sum_{1\le i<j\le N} v(|\mathbf{x}_i - \mathbf{x}_j|).$$

with

$$\mathbf{A}(\mathbf{x}) = \mathbf{\Omega} \times \mathbf{x} = \Omega r\, \mathbf{e}_\theta$$

and r=distance from the rotation axis.

This corresponds to the splitting of the rotational effects into Coriolis and centrifugal forces. The Coriolis force has formally the same effect as a constant magnetic field $\mathbf{B} = 2\mathbf{\Omega}$ with vector potential $\mathbf{A}(\mathbf{x})$.

Harmonic vs. Anharmonic Traps

If V is harmonic in the direction $\perp$ to $\mathbf{\Omega}$, i.e.,

$$V(\mathbf{x}) = \tfrac{1}{2}\Omega_{\rm trap} r^2 + V^{\|}(z)$$

then stability requires $\Omega < \Omega_{\rm trap}$. Rapid rotation means here that

$$\Omega \to \Omega_{\rm trap}$$

from below.

If V is anharmonic and increases faster than quadratically in the direction $\perp$ to $\mathbf{\Omega}$, e.g. $V \sim r^s$ with $s > 2$, then rapid rotation means simply $\Omega \to \infty$.

Gross-Pitaevskii Equation

Basic fact (E. Lieb and R. Seiringer, 2005) about the the many-body Hamiltonian for $N \to \infty$ with Na and Ω fixed, where a is the scattering length of the interaction potential v:

There is Bose-Einstein condensation in the many-body ground state as $N \to \infty$. Moreover the ground state is a superfluid, and the wave function of the condensate is the superfluid order parameter. It satisfies a non-linear Schrödinger equation, the Gross-Pitaevskii equation

$$\left[\tfrac{1}{2}(\mathrm{i}\nabla + \mathbf{A})^2 + (V - \tfrac{1}{2}\Omega^2 r^2) + 4\pi Na|\psi|^2\right]\psi = \mu\psi \ .$$

The limit $N \to \infty$ with Na and Ω fixed is called the GP limt.

Zero energy scattering equation for the two particle scattering with a potential v:

$$-\frac{\hbar^2}{m}\nabla^2\psi + v\psi = 0.$$

Writing $\psi(\mathbf{x}) = u(r)/r$ with $r = |\mathbf{x}|$ this is equivalent to

$$-\frac{\hbar^2}{m}u''(r) + v(r)u(r) = 0.$$

For r larger than the range of v the solution with $u(0) = 0$ has the form

$$u(r) = (\text{const.})(r - a)$$

with a constant a that is called the *scattering length* of v. Equivalently,

$$a = \lim_{r\to\infty}\left[r - \frac{u(r)}{u'(r)}\right]$$

and this is finite if v decreases at least as $r^{-(3+\varepsilon)}$ at infinity.

Scattering Lenght (cont.)

For $\psi(\mathbf{x}) = u(r)/r$ we have thus outside of the support of v

$$\psi(\mathbf{x}) = (\mathrm{const.}) \left(1 - \frac{a}{r}\right).$$

If $v \geq 0$, then also $a \geq 0$, but $a \leq \text{range of } v$. For a hard sphere potential a is equal to the radius of the sphere.

If v is not positive then a can be negative and if $-\frac{\hbar^2}{m}\nabla^2 + v$ has bound states, then a can be much larger than the range of v.

If $v \geq 0$ then

$$\frac{4\pi\hbar^2}{m} a = \inf_{\psi} \int \left\{ \frac{\hbar^2}{m} |\nabla\psi|^2 + |\psi|^2 v \right\} d^3\mathbf{x}$$

where the infimum is over all differentiable ψ that tend to 1 at infinity. This implies in particular

$$a \leq \frac{m}{4\pi\hbar^2} \int v(r)\, d^3\mathbf{x}.$$

Digression: The Concept of BEC

One-particle density matrix of an N-particle wave function Ψ:

$$\rho^{(1)}(\mathbf{x},\mathbf{x}') = N\int \Psi(\mathbf{x},\mathbf{x}_2,\dots\mathbf{x}_N)\Psi(\mathbf{x}',\mathbf{x}_2,\dots\mathbf{x}_N)^* d\mathbf{x}_2\cdots d\mathbf{x}_N.$$

Spectral decomposition:

$$\rho^{(1)}(\mathbf{x},\mathbf{x}') = \sum_i \lambda_i\,\psi_i(\mathbf{x})\psi_i^*(\mathbf{x}')$$

with $\lambda_0 \geq \lambda_1 \geq \dots$ and orthonormal ψ_i.

BEC in the state Ψ means, by definition, that $\lambda_0 = O(N)$.The function ψ_0 is the called the wave function of the condensate.

Digression: The Concept of BEC (cont.)

We shall always be considered with ground states of the many-body Hamiltonian in the rotating system, and a slight complication arises beacse the ground state is in general not unique in a rotationally symmetric potential. This is due to the appearance of vortices that break this symmetry if there is more than one vortex.

A generic ground state, therefore, need not show condensation into a single wave function, but rather a fragmented condensation, where many states may have an occupation $O(N)$. It can be shown, however, that in the GP limit, the limit points of one-particle density matrices of ground states form a simplex whose extremal points are projections onto solutions of the GP equation.

A slight perturbation of a rotationally symmetric potential can also render the ground state unique, in which case BEC in the usual sense of macroscopic occupation of a single state holds.

The Gross-Pitaevskii Energy Functional

The GP equation is obtained by minimizing the energy functional

$$\begin{aligned}\mathcal{E}^{\mathrm{GP}}[\psi] &= \int_{\mathbb{R}^3} \left\{ \tfrac{1}{2}|\nabla\psi|^2 + V|\psi|^2 - \psi^* \,\mathbf{\Omega}\cdot\mathbf{L}\psi + 2\pi Na|\psi|^4 \right\} d\mathbf{x} \\ &= \int_{\mathbb{R}^3} \left\{ \tfrac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\psi|^2 + (V - \tfrac{1}{2}\Omega^2 r^2)|\psi|^2 + 2\pi Na|\psi|^4 \right\} d\mathbf{x}\end{aligned}$$

with the normalization condition $\int_{\mathbb{R}^3}|\psi|^2 = 1$. A minimizer, i.e., a solution of the GP equation, will be denoted by ψ^{GP}.

Reduction to 2D

If the external potential is strongly confining in the direction of the rotational axis (z-direction), a 2D description is appropriate.

The same applies to the opposite case, i.e., when the trap potential is almost constant in the z-direction. In this case 2D GP functional describes the ground state energy per unit length in the z-direction.

The coupling constant in the 2D GP functional is in both cases

$$g = 2\pi Na/L$$

with L a length in the z-direction.

Reduction to 2D (cont.)

It is custormary and convenient to write

$$g = \frac{1}{\varepsilon^2}.$$

The 2D GP functional we consider is thus

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\psi|^2 + (V - \tfrac{1}{2}\Omega^2 r^2)|\psi|^2 + \frac{1}{\varepsilon^2}|\psi|^4 \right\} \mathrm{d}^2\mathbf{r}$$

We shall in particular be interested in large g which means small ε.

The Meaning of ε

The healing length ℓ_h is defined by the condition that the kinetic energy associated with ℓ_h equals the interaction energy per particle, i.e.,

$$\frac{1}{\ell_\mathrm{h}^2} \sim \frac{1}{\varepsilon^2}\int |\psi|^4.$$

In a trap of effective radius R we have $|\psi|^2 \sim R^{-2}$ by the normalization condition, and thus

$$\int |\psi|^4 \sim R^{-2}.$$

Hence

$$\varepsilon \sim \ell_\mathrm{h}/R.$$

The TF density and energy

Dropping the kinetic energy term $\frac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\psi|^2$ from the GP energy functional lead to the so-called TF functional of the density $\rho = |\psi|^2$:

$$\mathcal{E}^{\mathrm{TF}}[\rho] = \int \left\{ (V - \tfrac{1}{2}\Omega^2 r^2)\rho + \frac{1}{\varepsilon^2}\rho^2 \right\} \mathrm{d}^2\mathbf{r}$$

The minimizer under the normalization condition $\int \rho = 1$, denoted by ρ^{TF}, is explicitly given as

$$\rho^{\mathrm{TF}}(\mathbf{r}) = \frac{\varepsilon^2}{2}\left[\mu - V(\mathbf{r}) + \tfrac{1}{2}\Omega^2 r^2\right]_+$$

where μ is a chemical potential and $[\cdot]_+$ denotes the positive part. The corresponding energy is denoted by E^{TF}.

The density ρ^{TF} essentially determines the global profile of the condensate, while vortices are due to the term $\frac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\psi|^2$.

For small Ω the condensate is at rest in the inertial system and thus rotates (with angular velocity $-\mathbf{\Omega}$) in the rotating system. (This is due to the superfluidity of the conensate; a normal fluid would rotate with the trap and thus be at rest in the rotating system.)

The velocity operator in the rotating system is $-\mathrm{i}\nabla - \mathbf{A}(\mathbf{r})$. The constant function is for small Ω the ground state and has the velocity distribution

$$\mathbf{v}(\mathbf{r}) = -\mathbf{A}(\mathbf{r}) = -\mathbf{\Omega} \times \mathbf{r} = -\Omega r \, \mathbf{e}_\theta.$$

The corresponding kinetic energy is exacly compensated by the centrifugal energy $-\frac{1}{2}\Omega^2 r^2$.

The Emergence of Vortices (cont.)

At higher rotational velocities vortices may partly compensate the term $-\mathbf{A}$ of the velocity and hence reduce the kinetic energy. This reduction is necessarily accompanied by a redistribution of the density and hence some increase in interaction energy which determines the size of the vortex cores.

Consider the case of small ε and a trap with effective radius R. A *vortex of degree* d located at the origin, can be approximated by the ansatz

$$\psi(r,\theta) = f(r)\exp(\mathrm{i}\theta d)$$

with

$$f(r) \sim \begin{cases} r^d & \text{if} \quad 0 \leq r \lesssim r_{\mathrm{v}} \\ R^{-1} & \text{if} \quad r_{\mathrm{v}} \lesssim r \leq R \end{cases}$$

where r_{v} is the radius of the vortex core where the density is small.

The Emergence of Vortices (cont.)

The component of the velocity in the direction of $\mathbf{e}_\theta$ is

$$\mathbf{v}(\mathbf{r})_\theta = \left(\frac{d}{r} - \Omega\, r\right).$$

The change in kinetic energy compared to the vortex free case, $d = 0$, has a cost term and a gain term:

$$\Delta E_{\mathrm{kin}} \sim R^{-2} \int_{r_{\mathrm{v}}}^{R} [(d/r)^2 - 2d\,\Omega]\, r\, dr + O(1)$$

$$= R^{-2} d^2 |\log(r_{\mathrm{v}}/R)| - d\,\Omega + O(1).$$

The increase in interaction energy through the creation of the vortex is

$$\Delta E_{\mathrm{int}} \sim \frac{1}{\varepsilon^2}(r_{\mathrm{v}}/R)^2.$$

Optimizing the total energy change w.r.t. r_{v} gives

$$r_{\mathrm{v}} \sim \varepsilon R$$

and an interaction energy change $\Delta E_{\mathrm{int}} \sim R^{-2}$.

The total energy change is thus

$$\Delta E \sim R^{-2}d^2|\log\varepsilon| - d\,\Omega + O(1).$$

A vortex of degree $d = 1$ becomes energetically favorable when

$$R^{-2}|\log\varepsilon| - \Omega + O(1) < 0$$

which for $\varepsilon \ll 1$ means

$$\Omega \gtrsim R^{-2}|\log\varepsilon|.$$

We also see that d vortices of degree 1, ignoring their interaction, have energy $\sim d(R^{-2}|\log\varepsilon| - \Omega)$ while a vortex of degree d has energy $R^{-2}d^2|\log\varepsilon| - d\,\Omega$. Hence it is energetically favorable to 'split' a d-vortex into d pieces of 1-vortices, breaking the rotational symmetry.

Generalization:

Consider a disc of radius R, possibly much smaller than the radius of the condensate, and with center $\mathbf{r}_0$ somewhere in the trap. Shifting the origin to $\mathbf{r}_0$ the velocity is

$$\mathbf{v}(\mathbf{r}) = \left(\frac{d}{r} - \Omega\, r\right)\mathbf{e}_\theta - \Omega \mathbf{r}_0^\perp$$

with $\mathbf{r}_0^\perp \cdot \mathbf{r}_0 = 0$. If $|\psi|^2 = \rho$ on the disc we ask for the energy change if a vortex with core radius r_v is created at $\mathbf{r}_0$.

A computation analogous to the previous one, using that $\int (\mathbf{e}_\theta \cdot \mathbf{r}_0^\perp)\, \mathrm{d}\theta = 0$, leads to

$$r_\mathrm{v} = \varepsilon \rho^{-1/2},$$

and the condition for the creation of a vortex becomes

$$\Omega \gtrsim \rho\, |\log(\varepsilon R \rho^{1/2})|.$$

Remark:

While the preceeding heuristic discussion is adequate as a first orientation, it ignores some points that are important to take into account in a precise analysis:

- Inhomogeneities of the background density are in general significant.
- When there are several vortices their long-range interaction due to the nonlinearity of the GP functional may also be relevant.

In an inhomogeneous background, precisely defined cost functions that go beyond the rough approximation $\rho\,|\log\varepsilon| - \Omega + O(1)$ have to be considered.

The case of a 'Flat' Trap

Consider now a 2D 'flat', circular trap with rigid boundary at radius 1. The GP energy functional is then

$$\mathcal{E}^{\rm GP}[\psi] = \int_{\mathcal{B}} \left\{ \tfrac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\psi|^2 - \tfrac{1}{2}\Omega^2 r^2 |\psi|^2 + \frac{1}{\varepsilon^2}|\psi|^4 \right\} \mathrm{d}^2\mathbf{r}$$

where $\mathcal{B}$ is the unit disc and $\mathbf{A}(\mathbf{r}) = \Omega\, r\, \mathbf{e}_\theta$.

Vortices start to appear for $\Omega > \Omega_{\rm c1} = \omega_{\rm c1}|\log\varepsilon|$, and it can be proved that if $\Omega < \omega_{\rm c1}|\log\varepsilon| + O(\log|\log\varepsilon|)$ there is a finite number of vortices, even as $\varepsilon \to 0$. For larger Ω the number of vortices is unbounded as $\varepsilon \to 0$.

The vortex pattern in the regime $\Omega \sim |\log\varepsilon|$ bas been studied by M. Correggi and N. Rougerie (2013).

Effects of the Centrifugal Term

For $\Omega \gg |\log \varepsilon|$ a lattice of vortices emerges, but new phenomena appear at two additional critical velocities, namely for $\Omega \sim 1/\varepsilon$ and $\Omega \sim 1/(\varepsilon^2 |\log \varepsilon|)$ respectively.

If $\Omega = O(1/\varepsilon)$ the centrifugal term $-(\Omega^2/2) r^2 |\psi(\mathbf{r})|^2$ and the interaction term $(1/\varepsilon^2)|\psi(\mathbf{r})|^4$ become comparable in size and the centrifugal forces influence the bulk shape of the condensate.

For $\Omega > \Omega_{\mathrm{c2}} = \omega_{\mathrm{c2}}\, \varepsilon^{-1}$ the centrifugal forces deplete strongly the density in a 'hole' of radius

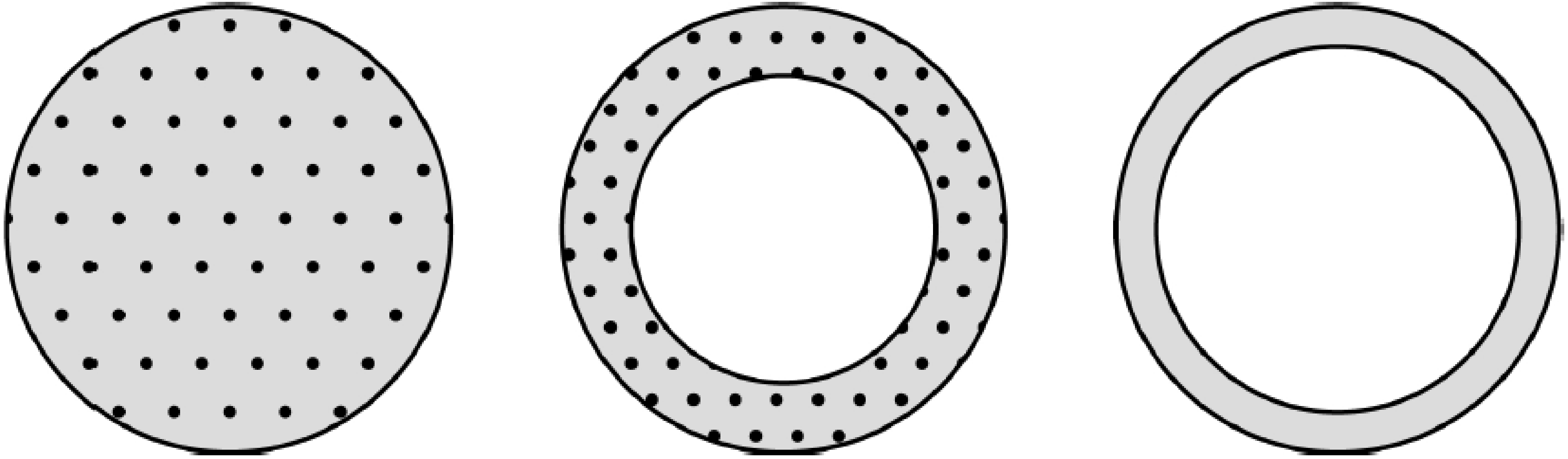
$$R_{\mathrm{h}} = 1 - c(\Omega\varepsilon)^{-1}$$

around the rotation axis and the bulk of the condensate is concentrated in an annulus of thickness $\sim (\varepsilon\Omega)^{-1}$.

Effects of the Centrifugal Term (cont.)

As long as $\Omega \ll 1/(\varepsilon^2|\log\varepsilon|)$, however, the annulus still contains a lattice of vortices, but if $\Omega > \Omega_{c3} = \omega_{c3}(\varepsilon^2|\log\varepsilon|)^{-1}$ the high density of the condensate in the annulus make vortices too costly. A transition to a *'giant vortex'* state takes place where all vorticity is concentrated in the 'hole' but the bulk of the condensate is vortex free.

We discuss these results in more detail in the sequel, starting with the parameter region $|\log\varepsilon| \ll \Omega \ll 1/(\varepsilon^2|\log\varepsilon|)$.



$|\log \varepsilon| \ll \Omega \ll 1/\varepsilon$ $\quad$ $1/\varepsilon \lesssim \Omega \ll 1/(\varepsilon^2|\log \varepsilon|)$ $\quad$ $1/(\varepsilon^2|\log \varepsilon|) \lesssim \Omega$

The TF density profile

The appearance of a'hole' due to the centrifugal forces and an annulus containing the bulk of the mass can be seen from the TF density profile which is

$$\rho^{\mathrm{TF}}(\mathbf{r}) = \frac{\varepsilon^2}{2}\left[\mu^{\mathrm{TF}} + \tfrac{1}{2}\Omega^2 r^2\right]_+$$

for $r \leq 1$ and zero otherwise. The chemical potential μ^{TF} is determined by the normalization and can be explicitly computed.

The important features of the profile for $\varepsilon \ll 1$ are:

- For $\Omega \ll \varepsilon^{-1}$ the profile is approximately flat, but for $\Omega \gtrsim \varepsilon^{-1}$ it becomes parabolic.
- A 'hole' appears for $\Omega \geq \Omega_{c2} = (4/\sqrt{\pi})\,\varepsilon^{-1}$.
- For $\Omega \gg \Omega_{c2}$ the support is contained in an annulus of thickness $\sim \varepsilon\Omega$ and in the radial variable the profile is approximately a triangle of height $\sim (\varepsilon\Omega)^{-1}$.

Energy for $|\log\varepsilon| \ll \Omega \ll (\varepsilon^2|\log\varepsilon|)^{-1}$

Theorem (M.Correggi, JY, 2008)

Let E^{GP} denote the GP energy, i.e., the minimum of the GP energy functional. Let E^{TF} denote the minimal energy of the GP functional without *the kinetic term.*

If $|\log\varepsilon| \ll \Omega \ll 1/\varepsilon$, then

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \tfrac{1}{2}\Omega|\log(\varepsilon^2\Omega)|(1+o(1)).$$

If $1/\varepsilon \lesssim \Omega \ll 1/(\varepsilon^2|\log\varepsilon|)$ then

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \tfrac{1}{2}\Omega|\log\varepsilon|(1+o(1)).$$

In both cases the energy corresponds to a uniform distribution of vorticity in a the form of a vortex lattice.

An electrostatic analogy

The upper bound to the energy in the Theorem is based on a variational ansatz that can be motivated by an electrostatic analogy.

We write points $\mathbf{r} = (x, y) \in \mathbb{R}^2$ as complex numbers, $\zeta = x + \mathrm{i}y$, and consider a lattice of points ζ_j. Placing a vortex of degree 1 at each point ζ_j leads to a trial function for the GP energy of the form

$$\psi(\mathbf{r}) = c\,\rho(\mathbf{r})^{1/2}\xi(\mathbf{r})\exp\{\mathrm{i}\varphi(\mathbf{r})\}$$

where ρ is (a possibly regularized version of) the TF density, ξ a function modelling vortex cores around the points ζ_j, and the phase factor is

$$\exp\{\mathrm{i}\varphi(\mathbf{r})\} = \prod_j \frac{\zeta - \zeta_j}{|\zeta - \zeta_j|}.$$

An Electrostatic Analogy (cont.)

Now

$$|(\mathrm{i}\nabla + \mathbf{A})\psi|^2 = |\nabla f|^2 + f^2|\mathbf{A} - \nabla\varphi|^2$$

and

$$\varphi = \sum_j \arg(\zeta - \zeta_j).$$

The phase $\arg z$ of a complex number is the imaginary part of the complex logarithm which is an analytic function on the complex plane (suitably cut). The Cauchy-Riemann equations for the real and imaginary part of an analytic functions imply

$$|\mathbf{A} - \nabla\varphi|^2 = |\Omega\, r\mathbf{e}_r - \nabla\chi|^2$$

where

$$\chi(\mathbf{r}) = \sum_j \log|\mathbf{r} - \mathbf{r}_j|.$$

An Electrostatic Analogy (cont.)

But

$$\mathbf{E}(\mathbf{r}) := \Omega\, r\mathbf{e}_r - \nabla\chi(\mathbf{r})$$

has a simple physical interpretation: It can be regarded as an ‘electric field’ generated by a uniform charge distribution of density Ω/π together with unit ‘charges’ of opposite sign at the positions of the vortices, $\mathbf{r}_j$. The integral of $|\mathbf{E}(\mathbf{r})|^2$ is the corresponding electrostatic energy.

Vortex Lattice

We now distribute the vortices over the unit disk so that the vorticity per unit area is Ω/π. (This is really $2\Omega \cdot m/h$.) Thus every vortex $\mathbf{r}_i$ sits at the center of lattice cell Q_i of area $|Q_i| = \pi/\Omega$, surrounded by a uniform charge distribution of the opposite sign so that the total charge in the cell is zero.

If the cells were disc-shaped, then Newton's theorem would imply that the 'electric field' generated by the cell would vanish outside the cell, i.e, there would be no interaction between the cells.

Vortex Lattice (cont.)

The energy of each cell, taking into account the factor $\xi(\mathbf{r})$ that cuts off the Coulomb field from the point charge (vortex) at a radius $r_{\rm v} \ll \Omega^{-1/2}$, is just

$$2\pi \int_{r_{\rm v}}^{\Omega^{-1/2}} (1/r)^2\, r\, dr + O(1) = \pi |\log(r_{\rm v}^2 \Omega)| + O(1).$$

We now multiply by the density of cells, $\Omega/(2\pi)$, obtaining

$$(\Omega/2)|\log(r_{\rm v}^2\Omega)|(1+o(1)).$$

$$r_{\rm v} = \varepsilon \cdot \rho^{-1/2} = \begin{cases} \varepsilon & \text{if} \quad \Omega \lesssim 1/\varepsilon \\ (\varepsilon/\Omega)^{1/2} & \text{if} \quad 1/\varepsilon \lesssim \Omega \end{cases}$$

Why hexagonal cells are optimal

The cells can, of course, not be disc shaped, but the closest approximation to that are hexagonal cells, giving the optimal energy. The vortices then sit on a triangular lattice. The interaction between the cells, although not zero, is small because the cells have no dipole moment.

Among the regular lattices the hexagonal cells have the lowest multipole moments which indicates why they are preferred.

This difference between hexagonal and other regular lattices (rectangular or triangular unit cells) is, however, a delicate higher order effect and only shows up in higher orders than stated in the Theorem.

The Lower Bound

The electrostatic analogy leads to a good trial function for an upper bound to the energy, but the lower bound is more complicated. It relies on results from Ginzburg-Landau Theory obtained by Etienne Sandier and Sylvia Serfaty, which in turn are based on the construction of "vortex balls" to enclose the regions where the GP minimizer is small, combined with so-called "jacobian estimates" on the curl of the superfluid current. Such constructions and estimates are originaly due to Etienne Sandier and, independently, Robert Jerrard and Halil Soner.

The Lower Bound (cont.)

Ginzburg-Landau Energy functional in a domain Q:

$$\mathcal{E}^{\mathrm{GL}}[u, \mathbf{A}] = \int_Q \mathrm{d}\mathbf{r} \, \left\{ |(\nabla - \mathrm{i}\mathbf{A})\, u|^2 + |\nabla \times \mathbf{A} - \mathbf{h}_{\mathrm{ex}}|^2 + \kappa^2 \left(1 - |u|^2\right)^2 \right\}.$$

Here $\mathbf{A}$ is a variable vector potential and $\mathbf{h}_{\mathrm{ex}}$ a fixed magnetic field.

Sandier and Serfaty prove for

$$\log \kappa \ll h_{\mathrm{ex}} \ll \kappa^2$$

the lower bound

$$E^{\mathrm{GL}} \geq (1 - o(1))|Q| h_{\mathrm{ex}} \log \frac{\kappa}{\sqrt{h_{\mathrm{ex}}}}.$$

The Lower Bound (cont.)

To use this for the GP problem, the first step is to write

$$\psi^{\mathrm{GP}}(\mathbf{r}) = u(\mathbf{r})\rho^{\mathrm{TF}}(r)^{1/2}$$

that is in any case possible for

$$\mathbf{r} \in \mathcal{T} \equiv \left\{\mathbf{r} \in \mathcal{B}_1 \mid \rho^{\mathrm{TF}}(r) \geq (\varepsilon\Omega)|\log\delta|^{-1}\right\}$$

with $\delta \equiv \varepsilon^2|\log\varepsilon|\Omega \ll 1$ and $\delta \gg \varepsilon^2|\log\varepsilon|^2$.
We then get

$$E^{\mathrm{GP}} \geq E^{\mathrm{TF}} + \tilde{\mathcal{E}}^{\mathrm{GP}}[u] - \mathrm{const.}(\varepsilon\Omega)|\log\varepsilon|.$$

with a weighted GL-type functional and $\mathbf{A} = \mathbf{x} \times \mathbf{\Omega}$:

$$\tilde{\mathcal{E}}^{\mathrm{GP}}[u] \equiv \int_{\mathcal{T}} \mathrm{d}\mathbf{r}\,\rho^{\mathrm{TF}}(r)\left\{|(\nabla - \mathrm{i}\mathbf{A})\,u|^2 + \varepsilon^{-2}\rho^{\mathrm{TF}}(r)\left(1 - |u|^2\right)^2\right\}.$$

The Lower Bound (cont.)

To estimate this further, introduce a square regular lattice $\mathcal{L}$ with side length ℓ satisfying

$$\sqrt{\frac{|\log \varepsilon|}{\Omega}} \ll \ell \ll \min\left[1, \frac{1}{(\varepsilon\Omega)|\log\delta|}\right].$$

One then obtains

$$\tilde{\mathcal{E}}^{\mathrm{GP}}[u] \geq (1 - o(1)) \sum_{\mathbf{r}_i \in \mathcal{L}} \rho^{\mathrm{TF}}(r_i)\, \mathcal{E}^{(i)}[u]$$

with

$$\mathcal{E}^{(i)}[u] \equiv \int_{\mathcal{Q}^i} \mathrm{d}\mathbf{r} \left\{ |(\nabla - \mathrm{i}\mathbf{A})\, u|^2 + \varepsilon^{-2} \rho^{\mathrm{TF}}(r_i) \left(1 - |u|^2\right)^2 \right\}.$$

The Lower Bound (cont.)

One might now be tempted to use the GL estimate of Sherfaty and Sandier on $\mathcal{E}^{(i)}[u]$, taking $\mathbf{h}_{\text{ex}} = \nabla \times \mathbf{A} = 2\Omega \mathbf{e}_z$ and $\kappa^2 = \varepsilon^{-2}\rho^{\text{TF}}(r_i)$. However, the bound $|\mathcal{Q}^i| h_{\text{ex}} \log(\kappa/\sqrt{h_{\text{ex}}})$ would be too small and not even applicable, because $|\mathcal{Q}^i|$ depends on ε and Ω while the SS estimate is for a fixed domain.

The way out is to blow up the size of the Q_i by scaling all lengths with ℓ^{-1}. This transforms the problem into an estimate for a GL functional on a unit square with

$$|\mathbf{h}_{\text{ex}}| = \ell^2 2\Omega \quad \text{and} \quad \kappa^2 = \varepsilon^{-2}\ell^2\rho^{\text{TF}}(r_i).$$

By the condition on ℓ we have $\log\kappa \ll |\mathbf{h}_{\text{ex}}| \ll \kappa^2$. Thus one can apply the estimate from GL theory and obtain a lower bound matching the upper bound, provided the condition $|\log\varepsilon| \ll \Omega \ll \varepsilon^{-2}|\log\varepsilon|^{-1}$ (that entered in the definition of ℓ) is satisfied.

Uniform Distribution of Vorticity

A further result that can be proved using GL techniques is

Theorem

Let ψ^{GP} be any GP minimizer and $\varepsilon > 0$ sufficiently small. If $|\log\varepsilon| \ll \Omega \ll \varepsilon^{-2}|\log\varepsilon|^{-1}$, there exists a finite family of disjoint discs $\{\mathcal{B}^i\} \subset \mathrm{supp}\,\rho^{\mathrm{TF}}$ with

- *the radius of any disc is smaller than $\Omega^{-1/2}$*
- *the sum of all the radii is much smaller than $\Omega^{1/2}$ and, denoting by $\mathbf{r}_i$ the center of each ball $\mathcal{B}^i$ and by d_i the winding number of ψ^{GP} on $\partial\mathcal{B}^i$,*

$$\frac{2\pi}{\Omega}\sum d_i \delta\left(\mathbf{r} - \mathbf{r}_i\right) \xrightarrow[\varepsilon\to 0]{\mathrm{w}} \chi^{\mathrm{TF}}(\mathbf{r})\,\mathrm{d}\mathbf{r},$$

in the sense of measures, with $\chi^{\mathrm{TF}}(\mathbf{r})$ the characteristic function of $\mathrm{supp}\,\rho^{\mathrm{TF}}$.

Reviews on Vortex Ball Constructions

Due to lack of time the vortex ball technique cannot be explained here but detailed information can be found in

E. Sandier and S. Serfaty, *Vortices in the Magnetic Ginzburg-Landau Model*, Birkhäuser 2007.

S. Serfaty, *Coulomb Gases and Ginzburg-Landau Vortices*, ArXiv:1403.6860v2, in particular Section 8.

The Emergence of a 'Giant Vortex'

For $\Omega > \omega_{c3}(\varepsilon^2|\log\varepsilon|)^{-1}$ a variational ansatz of the form

$$\psi(\mathbf{r}) = f(\mathbf{r})\exp(\mathrm{i}\hat{\Omega}\theta)$$

with a real valued function f and

$$\hat{\Omega} = \Omega - O(\varepsilon^{-1})$$

gives a lower energy than the vortex lattice ansatz, namely a correction $O(\varepsilon^{-2})$ to the TF energy, which is $\lesssim \Omega|\log\varepsilon|$ for $\Omega \gtrsim (\varepsilon^2|\log\varepsilon|)^{-1}$.

This does not prove, however, that the energy E^{gv} of the 'giant vortex ansatz' gives a good approximation to the energy of the true minimizer, nor that the latter is free of vortices in the bulk. That both statements *are* true is the content of the two theorems, proved by Michele Correggi, Nicolas Rougerie and JY in 2011.

Theorem (Energy in the giant vortex regime)

For $\Omega = \omega\,(\varepsilon^2 |\log\varepsilon|)^{-1}$ *with* $\omega > \omega_{c3} = 2/(3\pi)$ *the ground state energy is*

$$E^{\mathrm{GP}} = E^{\mathrm{gv}} - O(|\log\varepsilon|^{3/2}/\varepsilon^2).$$

Theorem (Absence of vortices in the bulk)

There is an annulus $\mathcal{A}$ *of width* $O((\varepsilon\Omega)^{-1})$ *with* $\int_{\mathcal{A}} |\psi^{\mathrm{GP}}|^2 = 1 - o(1)$ *such that for* Ω *as above and* ε *sufficiently small the minimizer* ψ^{GP} *is free of zeros in the annulus.*

More precisely, on the annulus

$$\left| |\psi^{\mathrm{GP}}(\mathbf{r})|^2 - \rho^{\mathrm{TF}}(\mathbf{r}) \right| \leq C |\log\varepsilon|^2 \varepsilon^{-3/4} \ll \rho^{\mathrm{TF}}(\mathbf{r})$$

Heuristics for the Giant Vortex

The proof, in particular of the latter theorem, is surprisingly difficult but a heuristic explanation for the transition at $\Omega \sim 1/(\varepsilon^2 |\log \varepsilon|)$ can be given by exploiting the electrostatic analogy.

Consider the giant vortex variational ansatz and interpret $\hat{\Omega}$ as a 'charge' situated at the origin. The 'electric field' generated this charge exactly cancels, in the annulus $\mathcal{A}$, the 'electric field' generated in the annulus by the uniform charge density Ω/π of the 'hole' (by Newton's theorem), due to the vector potential.

The 'charge' corresponding to the vector potential in the annulus, however, is not cancelled, and this 'residual charge' is

$$\sim \Omega \times (\varepsilon\Omega)^{-1} = \varepsilon^{-1}.$$

The electrostatic energy of this residual charge distribution is $\sim \varepsilon^{-2}$.

Heuristics for the Giant Vortex (cont.)

Creating a vortex in the annulus neutralizes one charge unit and thus reduces the electrostatic energy by ε^{-1}.

On the other hand, the *cost* of a vortex is $\sim f^2\,|\log\varepsilon|$, and we have $f^2 \sim (\varepsilon\Omega)$, so the cost of a single vortex in the bulk is

$$\sim \varepsilon\Omega\,|\log\varepsilon|\,.$$

Gain and cost are comparable if $\varepsilon^{-1} \sim \varepsilon\Omega\,|\log\varepsilon|$, i.e., for

$$\Omega \sim \frac{1}{\varepsilon^2|\log\varepsilon|}.$$

If Ω is smaller it still pays to create vortices also in the annulus, but if Ω is larger, the cost outweighs the gain and the annulus is vortex free.

Remark: These heuristics considerations are merely a plausibility argument. They are quite far from the rigorous proof, which is a great deal more complicated!

Comments on the Proof of the GV Transition

The first step is an energy splitting. The GV variational ansatz

$$\psi(\mathbf{r}) = f(\mathbf{r})\exp(\mathrm{i}\hat{\Omega}\theta)$$

leads to a minimization problem for $\hat{\Omega}$ and the functional

$$\hat{\mathcal{E}}^{\mathrm{GP}} = \int \left\{\tfrac{1}{2}|\nabla f|^2 - \tfrac{1}{2}\Omega^2 r^2 f^2 + \tfrac{1}{2}B^2 f^2 + \varepsilon^{-2} f^4\right\}$$

with

$$B(r) = \Omega r - \hat{\Omega} r^{-1}.$$

Comments on the Proof of the GV Transition (cont.)

Writing

$$\psi = g\exp(\mathrm{i}\hat{\Omega}\theta)u$$

with g the minimizer of $\hat{\mathcal{E}}^{\mathrm{GP}}$ and using the variational equation for g we obtain a splitting of the energy functional:

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \hat{E}^{\mathrm{GP}} + \mathcal{E}[u]$$

with a weighted GL type functional

$$\mathcal{E}[u] = \int g^2 \left\{ \tfrac{1}{2}|\nabla u|^2 - \mathbf{B}\cdot\mathbf{J}(u) + g^2\varepsilon^{-2}(1-|u|^2)^2 \right\}$$

and $\mathbf{B} = B\mathbf{e}_\theta$.

Comments on the Proof of the GV Transition (cont.)

The functional $\mathcal{E}[u]$ is then studied by GL techniques, but a serious complication arises because the support of g is essentially concentrated in an annulus of shrinking width $\varepsilon\Omega$ as $\varepsilon \to 0$. Blowing up the width of the annulus, in a similar way as was done for the cubes Q_i in the vortex lattice regime, looks at first promising but still does not reduce the problem to known results. The reason is that even if the width of the blown up annulus is fixed, its radius is not.

A decomposition of the annulus into cells of fixed size (but increasing number) and an inductive application of the vortex ball techniques is needed to achieve the desired results.

In a paper published in 2012 Nicolas Rougerie refined this method even further. He showed in particular that the value $\omega_{c3} = 2/(3\pi)$ is optimal and that the 'last' vortices arrange themselves in a *vortex ring*.

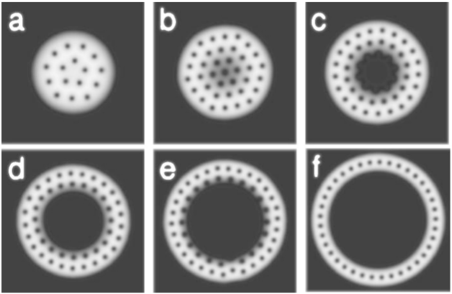


FIG. 2. Density profiles of a rotating condensate at $g = 1000$ and $\lambda = 0.5$, for $\Omega =$ (a) 2.0, (b) 3.0, (c) 3.5, (d) 4.0, (e) 4.5, and (f) 5.0, showing the stable vortice lattice configurations. The scale of each figure is 6×6 in units of $d_\perp$.

Summary on Vortices in a 'Flat' Trap.

When both the coupling constant, $1/\varepsilon^2$, and the rotational velocity, Ω, are large the picture in a 'flat' trap is as follows:

- Vortices begin to appear for $\Omega \sim |\log \varepsilon|$
- For $|\log \varepsilon| \ll \Omega \ll 1/(\varepsilon^2 |\log \varepsilon|)$ the bulk is covered by a vortex lattice.
- A 'hole' due to centrifugal forces appears for $1/\varepsilon \lesssim \Omega$.
- For $1/(\varepsilon^2 |\log \varepsilon|) \lesssim \Omega$ a 'giant vortex' around the origin gives the right energy and there are no vortices in the bulk.

'Soft' Anharmonic Traps

Consider now the 2D GP functional

$$\mathcal{E}^{\mathrm{GP}}_{\mathrm{phys}}[\Psi] = \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2} \left|(\mathrm{i}\nabla + \mathbf{A}_{\mathrm{phys}})\,\Psi\right|^2 + (V - \tfrac{1}{2}\Omega^2_{\mathrm{phys}} r^2)|\Psi|^2 + \frac{|\Psi|^4}{\varepsilon^2} \right\}$$

with a trap potential of the form

$$V(r) = kr^s$$

with $s > 2$, $k > 0$. The limiting case $s \to \infty$ corresponds to a 'flat' trap with fixed boundary at $r = 1$.

We have used the subscript 'phys' to distinguish the original quantities some some scaled versions that appear naturally if $s < \infty$.

Need for Scaling(s)

Contrary to the flat trap, where the extension of the condensate is limited by the trap, the repulsive terms in the energy functional may cause an expansion of the condensate in a 'soft' trap.

For suffciently slow rotation the interaction term dominates the centrifugal term and determines the extension. The effective radius, R, is determined by

$$R^s \sim \varepsilon^{-2} R^{-2} \quad i.e. \quad R \sim \varepsilon^{-2/(2+s)}$$

In particular, the first vortex therefore appears for

$$\Omega_{\rm phys} \sim \varepsilon^{4/(2+s)} |\log \varepsilon|$$

contrary to $\Omega_{\rm phys} \sim |\log \varepsilon|$ for a flat trap.
For faster rotation the extension is mainly determined by the centrifugal force and the effective radius is different.

Scaling for 'slow' rotation

If

$$\Omega_{\text{phys}} \lesssim \frac{1}{\varepsilon^{(s-2)/(s+2)}}$$

we define

$$R_\varepsilon = (k\varepsilon^2)^{-1/(s+2)}, \quad \mathbf{x} = R_\varepsilon^{-1}\mathbf{r},$$

$$\psi(\mathbf{x}) = R_\varepsilon \Psi(R_\varepsilon \mathbf{x}), \quad \Omega = R_\varepsilon^2 \Omega_{\text{phys}}, \quad \mathbf{A} = \Omega x \mathbf{e}_\theta$$

and obtain

$$\mathcal{E}^{\text{GP}}_{\text{phys}}[\Psi] = R_\varepsilon^{-2} \mathcal{E}^{\text{GP}}[\psi]$$

with

$$\mathcal{E}^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2} |(\mathrm{i}\nabla + \mathbf{A})\psi)|^2 + \varepsilon^{-2} \left[x^s |\psi|^2 - \tfrac{1}{2}\varepsilon^2 \Omega^2 |\psi|^2 + |\psi|^4 \right] \right\} \mathrm{d}^2\mathbf{x}$$

Energy in the regime $|\log \varepsilon| \ll \Omega \ll \varepsilon^{-1}$

By the same methods as in the 'flat' case one obtains a formula for the energy:

Theorem (M. Correggi, F. Pinsker, N. Rougerie, JY, 2012)

If $|\log \varepsilon| \lesssim \Omega \ll \varepsilon^{-1}$ *as* $\varepsilon \to 0$, *then*

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \tfrac{1}{2}\Omega |\log(\varepsilon^2\Omega)|(1 + o(1)).$$

Scaling for 'fast' rotation

For

$$\Omega_{\rm phys} \gtrsim \frac{1}{\varepsilon^{(s-2)/(s+2)}}$$

we use a different scaling than for 'slow' rotation:

The effective potential $(kr^s - \frac{1}{2}\Omega_{\rm phys}^2 r^2)$ has a unique minimum at $r = R_{\rm m} = (\Omega_{\rm phys}^2/(sk))^{1/(s-2)}$. Taking this as a length unit rather than R_ε we obtain the scaled energy functional

$$\mathcal{E}^{\rm GP}[\psi] = \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2}|({\rm i}\nabla + \mathbf{A})\psi|^2 + \Omega^2 W(x)|\psi|^2 + \varepsilon^{-2}|\psi|^4 \right\} {\rm d}^2\mathbf{x}$$

where

$$\Omega = R_{\rm m}^2 \Omega_{\rm phys} \sim \Omega_{\rm phys}^{(s+2)/(s-2)}, \quad \text{and} \quad W(x) = \left(\tfrac{1}{s}x^s - \tfrac{1}{2}x^2\right).$$

Scaling for 'fast' rotation (cont.)

The scaled effective potential $\Omega^2 W$ has a minimum at $x = 1$ independently of Ω but the value at the minimum is proportional to Ω^2.

The energy of the scaled functional is related to the original energy by

$$E^{\rm GP}_{\rm phys} = R_{\rm m}^{-2} \left[E^{\rm GP} + \left(\tfrac{1}{s} - \tfrac{1}{2}\right) \Omega^2 \right].$$

The case $\Omega_{\rm phys} \sim \frac{1}{\varepsilon^{(s-2)/(s+2)}}$ corresponds to $\Omega \sim \varepsilon^{-1}$. This holds also when Ω is defined by scaling with R_ε rather than $R_{\rm m}$.

The TF density profile

In the parameter range $\Omega \ll \varepsilon^{-4}$ the bulk density profile of ψ^{GP} can be approximately described by the Thomas-Fermi (TF) density

$$\rho^{\mathrm{TF}}(x) = \frac{\varepsilon^2}{2}\left[\mu^{\mathrm{TF}} - \Omega^2 W(x)\right]_+$$

The density ρ^{TF} vanishes at the origin for $\mu^{\mathrm{TF}} = 0$ and a hole of fine radius forms when $\mu^{\mathrm{TF}} < 0$. The critical velocity for the appearance of the hole is given by

$$\Omega_{c2} = \varepsilon^{-1}\left(2\int[-W]_+\right)^{-1/2}.$$

The TF density profile (cont.)

As $(\varepsilon\Omega) \to \infty$ we have $\mu^{\rm TF}/(\Omega^2) \to (s-2)/2s$ and the density $\rho^{\rm TF}$ becomes concentrated around $x = 1$. The inner and outer radii, $x_{\rm in} < 1$ and $x_{\rm out} > 1$ of the support, as well as $\mu^{\rm TF}$, are determined by

$$\rho^{\rm TF}(x_{\rm in}) = \rho^{\rm TF}(x_{\rm out}) = 0, \quad 2\pi \int_{x_{\rm in}}^{x_{\rm out}} \rho^{\rm TF}(x)\, x\, dx = 1.$$

A Taylor expansion of W around $x = 1$ gives the thickness of the support:

$$x_{\rm out} - x_{\rm in} \sim (\varepsilon\Omega)^{-2/3}.$$

By the normalization of $\rho^{\rm TF}$ it follows that its maximum is $O((\varepsilon\Omega)^{2/3})$.

Limits cannot be interchanged!

In a 'flat' trap, corresponding formally to $s = \infty$, the annulus has thickness $O((\varepsilon\Omega)^{-1})$ and the density is $O(\varepsilon\Omega)$.

Reason for the different powers of $\varepsilon\Omega$: The Taylor expansion is only justified as long as the turning point $x_{\rm turn}$ where $W''(x_{\rm turn}) = 0$ is much farther from 1 than $x_{\rm in}$ and $x_{\rm out}$, i.e., if

$$1 - x_{\rm turn} \gg (\varepsilon\Omega)^{-2/3}(s-1)^{-1/3}.$$

For large s this is equivalent to

$$\varepsilon\Omega \gg s/(\log s)^{3/2}$$

This is always fulfilled for each finite s if $\varepsilon\Omega$ is large enough (hence the independence of s) but violated for every fixed value of $\varepsilon\Omega$ if $s \to \infty$.

The regime $\varepsilon^{-1} \lesssim \Omega \ll \varepsilon^{-4}$

This regime has has analogous features to the regime $\varepsilon^{-1} \lesssim \Omega \ll 1/(\varepsilon^2|\log\varepsilon|)$ in a 'flat' trap: A 'hole' appears for $\Omega > \Omega_{2c} = \omega_{2c}\varepsilon^{-1}$, but there is still a lattice of vortices in the annulus where the bulk of the mass is concentrated.

By the same methods as in the 'flat' case and for $|\log\varepsilon| \ll \Omega \ll \varepsilon^{-1}$ (vortex lattice ansatz for the upper bound, GL estimates for the lower bound) one obtains in particular a formula for the energy:

Theorem (Energy between Ω_{c2} and Ω_{c3})

If $\varepsilon^{-1} \lesssim \Omega \ll \varepsilon^{-4}$ *as* $\varepsilon \to 0$, *then*

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \tfrac{1}{6}\Omega|\log(\varepsilon^4\Omega)|(1 + o(1)).$$

The Giant Vortex Regime $\Omega \gtrsim \varepsilon^{-4}$

Consider a variational ansatz for the wave function of the form

$$\psi(\mathbf{x}) = g(\mathbf{x}) \exp(\mathrm{i}\Omega\vartheta)$$

with a real valued function g, normalized such that $\int g^2 = 1$. (Assume that Ω is an integer). This gives

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2}|\nabla g|^2 + \tfrac{1}{2}\Omega^2(x - x^{-1})^2 g^2 \right.$$
$$\left. + \Omega^2 \left(\tfrac{1}{s} x^s - \tfrac{1}{2} x^2 \right) g^2 + \varepsilon^{-2} g^4 \right\} \equiv \mathcal{E}^{\mathrm{gv}}[g].$$

The unique positive minimizer g_{gv} of $\mathcal{E}^{\mathrm{gv}}$ is rotationally symmetric, Corresponding energy: E^{gv}.

Rough upper bound

Taking for g a regularizion of $\sqrt{\rho^{\mathrm{TF}}}$ we obtain

$$E^{\mathrm{GP}} \leq \mathcal{E}^{\mathrm{gv}}[g] = E^{\mathrm{TF}} + O(\varepsilon^{-4/3}\Omega^{2/3}) + O((\varepsilon\Omega)^{4/3}).$$

From now on we shall always assume that

$$\Omega = \omega\,\varepsilon^{-4}$$

with some *fixed* $\omega > 0$ while $\varepsilon \to 0$.

Then the second term in the energy upper bound is then $O(\omega^{2/3}\varepsilon^{-4})$ while the previous vortex lattice kinetic energy is $O(\omega|\log\omega|\varepsilon^{-4})$ and thus larger if ω is sufficiently large.

Bottom line: For large ω the giant vortex ansatz is energetically favorable to a vortex lattice.

The Gaussian Density Profile

In contrast to the regime $\Omega \ll \varepsilon^{-4}$ the TF profile is not a good approximation if $\Omega \sim \varepsilon^{-4}$.

$$\mathcal{E}^{\mathrm{gv}}[g] = -\frac{(s-2)}{2s}\Omega^2 + \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2}|\nabla g|^2 + \Omega^2 U(x) g^2 + \varepsilon^{-2} g^4 \right\} \mathrm{d}^2\mathbf{x}$$

with

$$U(x) = \tfrac{1}{2}(x - x^{-1})^2 + \left(\tfrac{1}{s}x^s - \tfrac{1}{2}x^2\right) + (s-2)/(2s).$$

Taylor expansion of U around $x = 1$ gives

$$U(x) = \tfrac{1}{2}\alpha^2 (x-1)^2 + O((x-1)^3)$$

with $\alpha^2 = 4 + (s-2)$.

The Gaussian Density Profile (cont.)

After scaling and translation $x \to y = \Omega^{1/2}(x-1)$ we are, for $\Omega = \omega\,\varepsilon^{-4}$, led to the functional

$$\mathcal{E}^{\rm aux}[f] = \Omega \int_{\mathbb{R}} \left\{ \tfrac{1}{2}|f'|^2 + \tfrac{1}{2}\alpha^2 y^2 f^2 + \omega^{-1/2} f^4 \right\} \mathrm{d}y$$

All three terms are of the same order of magnitude, and we cannot ignore the gradient term as in the TF approximation!

Without the last term the minimizer is the gaussian

$$f_{\rm osc}(y) = \pi^{-1/4}\alpha^{1/4}\exp\{-\tfrac{1}{2}\alpha y^2\}.$$

Hence the minimizer is approximately gaussian if ω is large.

The length scale is $\Omega^{-1/2} = \varepsilon^2\omega^{-1/2}$ that is, for large ω, much larger than the thickness of the TF annulus $\sim (\varepsilon\Omega)^{-2/3} = \varepsilon^2\omega^{-2/3}$.

The Gaussian Density Profile (cont.)

For large ω

$$g_{\text{gv}}(x) \approx g_{\text{osc}}(x) = \Omega^{1/4} f_{\text{osc}}(\Omega^{1/2}(x-1)).$$

In particular, the integral of g_{gv}^2 over an annulus

$$\mathcal{A}_\eta = \{\mathbf{x} : |x-1| \leq \Omega^{-1/2}\eta\}$$

tends to 1 if and only if $\eta \to \infty$. (Even though $\Omega^{-1/2}\eta \to 0$.) The same holds for the density $|\psi^{\text{GP}}|^2$ so any such annulus contains the bulk of the density if $\eta \to \infty$.

For the proof of absence of vortices in $\mathcal{A}_\eta$ it is, however, necessary to restrict η. In fact, we prove that the annulus is vortex free if $\eta = O(|\log \varepsilon|^{1/2})$.

Energy Estimates and Absence of Vortices

THEOREM [Energy in the giant vortex regime]

There is a constant $0 < \omega_{c3} < \infty$ *such that for* $\Omega = \omega\,\varepsilon^{-4}$ *with* $\omega > \omega_{c3}$ *the ground state energy is*

$$E^{\mathrm{GP}} = E^{\mathrm{gv}} + O(|\log\varepsilon|^{9/2}).$$

THEOREM [Absence of vortices in the bulk]

There is a constant $c > 0$ *such that for* $\Omega = \omega\,\varepsilon^{-4}$ *with* $\omega > \omega_{c3}$ *and* ε *sufficiently small the minimizer* ψ^{GP} *is free of zeros in the annulus*

$$\mathcal{A} = \{\mathbf{x} : \; |1 - x| \leq c\Omega^{-1/2}|\log\varepsilon|^{1/2}\}.$$

On the proofs

The main issue is the lower bound. Restrict $\mathcal{E}^{\rm gv}$ to the annulus $\mathcal{A}$, obtaining a positive minimizer g. Define $u(\mathbf{x})$ on the annulus by writing

$$\psi^{\rm GP}(\mathbf{x}) = g(x)u(\mathbf{x})\exp(\mathrm{i}\Omega\vartheta).$$

The function u contains all possible zeros of $\psi^{\rm GP}$ in the annulus. The variational equation for g leads to the lower bound

$$E^{\rm GP} \geq E^{\rm gv}_{\mathcal{A}} + \mathcal{E}_{\mathcal{A}}[u]$$

with a functional of Ginzburg-Landau type with g^2 as weight

$$\mathcal{E}_{\mathcal{A}}[u] = \int_{\mathcal{A}} g^2 \left\{ \tfrac{1}{2}|\nabla u|^2 - \mathbf{B}\cdot\mathbf{J}(u) + \varepsilon^{-2} g^2 (1-|u|^2)^2 \right\}$$

where $\mathbf{B} = \Omega(x - x^{-1})\,\mathbf{e}_\vartheta$ and $\mathbf{J}(u) = \frac{\mathrm{i}}{2}(u\nabla u^* - u^*\nabla u)$.

One needs to estimate the negative term $-\int g^2\mathbf{B}\cdot\mathbf{J}(u)$.

On the proofs (cont.)

Write $g^2\mathbf{B} = \nabla^\perp F$ with $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ and a potential function F. Integartion by parts and estimates of F (key point!) give

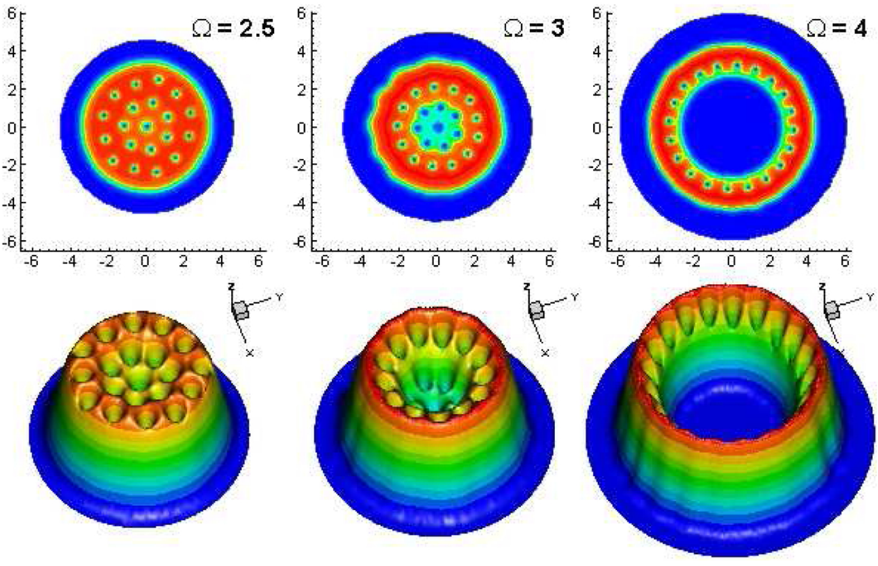
$$\int_{\mathcal{A}} g^2 \left\{ \tfrac{1}{2}|\nabla u|^2 - \mathbf{B}\cdot\mathbf{J}(u) \right\} \geq -C\omega^2 |\log\varepsilon|^{3/2}$$

leading to the lower energy bound.

A consequence of this bound, combined with the variational upper bound $E^{\mathrm{gv}}_{\mathcal{A}} \leq 0$ is an upper bound on the interaction term for large ω:

$$\int_{\mathcal{A}} \varepsilon^{-2} g^4 (1-|u|^2)^2 \leq C\omega^2 |\log\varepsilon|^{3/2}$$

Together with the upper bound and the Gagliardo-Nirenberg inequality this implies that u must be close to 1, in particular free of zeros.



Ω = 2.5
Ω = 3
Ω = 4

Comparison with the 'flat' case

The flat case, $s = \infty$, differs from the case $s < \infty$ in several respects:

- The GV transition takes place at $\Omega \sim \varepsilon^{-2}|\log \varepsilon|^{-1}$ rather than $\Omega \sim \varepsilon^{-4}$
- The density profile in the GV regime is of TF type
- The 'last' vortices before the GV transition have size $\sim \varepsilon^{3/2}$ that is much smaller than the thickness of the annulus $\sim \varepsilon|\log \varepsilon|$. For $s < \infty$ the size of vortices, $\sim \varepsilon^2$ and the size of the annulus, $\sim \varepsilon^2|\log \varepsilon|^{1/2}$, are almost comparable.

The techniques of proof in the two cases are also by necessity different: While *vortex ball constructions* and subsequent *jacobian estimates* for the potential function are applicable for the 'small' vortices in a 'flat' trap they are useless for $s < \infty$ and new ideas are required.

Circulation and symmetry breaking

Below the onset of the second vortex the GP minimizer has rotationally symmetric density, but a vortex lattice clearly breaks the symmetry. On the other hand, the giant vortex variational ansatz, that gives an excellent approximation to the energy for $\omega > \bar{\omega}$, is an eigenfunction of angular momentum. A true minimizer does not have this property, however:

THEOREM (Circulation and rotational symmetry breaking)

In the giant vortex regime $\omega > \bar{\omega}$ the circulation of any GP minimizer is $2\pi\,\Omega + O(\omega\,|\log\varepsilon|^{9/4})$, but no minimizer is an eigenfunction of angular momentum.

These result holds both for $s < \infty$ and $s = \infty$.

Summary

The study of the GP equation for dilute Bose gases in rapidly rotating, anharmonic traps reveals a surprising rich landscape, both from the mathematical and physical point of view. Detailed analysis can be carried out in an asymptotic regime where both the coupling constant and the rotational speed are large.

Among the results found are:

- Energy asymptotics corresponding to a distribution of vorticity in a lattice of vortices for $\Omega_{c1} \ll \Omega \ll \Omega_{c3}$.
- Emergence of a 'hole' with strongly depleted density above a critical rotation speed $\Omega_{c2} \sim \varepsilon^{-1}$.
- Transition to a 'giant vortex' state above $\Omega_{c3} \sim \varepsilon^{-4}$ where the vortex lattice disappears from the bulk and all vorticity resides in the 'hole', creating a macroscopic circulation in the bulk.
- Breaking of rotational symmetry, also in the giant vortex regime.

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