Spectral properties of the QREM

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Preface to the promotional trip which you booked . . .

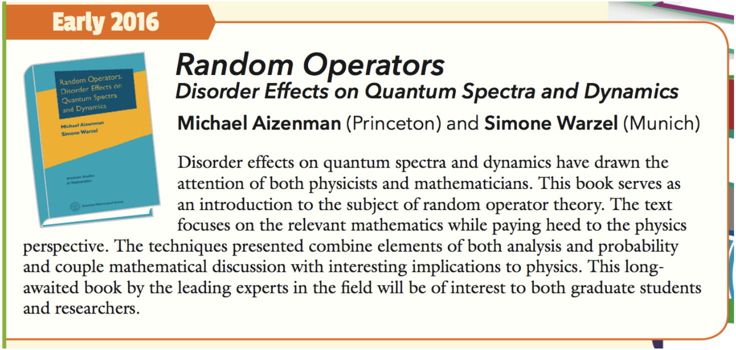
These four lectures are meant as an invitation to the mathematically largely unexplored playing field of quantum spin glasses. The QREM is the simplest mean-field quantum spin glass and we will explore its low-energy properties – in particular, the quantum phase transition at its ground state and questions this connects to.

Selected references to these lectures:

- A. Bovier. Statistical Mechanics of Disordered Systems: A Mathematical Perspective. Cambridge University Press, 2006.
- M. Ledoux. The Concentration of Measure Phenomenon. AMS 2001.
- T. Jörg, F. Krzakala, J. Kurchan, A. C. Maggs, Simple Glass Models and Their Quantum Annealing. Phys. Rev. Lett. 101, 147204 (2008).
- S. Warzel. Low-energy properties and the ground-state phase transition in the QREM. In preparation.
- E. Farhi, J. Goldstone, S. Gutmann, and D. Nagaj . How To Make the Quantum Adiabatic Algorithm Fail. Int. J. Quant. Inf. 6, 503-516 (2008).
- E. Fahri, J. Goldstone, D. Gosset, S. Gutmann, and P. Shor. Unstructured Randomness, Small Gaps and Localization. J. Quant. Inf. Comp. 11, 840-854 (2011).
- J. Adame, S. Warzel, Exponential vanishing of the ground-state gap of the QREM via adiabatic quantum computing. arXiv:1412.8342.

Preface to the promotional trip which you booked . . .

More generally, mathematical analysis of disordered quantum systems includes the theory of random matrices and random operators. Background on the latter can be found in:



Outline

I. **Motivations and a common theme**
II. **Anderson Model on the hypercube**
III. **More on adiabatic quantum computing**
IV. **Interesting directions to explore**

I. Motivations and a common theme

Schuster/Eigner ’77, ...

Simple organism, whose genetic information is encoded in genotyps of length N, i.e., in a binary vector from $\{0,1\}^N$.

- Total number of genotyps: 2^N
- Mean number of genotype α in sample: $n_\alpha \in \mathbb{R}$

Evolution:

$$\frac{d}{dt} n_\alpha(t) = \sum_{\beta=1}^{2^N} H_{\alpha\beta}\, n_\beta(t) - n_\alpha(t)\, J(t)\,, \ \alpha \in \{1, \ldots, 2^N\}\,.$$

- $H_{\alpha\beta}$ transition rate (by mutation and selection) trom type β zu α.
- $J(t)$ death rate, i.e. due to overpopulation

  $J(t) = J_0 \sum_{\alpha=1}^{2^N} n_\alpha(t)$, $J_0 > 0$.

Mutation and selection model

Baake/Wagner '01, ...

$$H_{\alpha\beta} = U_\alpha \delta_{\alpha\beta} + \kappa^{-1} \Delta_{\alpha\beta}$$

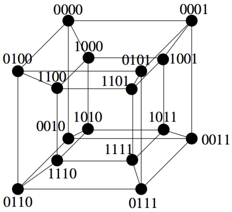
- Graph-Laplacian Δ on Hamming cube $\{0,1\}^N$, i.e.

$$(\Delta n)_\alpha = \sum_{\beta\sim\alpha} n_\beta - N\, n_\alpha$$

(Excursion: properties of Δ)

- Mutation rate $\kappa^{-1} > 0$
- Growth rate, i.e., 'fitness' of genotype α:

$$U_\alpha = \sum_\beta H_{\alpha\beta}$$



Hammingcube in case $N = 4$

Rough fitnesss landscape: $\{U_\alpha\}_{\alpha\in\{1,\dots,2^N\}}$ i.i.d. random variables

Evolution in this model

$$\frac{d}{dt} n_\alpha(t) = \sum_{\beta=1}^{2^N} H_{\alpha\beta}\, n_\beta(t) - n_\alpha(t)\, J(t)$$

Basic question:

relative number of genotypes for $t \to \infty$?

Trick: $r_\alpha(t) := n_\alpha(t) \exp\left(\int_0^t J(s)\, ds\right)$ solves $\frac{d}{dt} r_\alpha(t) = \sum_\beta H_{\alpha\beta} r_\beta(t)$ s.t.

$$r_\alpha(t) = \left(e^{tH} r(0)\right)_\alpha .$$

Evolution in this model

$$\boxed{\frac{d}{dt} n_\alpha(t) = \sum_{\beta=1}^{2^N} H_{\alpha\beta}\, n_\beta(t) - n_\alpha(t)\, J(t)}$$

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$$\boxed{r_\alpha(t) = \left(e^{tH} r(0)\right)_\alpha \ .}$$

Summary:

- Largest eigenvalue $\lambda_1 > \lambda_2 > \dots$ of $H = (H_{\alpha\beta})$ and eigenvector ψ_1 dominate the long-time behavior:

$$r(t) \approx e^{t\lambda_1} \langle \psi_1 \, , \, r(0) \rangle \, \psi_1 \qquad (t \to \infty) \, .$$

- ψ_1 determines relative number of genotypes for $t \to \infty$:
  - If ψ_1 sharpely **localized** in one entry, then this genotype dominates after evolution.
  - If ψ_1 **delocalized**, the evolution does not create a dominant genotype.

Excursion: Laplacian on the Hammingcube

Laplacian on $\ell^2(\{-1,1\}^N)$:

$$(-\Delta\psi)(\sigma) := N\psi(\sigma) - \sum_{j=1}^{N} \psi(F_j\sigma)$$

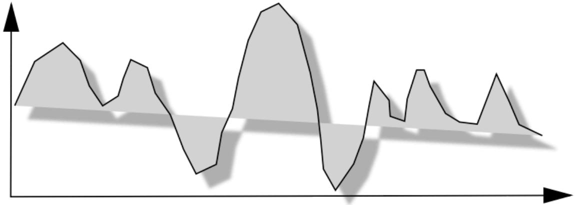
- 'Spin' Flip Operator: $F_j\sigma = (\sigma_1, \dots, -\sigma_j, \dots, \sigma_N)$

  Spin Flip Operators commute for different j. Hence Laplacian is a direct sum of N commuting operators!

- Eigenvalues: $2|A|, \quad A \subset \{1, \dots, N\}$ Degeneracy: $\binom{N}{|A|}$
- Normalized eigenvektors: $f_A(\sigma) = \frac{1}{\sqrt{2}^N} \prod_{j \in A} \sigma_j$

Problem: Find minimum in a complex energy landscape $(M = 2^N)$

$$u : \{1, \dots, M\} \to \mathbb{R}$$



Classical algorithms ... succeed in $\mathcal{O}(M)$ steps.

Idea for speed-up: *Quantum Computation by Adiabatic Evolution*

E. Farhi, J. Goldstone, S. Gutmann, M. Sipser:

arXiv:quant-ph/0001106

The energy landscape $u : \{1, \ldots, M\} \to \mathbb{R}$ defines a 'Problem-Hamiltonian':

$$U = \mathrm{diag}\,(u(1), \ldots, u(M)) \; .$$

Consider the quantum-time evolution on $\mathbb{C}^M$ generated by

$$\boxed{h(s) := h_D(s) + c(s)\, U}$$

where

- $c : [0,1] \to [0,1]$ is a continuous coupling, $c(0) = 0$, $c(1) = 1$.
- 'Driving-Hamiltonian' $h_D : [0,1] \to \mathrm{Herm}(\mathbb{C}^{M \times M})$ is continuous, $h_D(1) = 0$

Initial value problem: $$\boxed{i \frac{d}{dt} \psi(t) = h(t/T)\psi(t) \qquad \psi(0) \in \mathbb{C}^M \, .}$$

Hope:

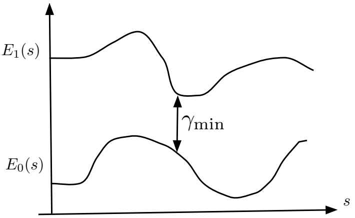
Interpolate between known ground-state of $h(0) = h_D(0)$ and $h(1) = U$ in time T.

Rule of thumb:

Time it takes $T \approx c \,/\, \gamma_{min}^2$

where γ_{min} is the lowest spectral gap of $h(s)$ minimized wrt s.

...more later in Part III.



Summary: Scaling of the lowest spectral gap of $h(s)$ with N decides, whether the problem remains 'hard' on a quantum computer as well.

Guiding theme: Anderson Modell on the Hamming cube

$$H = -\Delta + \kappa\, U \quad \text{on } \ell^2(\{-1,1\}^N)$$

- $\kappa \geq 0$ disorder parameter
- $U(\sigma)$ i.i.d. random variables

interesting regime: $\|U\|_\infty \approx \mathcal{O}(N)$

In these lectures: $$U(\sigma) = \sqrt{N}\, g(\sigma)$$

with $g(\sigma)$ i.i.d. Gaussian random variables.

Physics literature: **Quantum Random Energy Model**

Reference: Bovier, Statistical Mechanics of Disordered Systems, CUP 2006.

Let $u_N(x)$ für $x > -\frac{\ln N}{\ln 2}$ be unique solution of $\frac{2^N}{\sqrt{2\pi}} \int_{\sqrt{N}\, u_N(x)}^{\infty} e^{-y^2/2} dy = e^{-x}$.

Then: $u_N(x) = \frac{1}{\kappa_c} + \frac{\kappa_c}{N}\left(x - \frac{\ln\left(4\pi \ln 2^N\right)}{2}\right) + o\left(\frac{1}{N^{\frac{3}{2}}}\right)$ mit $\boxed{\kappa_c = \frac{1}{\sqrt{2\ln 2}}}$.

Lemma (Extremal value statistics I)

For all $x > -\frac{\ln N}{\ln 2}$*:*

$$\mathbb{P}\left(\min U \geq -N\, u_N(x)\right) = \left(1 - 2^{-N} e^{-x}\right)^{2^N} \to e^{-e^{-x}}\,.$$

E.g. for $x = \varepsilon N/\kappa_c^2$ mit $\varepsilon > 0$ with **asympt. (exp.) full probability**:

$$\boxed{\min U \geq -N u_N(\varepsilon N/\kappa_c^2) \geq -\kappa_c^{-1}\,(1+\varepsilon)\, N\,,}$$

$$\boxed{\|U\|_\infty \leq \kappa_c^{-1}\,(1+\varepsilon)\, N\,.}$$

More is known, e.g.:

Extremal values $U_{\min} =: U_0 \leq U_1 \leq \dots$ consitute a **Poisson process** with exponentially increasing intensity:

Lemma (Extremal value statistics II)

The point process

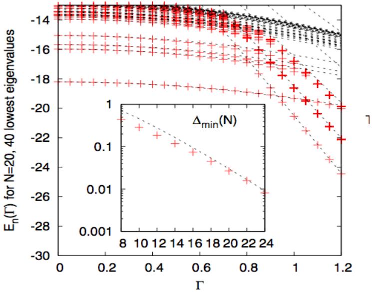
$$\sum_{\sigma} \delta_{u_N^{-1}(U(\sigma)/N)}$$

converges weakly for $N \to \infty$ *to the Poisson process with intensity measure* $e^{-x} dx$.

Prediction for QREM

Low-energy spectrum:

Jörg/Krzakala/Kurchan/Maggs,
PRL 101, 147204 (2008)



$$\Gamma = \kappa^{-1}, \quad \widehat{H} = \kappa^{-1}(\tfrac{1}{2}(-\Delta - N) + \kappa\, U)$$

Quantum phase transition of the ground-state at $\kappa_c = \frac{1}{\sqrt{2\ln 2}}$:

$\kappa < \kappa_c$: delocalized ground state and low-energy states

$\kappa > \kappa_c$: ground-state is localized mostly in lowest value of U.

$\kappa = \kappa_c$: $\gamma_{min} = E_1 - E_0$ is typically exponentially small N

Prediction for QREM

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Jörg/Krzakala/Kurchan/Maggs,
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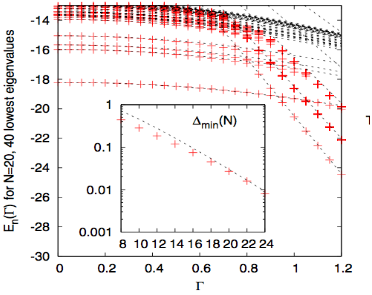


$$\Gamma = \kappa^{-1}, \quad \widehat{H} = \kappa^{-1}(\tfrac{1}{2}(-\Delta - N) + \kappa\, U)$$

Back of the envelop calculations for the ground state: *1st perturbation theory*

- Fate of localized states: $\langle \delta_\sigma , H\delta_\sigma \rangle = N + \kappa\, U(\sigma).$
- Fate of delocalized states:

$$\langle f_A \, , U \, f_A \rangle = \frac{1}{2^N} \sum_\sigma U(\sigma) = \mathcal{O}(\sqrt{N}\, 2^{-N/2}) \, ,$$

II. Anderson Model on the Hamming cube

Spectral properties near the ground state and some math methods to take home...

QREM and predictions

$$H = -\Delta + \kappa\, U \quad \text{on } \ell^2(\{-1,1\}^N)$$

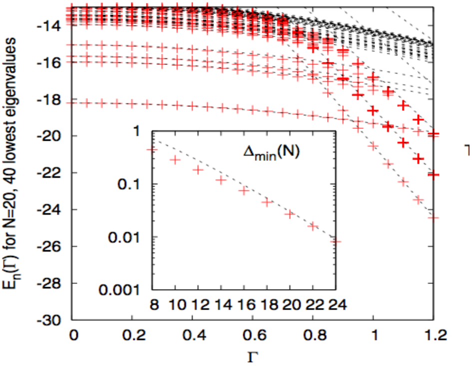
- $\kappa \geq 0$ disorder strength
- $U(\sigma) = \sqrt{N}\, g(\sigma)$, where $g(\sigma)$ i.i.d. standard Gaussian random variables.

Low-energy spectrum:

Jörg/Krzakala/Kurchan/Maggs, PRL 101, 147204 (2008)

$\Gamma = \kappa^{-1}$,

$\widehat{H} = \kappa^{-1}(\frac{1}{2}(-\Delta - N) + \kappa\, U)$



Critical disorder parameter: $\kappa_c = \frac{1}{\sqrt{2\ln 2}}$.

II.1. Spectral properties in case $\kappa < \kappa_c$

Theorem ($\kappa < \kappa_c$)

In case $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$, s.t. with asympt. (exp.) full probability and all $N \geq N_\varepsilon$, the eigenvalues E of H with $E \leq \left(1 - \frac{\kappa}{\kappa_c} - 3\varepsilon\right) N$ are found in intervals centered at

$$\boxed{2n - \frac{\kappa^2}{1 - \frac{2n}{N}}\,, \qquad n \in \{0, 1, \dots\}}\,,$$

with radius $\mathcal{O}\left(\sqrt{\frac{\ln N}{N}}\right)$.

There are exactly $\binom{N}{n}$ eigenvalues in each ball and the corresponding normalized eigenfunctions ψ_E are delocalized:

$$\boxed{\|\psi_E\|_\infty^2 \;\leq\; 2^{-N}\, e^{\Gamma\left(\frac{x_E}{2}\right) N}}$$

where $\Gamma(x) := -x \ln x - (1 - x)\ln(1 - x)$ and $x_E := \frac{E}{N} - \frac{\kappa U_{\min}}{N}$.

Step 1: *Hypercontractivity of the Laplacian*

Integral kernel of semigroup: $\langle \delta_\sigma \,, e^{t\Delta}\, \delta_{\sigma'} \rangle = e^{-tN} \cosh(t)^N \tanh(t)^{d(\sigma,\sigma')}$.

Hypercontractivity:

$$\left\| e^{t\Delta} \right\|_{2\to\infty} = \sup_{\|\psi\|=1} \sup_{\sigma} \left| \langle \delta_\sigma \,, e^{t\Delta}\, \psi \rangle \right| \leq \left| \langle \delta_\sigma \,, e^{2t\Delta}\, \delta_\sigma \rangle \right|^{1/2} = \left(\frac{1+e^{-2t}}{2} \right)^{N/2} .$$

Estimate of eigenfunctions:

Lemma (Delocalization)

The ℓ^2-normalized eigenfunctions ψ_E of $H = -\Delta + \kappa U$ corresponding to eigenvalues $E \leq 2N + \kappa U_{\min}$ satisfy for all σ: $|\psi_E(\sigma)|^2 \leq 2^{-N}\, e^{\Gamma(\frac{x_E}{2})N}$, *with* $x_E := \frac{E}{N} - \frac{\kappa U_{\min}}{N} \leq 2$.

Proof:

$$|\psi_E(\sigma)|^2 \leq \langle \delta_\sigma \,, P_{(-\infty,E]}(H)\, \delta_\sigma \rangle \leq \inf_{t>0} e^{tE} \langle \delta_\sigma \,, e^{-tH}\, \delta_\sigma \rangle$$

$$\leq \inf_{t>0} e^{t(E-\kappa U_{\min})} \langle \delta_\sigma \,, e^{t\Delta}\, \delta_\sigma \rangle = 2^{-N}\, e^{\Gamma(\frac{x_E}{2})N} . \quad \square$$

Step 2: *Concentration of measure*

Spectral projection onto center of the band and its complement: $\delta > 0$.

$$Q_\delta := 1 - P_\delta \; := \; 1_{[N(1-\delta), N(1+\delta)]}(-\Delta) \, .$$

Chernoff estimate $\sum_{n=0}^{(N-a)/2} \binom{N}{n} \leq 2^N \exp\left(-a^2/2N\right)$, $a \in (0, N)$, folgt:

$$\boxed{\dim P_\delta \leq 2^{N+1} \, e^{-\delta^2 N/2} \, .}$$

Lemma (Concentarion of measure I)

Consider $\{W(\sigma)\}$ i.i.d. r.v., which are bouned, $\|W\|_\infty \leq 1$. Then for all $\delta > 0$, $\lambda > 0$:

$$\mathbb{P}\left(\big| \|P_\delta W P_\delta\| - \mathbb{E}\left[\|P_\delta W P_\delta\|\right] \big| > \lambda \sqrt{\frac{\dim P_\delta}{2^N}} \right) \leq \; C \, e^{-c\lambda^2} \, ,$$

where $C, c \in (0, \infty)$ are numerical constants.

Talagrand, Publ. Math. IHES 81, 73-205 (1995)

Lemma (Talagrand inequality)

Let $K > 0$ and $X_1, \dots, X_n$ independent complex-valued. r.v.'s, which are bounded by K. Let $F : \mathbb{C}^n \to \mathbb{R}$ be a 1-Lipschitz convex function. Then:

$$\mathbb{P}\left(|F(X) - \mathbb{E}\left[F(X)\right]| \geq \lambda K\right) \leq C\, e^{-c\lambda^2},$$

where $C, c \in (0, \infty)$ are numerical constants.

Application $F : \mathbb{R}^{\mathcal{Q}_N} \to \mathbb{R}$, $\quad F(W) := \|P_\delta W P_\delta\|$:

Buondedness and convexity are evident. (Triangle inequality).

Lipschitz continuity: Pick $\psi \in P_\delta \ell^2(\mathcal{Q}_N)$ normalized and $F(W) = \langle \psi, W\psi\rangle$.

$$F(W) - F(W') \leq \langle \psi, W\psi\rangle - \langle \psi, W'\psi\rangle \leq \|W - W'\|_2 \|\psi\|_\infty$$

$$\leq \|W - W'\|_2 \sqrt{\langle \delta_\sigma, P_\delta \delta_\sigma\rangle}\, \|\psi\|_2 = \|W - W'\|_2 \sqrt{\frac{\dim P_\delta}{2^N}}\,.$$

Proof of Talagrand's inequality in simplified Gaussian set-up[1]

Wlog. $\mathbb{E}[F(X)] = 0$ and F smooth — after Maurier, Pisier

Estimate on exponential moment is enough: $\mathbb{E}\left[e^{tF(X)}\right] \leq e^{ct^2}$ for all $t > 0$.

Inserting an independent copy Y of X results in an upper bound by Jensen's inequality:

$$\mathbb{E}\left[e^{tF(X)}\right] \leq \mathbb{E}\left[e^{t(F(X)-F(Y))}\right] = \mathbb{E}\left[\exp\left(t\int_0^{\pi/2} \frac{d}{d\theta}F(X\cos\theta + Y\sin\theta)d\theta\right)\right]$$

$$\leq \frac{2}{\pi}\int_0^{\pi/2} \mathbb{E}\left[\exp\left(t\frac{\pi}{2}(\nabla F)(X\cos\theta + Y\sin\theta)\cdot(-X\sin\theta + Y\cos\theta)\right)\right] d\theta$$

Conditioning on Gaussian rv's $X\cos\theta + Y\sin\theta$, the rv's $-X\sin\theta + Y\cos\theta$ are independent and Gaussian! Integrating out the latter, and using $\|\nabla F\| \leq 1$ yields the result.

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[1] For a complete proof, see also: Tao, Topics in random matrix theory, AMS 2012

Methods of proof – delocalization regime

Lemma (Concentarion of measure II)

Consider $\{W(\sigma)\}_{\sigma \in \mathcal{Q}_N}$ *i.i.d. r.v. with the following properties:*

1. *centered,* $\mathbb{E}\left[W(\sigma)\right] = 0$,
2. *bounded variance,* $\mathbb{E}\left[W(\sigma)^2\right] \leq 1$, *and*
3. *bounded* $\|W\|_\infty \leq p_N$, *where* p_N *is a polynomial in* N.

Then for all $\delta > 0$ *and all* N *with* $p_N^2 \exp(-\delta^2 N/2) \leq 1$ *(i.e. all* N *sufficiently large):*

$$\mathbb{E}\left[\|P_\delta W P_\delta\|\right] \leq 2N\, e^{-\delta^2 N/4}\,.$$

Upper bound: $\mathbb{E}\left[\|P_\delta W P_\delta\|\right] \leq \left(\mathbb{E}\left[\mathrm{Tr}(P_\delta W P_\delta)^{2N}\right]\right)^{1/2N}$

Estimate Schatten norms by **method of moments** . . .

Application: For any $\varepsilon > 0$ with asympt. (exp.) full probability:

$$\kappa_c \|U\|_\infty \le (1+\varepsilon)N$$

Effective truncation of the potential, s.t. for all $\varepsilon \in (0,1)$ with **asympt. (exp.) full probability** for all $\delta > 0$ and all N sufficiently large (only depending on ε):

- $\kappa_c \|P_\delta U P_\delta\| \le 4N^{3/2} e^{-\delta^2 N/4}$.

  Concentration I: $W = \kappa_c \frac{U}{(1+\varepsilon)N}$, $\lambda = \sqrt{N}/2$

  Concentration II: $W = U/\sqrt{N}$.

- $\kappa_c^2 \|P_\delta (U^2 - N) P_\delta\| \le 8N^2 e^{-\delta^2 N/4}$.

  Concentration I: $W = \kappa_c^2 \frac{U^2 - N}{(1+\varepsilon)^2 N^2}$, $\lambda = \sqrt{N}/8$

  Concentration II: $W = (U^2 - N)/N$.

- $\kappa_c^4 \|P_\delta (U^4 - cN^2) P_\delta\| \le 8N^3 e^{-\delta^2 N/4}$.

  Concentration I: $W = \kappa_c^2 \frac{U^4 - cN^2}{(1+\varepsilon)^4 N^4}$, $\lambda = \sqrt{N}/8$

  Concentration II: $W = (U^4 - cN^2)/N^2$.

Step 3: *Rigorous perturbation theory*

Lemma (Krein-Feshbach-Schur)

For all $E < \inf \sigma(QHQ)$ and $R(E) := (Q(H-E)Q)^{-1}$ (on the subspace corresponding to Q):

1. $E \in \sigma(H)$ *iff* $0 \in \sigma\,(PHP - E - PHR(E)HP)$.
2. $H\psi = E\psi$ *with* $\psi = (\psi_1, \psi_2)^T$ *iff:*

$$\begin{aligned} &(PHP - E - PHR(E)HP)\,\psi_1 = 0 \\ \textit{und} \quad &\psi_2 = -R(E)QHP\psi_1\,. \end{aligned}$$

Proof idea of theorem in case $\kappa < \kappa_c$:

- Lower bound on $Q_\delta H Q_\delta$ on $Q_\delta \ell^2(\mathcal{Q}_N)$ with asympt. (exp.) full probability:

$$-Q_\delta \Delta Q_\delta + \kappa\, Q_\delta U Q_\delta \geq (1-\delta)\, N - (1+\varepsilon)\, N \frac{\kappa}{\kappa_c} \geq \left(1 - \frac{\kappa}{\kappa_c} - 2\varepsilon\right) N$$

  where $0 < \delta \leq \varepsilon$ and N is sufficiently large .

  Hence for all $E \leq \left(1 - \frac{\kappa}{\kappa_c} - 3\varepsilon\right) N$: $\quad \|R_\delta(E)\| \leq \frac{1}{\varepsilon N}$.

- resolvent equation:

$$R_\delta(E) - \frac{Q_\delta}{N-E} = R_\delta(E)\,(NQ_\delta - Q_\delta H Q_\delta)\,\frac{Q_\delta}{N-E}\,.$$

  and hence:

$$P_\delta U R_\delta(E) U P_\delta - P_\delta \frac{N}{N-E}$$
$$= P_\delta U R_\delta(E)\,(NQ_\delta - Q_\delta H Q_\delta)\, U P_\delta \frac{1}{N-E} + P_\delta\,(U Q_\delta U - N)\, P_\delta \frac{1}{N-E}\,.$$

Beweisidee des Theorems $\kappa < \kappa_c$:

From concentration of measure estimates:

$$\|P_\delta\,(UQ_\delta U - N)\,P_\delta\| \le c\,N^2\,e^{-\delta^2 N/4}\,,$$

$$\|P_\delta UR_\delta(E)\,(NQ_\delta + Q_\delta \Delta Q_\delta)\,\frac{1}{N-E}UP_\delta\|$$

$$\le c\,\delta\|R_\delta(E)\|\|P_\delta U^2 P_\delta\| \le c\,\frac{\delta}{\varepsilon}\left(1 + N\,e^{-\delta^2 N/4}\right)\,,$$

$$\|\|P_\delta UR_\delta(E)Q_\delta UQ_\delta\frac{1}{N-E}UP_\delta\| \le \frac{\|R_\delta(E)\|}{N-E}\|UP_\delta\|\|UQ_\delta UP_\delta\|$$

$$\le \frac{c}{\varepsilon N^2}\left(N + N^2\,e^{-\delta^2 N/4}\right)^{\frac{1}{2}}\left(N^2 + N^3\,e^{-\delta^2 N/4}\right)^{\frac{1}{2}}\,.$$

Choice of $\delta = \mathcal{O}\left(\sqrt{\frac{\ln N}{N}}\right)$.

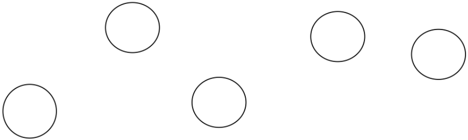
Insert into Krein-Feshbach-Schur formula . . . □

II. 3. Energies above the tips of the REM – some cherries from the pie

Idea: *Geometric decomposition of Hamming cube*

Eigenvalues below $E_\epsilon := \left(1 - \frac{\kappa}{\kappa_C} + \epsilon\right) N$ with $\epsilon > 0$ small, stem from **large negative deviations** of REM:

$$\boxed{X_\epsilon := \left\{\sigma \,|\, \kappa\, U(\sigma) < -\frac{\kappa}{\kappa_C} N + \epsilon N\right\}}$$



For $\epsilon > 0$ small enough $\gamma > 0$ and $0 < \nu < \frac{\kappa}{\kappa_C} - \epsilon$, s.t. with asmpt. (exp.) full probability:

- X_ϵ consists of isolated points, separated by $2\gamma N$ steps.
- On balls $B_\sigma := \{\sigma' \mid \mathrm{dist}(\sigma, \sigma') < \gamma N\}$ the potential $\kappa U(\sigma')$ is larger or equal to $-\nu N$ for all $\sigma' \neq \sigma$.

Energy adapted decomposition

Let $R := \mathcal{Q}_N \setminus \bigcup_{\sigma \in X_\epsilon} B_\sigma$ and

$$H_{B_\sigma} := 1_{B_\sigma} H 1_{B_\sigma}, \qquad \text{auf } \ell^2(B_\sigma),$$
$$H_R := 1_R H 1_R \qquad \text{auf } \ell^2(R).$$

Consider $\boxed{\widehat{H} := H - T := \bigoplus_{\sigma \in X_\epsilon} H_{B_\sigma} \oplus H_R}$.

Naive estimate: $\|T\| \le \sqrt{\gamma(1-\gamma)} N + o(N)$.

Energy adapted decomposition

Let $R := \mathcal{Q}_N \setminus \bigcup_{\sigma\in X_\epsilon} B_\sigma$ and

$$\begin{aligned} H_{B_\sigma} &:= 1_{B_\sigma} H 1_{B_\sigma}, & \text{auf } \ell^2(B_\sigma), \\ H_R &:= 1_R H 1_R & \text{auf } \ell^2(R). \end{aligned}$$

Consider $\boxed{\widehat{H} := H - T := \bigoplus_{\sigma\in X_\epsilon} H_{B_\sigma} \oplus H_R}.$

Better: $P_E := 1_{(-\infty,E]}(\widehat{H})$ mit $E = E_\epsilon + \|T\|$.

$$H = \widehat{H}_E + \widehat{T}_E \tag{1}$$

mit $\widehat{H}_E \equiv \begin{pmatrix} \widehat{H} P_E & 0 \\ 0 & \widehat{H} Q_E + Q_E T Q_E \end{pmatrix}$, und $\widehat{T}_E \equiv \begin{pmatrix} P_E T P_E & P_E T Q_E \\ Q_E T P_E & 0 \end{pmatrix}$.

Main message: $\|P_E T\| \leq e^{-c_\epsilon N}$.

- Spectrum of H_R below E_ε resembles H in delocalization regime.
- Spectrum of H_{B_σ} below E_ε can be computed explicitly . . . next page

Ground state of Laplacian on ball B_σ:

$$\boxed{E_0(-\Delta_{B_\sigma}) = N(1 - 2\sqrt{\gamma(1-\gamma)}) + o(N)}$$

Add rank-one perturbation $\kappa U(\sigma)$ plus moderate background potential:

- $\boxed{E_0(H_{B_\sigma}) \; = \; N + \kappa U(\sigma) + \mathcal{O}(1)}$
- The normalized ground state satisfies:

$$\sum_{\sigma' \in \partial B_\sigma} \left|\psi_0(\sigma')\right|^2 \; \leq \; e^{-L_\gamma N} \qquad \text{mit } L_\gamma > 0.$$

$$|\psi_0(\sigma)|^2 \geq 1 - \mathcal{O}(N^{-1})$$

- H_{B_σ} has a spectral gap $\mathcal{O}(N)$ above its ground state.

III. Adiabatic quantum computing and a gap estimate

Adiabatic quantum computing

Consider an energy landscape $u : \{1, \ldots, M\} \to \mathbb{R}$, which defines a 'Problem-Hamiltonian'

$$U = \mathrm{diag}\,(u(1), \ldots, u(M))$$

on $\mathbb{C}^M$. Consider the time-evolution generated by

$$\boxed{h(s) := h_D(s) + c(s)\,U}$$

on $\mathbb{C}^M$, where:

- $c : \mathbb{R} \to [0, 1]$ continuous coupling, $c(0) = 0$, $c(1) = 1$
- 'Driving-Hamiltonian' $h_D : \mathbb{R} \to \mathrm{Herm}(\mathbb{C}^{M \times M})$ continuous, $h_D(1) = 0$.

Initial value problem: $$\boxed{i\frac{d}{dt}\psi(t) = h(t/T)\psi(t) \qquad \psi(0) \in \mathbb{C}^M\,.}$$

Aim: Compute the mimimum location $j_0 \in \{1, \ldots, M\}$ of U!

One wants the quantum adiabatic algorithm to succeed not only for one energy landscape but for many. Consider the ensemble of **scambled** problems ...

III.1. Lower bounds on run time for scambled problem

Let $\pi \in \mathcal{S}_M$ be a permutation on M elements and define

$$U_\pi = \mathrm{diag}\left(u(\pi^{-1}(1)), \ldots, u(\pi^{-1}(M))\right)$$

and $h_\pi(t) := h_D(t/T) + c(t/T)U_\pi$, and denote by $\psi_\pi(t)$ the solution of the corresponding initial value problem starting from a **common** initial state $\psi(0)$.

Success probability for search after run-time T:

$$|\psi_\pi(\pi(j_0); T)|^2 =: b \quad (*)$$

Farhi, Goldstone, Gutmann, Nagaj Int. J. Quant. Inf., 503-516 (2008), 503-516

Theorem (Scambling theorem)

Let $\varepsilon > 0$ and suppose that (*) holds for a set of $\varepsilon M!$ permutations. Then for all M:

$$T \geq \frac{\varepsilon^2 bM - 4\varepsilon\sqrt{\frac{\varepsilon}{2}M}}{16\sigma_M(u)} \quad [=: T_M(b, \varepsilon)]$$

where $\sigma_M(u)^2 := \sum_{k=1}^M (u(k) - u(j_0))^2$ is assumed to be strictly positive.

Typically for large M: $\quad T \geq \mathcal{O}(\sqrt{M})$.

Timescale of Grover search algorithm!

III.2. A gap estimate via the run time

Adiabatic theorem: Jansen/Ruskai/Seiler, J. Math. Phys. 48, 102111 (2007)

Theorem (Kato)

Let $h(s)$, $s \in [0,1]$ be a family of twice continuously differentiable hermitian matrices on $\mathbb{C}^M$ with non-degenerate ground-state $\phi(s) \in \mathbb{C}^M$ and gap $\gamma(s) > 0$. Then the unique solution of the initial value problem

$$i \frac{d}{dt}\psi(t) = h(t/T)\,\psi(t)\,, \qquad \psi(0) = \phi(0)\,,$$

satisfies

$$\sqrt{1 - |\langle\psi(T), \phi(1)\rangle|^2} \leq \frac{1}{T}\left[\frac{1}{\gamma(0)^2}\|h'(0)\| + \frac{1}{\gamma(1)^2}\|h'(1)\| + \int_0^1 \frac{7}{\gamma(s)^3}\|h'(s)\|^2 + \frac{1}{\gamma(s)^2}\|h''(s)\|ds\right].$$

Generalization to infinite-dimensional Hilbertspaces

Application to the scrambled problem

Suppose $h_\pi(s) = H_D(sT) + c(sT)\, U_\pi$, $s \in [0,1]$, satisfies assumptions in adiabatic theorem with $\gamma_\pi(s) > 0$ gap above the ground-state and set

$$\gamma^{\#}_{\min,\pi} := \min_{s\in[0,1]} \left\{ \gamma_\pi(s)^2, \gamma_\pi(s)^3 \right\} .$$

Suppose that for some $C_M < \infty$ one has $\max\{\|h'_\pi(s)\|, \|h'_\pi(s)\|^2, \|h''_\pi(s)\|\} \leq C_M$ for all $s \in [0,1]$ and all π.

Application: scrambled QREM $M = 2^N$

- $h_\pi(s) = -(1-s)\Delta + sU_\pi$
- $\|h'_\pi(s)\| \leq \|\Delta\| + \kappa\|U\| \leq 2N + \frac{2}{\kappa_c}N$, $\|h''_\pi(s)\| = 0$
- $\sigma_M(u)^2 \leq M\, 2\|U\| \leq M\frac{4}{\kappa_c}N$.

Then by the adiabatic theorem:

$$\sqrt{1 - |\psi_\pi(\pi(j_0); T)|^2} \leq \frac{10 C_M}{T\, \gamma^{\#}_{\min,\pi}} . \quad (**)$$

for all $T > 0$.

A gap estimate via the run time

For $\varepsilon \in (0,1]$ take M large enough st $T_M(\frac{1}{2},\varepsilon) > 0$ and consider the set

$$\mathcal{G}_M(\varepsilon) := \left\{ \pi \mid \gamma^{\#}_{\min,\pi} \geq \frac{20\sqrt{2}C_M}{T_M(\frac{1}{2},\varepsilon)} \right\}$$

and

$$T = T_M(\tfrac{1}{2},\varepsilon)/2 \,.$$

By (**) for any $\pi \in \mathcal{G}_M(\varepsilon)$: $\quad |\psi_\pi(\pi(j_0);T)|^2 \geq \frac{1}{2}$.

The scambling theorem then implies that for all M large enough:

$$\boxed{|\mathcal{G}_M(\varepsilon)| < \varepsilon M! \,.}$$

Using permutation invariance of the REM distribution this yields with $\varepsilon = N^{-1}$:

Corollary

There is some constant $C < \infty$ such that for the QREM

$$\lim_{N\to\infty} \mathbb{P}\left(\gamma^{\#}_{\min} \leq CN^4\, 2^{-N/2}\right) = 1\,.$$

II.2. Proof of the lower bound on the run time

Let $k \in \{1, \ldots, M\}$ and $\boxed{\pi_k = \pi \circ \tau_{j_0,k}}$, where $\tau_{j,k} \in \mathcal{S}_M$ is the transposition of j and k.

Lemma ('scambling')

For all $T \geq 0$ and all k:

$$\frac{1}{M!} \sum_{\pi \in \mathcal{S}_M} \sum_{k=1}^{M} \|\psi_\pi(T) - \psi_{\pi_k}(T)\|^2 \leq 4\, T\, \sigma_M(u)\,.$$

Lemma ('geometry in Hilbert space')

Let $v_1, \ldots, v_L \in \mathbb{C}^M$ orthonormal vectors and $\psi_1, \ldots, \psi_L \in \mathbb{C}^M$ normalized vectors, which satisfy

for all $k \in \{1, \ldots, L\}$: $$|\langle v_k\,, \psi_k\rangle|^2 \geq b > 0\,.$$

Then for all normalized $\varphi \in \mathbb{C}^M$:

$$\sum_{k=1}^{L} \|\psi_k - \varphi\|^2 \geq b\, L - 2\sqrt{L}\,.$$

Proof of the lower bound in scrambling theorem

Fix $\pi \in \mathcal{S}_M$ and let

$$G_\pi := \left\{ k \in \{1, \dots, M\} \mid |\psi_{\pi_k}(\pi(k); T)|^2 \geq b \right\} .$$

Lemma 2 with $L = |G_\pi|$ und $v_k = e_{\pi(k)}$ with $k \in G_\pi$, and $\psi_k = \psi_{\pi_k}(T)$ and $\varphi = \psi_\pi(T)$ yields:

$$\sum_{k \in G_\pi} \|\psi_\pi(T) - \psi_{\pi_k}(T)\|^2 \geq b\,|G_\pi| - 2\sqrt{|G_\pi|}\,, \quad (*)$$

Estimate on $|G_\pi|$ starts from observation that by assumption:

$$\sum_{\pi \in \mathcal{S}_M} |G_\pi| = \sum_{k=1}^{M} \sum_{\pi \in \mathcal{S}_M} 1[|\psi_{\pi_k}(\pi_k(j_0)); T)|^2 \geq b] \geq \varepsilon M! M\,.$$

This implies: $\dfrac{1}{M!} \displaystyle\sum_{\pi \in \mathcal{S}_M} 1_{|G_\pi| \geq \frac{\varepsilon}{2} M} \geq \dfrac{\varepsilon}{2}.$

Apply Lemma 1 and use $(*)$:

$$T \geq \frac{1}{M!\, 4\sigma_M(u)} \sum_{\pi} \sum_{k \in G_\pi} \|\psi_\pi(T) - \psi_{\pi_k}(T)\|^2 \geq \frac{\varepsilon^2 bM - 4\varepsilon\sqrt{\frac{\varepsilon}{2} M}}{16\sigma_M(u)}\,.$$

Proof of Lemma 1

Just a calculation:

$$
\begin{aligned}
&\frac{d}{dt}\|\psi_\pi(t) - \psi_{\pi_k}(t)\|^2 \\
=& -2\frac{d}{dt}\,\mathrm{Re}\langle\psi_\pi(t), \psi_{\pi_k}(t)\rangle \\
=& -2\,\mathrm{Re}\left[i\langle H_\pi(t)\psi_\pi(t), \psi_{\pi_k}(t)\rangle - i\langle\psi_\pi(t), H_{\pi_k}(t)\psi_{\pi_k}(t)\rangle\right] \\
=& 2\,\mathrm{Im}\langle\psi_\pi(t), [H_\pi(t) - H_{\pi_k}(t)]\,\psi_{\pi_k}(t)\rangle \\
=& 2c(t)\ \mathrm{Im}\langle\psi_\pi(t), [U_\pi - U_{\pi_k}]\,\psi_{\pi_k}(t)\rangle \\
=& 2c(t)(u(k) - u(j_0))\,\mathrm{Im}\left[\langle\psi_\pi(t), e_{\pi(k)}\rangle\langle e_{\pi(k)}, \psi_{\pi_k}(t)\rangle - \langle\psi_\pi(t), e_{\pi_k(k)}\rangle\langle e_{\pi_k(k)}, \psi_{\pi_k}(t)\rangle\right] \\
\leq& 2\,|c(t)|\ |u(k) - u(j_0))|\ \left(\left|\langle\psi_\pi(t), e_{\pi(k)}\rangle\right| + \left|\langle e_{\pi_k(k)}, \psi_{\pi_k}(t)\rangle\right|\right)
\end{aligned}
$$

Proof of Lemma 1

Just a calculation:

$$\sum_{\pi\in\mathcal{S}_M}\frac{d}{dt}\|\psi_\pi(t)-\psi_{\pi_k}(t)\|^2\le 4|c(t)|\sum_{\pi\in\mathcal{S}_M}|u(k)-u(j_0))|\,\big|\langle\psi_\pi(t),e_{\pi(k)}\rangle\big|\ .$$

Cauchy-Schwarz for the k-sum:

$$\begin{aligned}\frac{d}{dt}\sum_{\pi\in\mathcal{S}_M}\sum_{k=1}^{M}\|\psi_\pi(t)-\psi_{\pi_k}(t)\|^2 &\le 4\sum_{\pi\in\mathcal{S}_M}\sqrt{\sum_{k=1}^{M}|u(k)-u(j_0))|^2}\sqrt{\sum_{k=1}^{M}\big|\langle\psi(t),e_{\pi(k)}\rangle\big|^2}\\ &= 4\,M!\,\sigma_M(u)\,.\end{aligned}$$

Integration using $\psi_\pi(0)=\psi(0)$ yields:

$$\begin{aligned}\sum_{\pi\in\mathcal{S}_M}\sum_{j=1}^{M}\|\psi_\pi(T)-\psi_{\pi_k}(T)\|^2 &= \int_0^T\sum_{\pi\in\mathcal{S}_M}\frac{d}{dt}\sum_{k=1}^{M}\|\psi_\pi(t)-\psi_{\pi_k}(t)\|^2 dt\\ &\le 4T\,M!\,\sigma_M(u)\,.\end{aligned}$$

□

Proof of Lemma 2

Complete $v_1, \dots, v_L$ to ONB of $\mathbb{C}^M$:

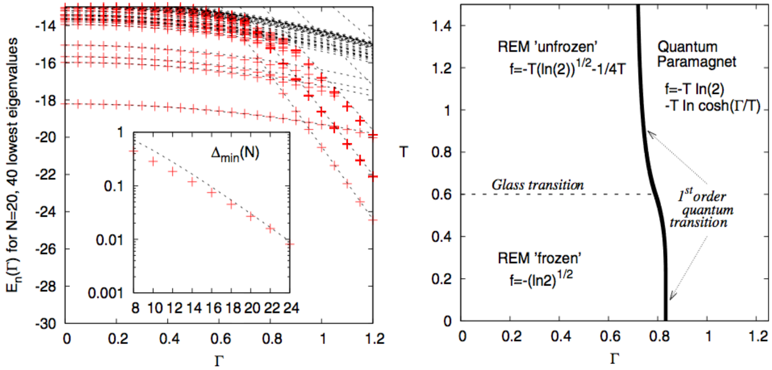
$$
\begin{aligned}
\sum_{k=1}^{L} \|\psi_k - \varphi\|^2 &= \sum_{k=1}^{L} \sum_{j=1}^{M} |\langle v_j, \psi_k \rangle - \langle v_j, \varphi \rangle|^2 \\
&\geq \sum_{k=1}^{L} |\langle v_k, \psi_k \rangle - \langle v_k, \varphi \rangle|^2 \\
&= \sum_{k=1}^{L} \Big[|\langle v_k, \psi_k \rangle|^2 + |\langle v_k, \varphi \rangle|^2 - 2\,\mathrm{Re} \langle \psi_k, v_k \rangle \langle v_k, \varphi \rangle \Big] \\
&\geq L\,b - 2 \sqrt{\sum_{k=1}^{L} |\langle v_k, \psi_k \rangle|^2} \sqrt{\sum_{k=1}^{L} |\langle v_k, \varphi \rangle|^2} \\
&\geq L\,b - 2\sqrt{L}.
\end{aligned}
$$

□

IV. Interesting directions to explore

- Thermodynamic phase transitions of QREM:

Jörg/Krzakala/Kurchan/Maggs, PRL 101, 147204 (2008)



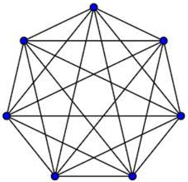
- **Resonant delocalization** of eigenfunctions closer to center of the band Bandmitte and in band-gaps of Laplacian.

  Numerical results: Laumann/Pal/Scardicchio '14

IV. Interesting directions to explore

Toy model for resonant delocalization in QREM:

Anderson Modell on the complete graph on M vertices



$$\boxed{H = -|\varphi_0\rangle\langle\varphi_0| + \kappa_M \, g}$$

with $\langle\varphi_0| = (1, 1, \ldots, 1)/\sqrt{M}$ und $\kappa_M := \frac{\lambda}{\sqrt{2 \log M}}$.

Surprising result M.Aizenman, M. Shamis, S.W. (2014)

Band of ℓ^1-delocalized states near $E \approx -1$ in case $\lambda > \sqrt{2}$.