Results in Contemporary Mathematical Physics

Conference in honor of Rafael Benguria

Book of abstracts

Index

Talenti's comparison theorem revisited

Mark S. Ashbaugh, Universidad de Concepción
and University of Missouri - Columbia

Abstract

We present some extensions of Talenti's comparison theorem for solutions of Poisson's equation on domains in Euclidean space under Dirichlet boundary conditions. We show how these results can be particularly useful in proving isoperimetric inequalities for eigenvalues.

The norm of the discrete Hilbert transform

Rodrigo Bañuelos, Purdue University

Abstract

The discrete Hilbert transform was introduced by David Hilbert at the beginning of the 20th century. It is the discrete analogue of the continuous Hilbert transform (conjugate function). In late 1923, M. Riesz proved the L^p boundedness, $1 < p < \infty$, of the continuous version, thereby solving a problem of considerable interest at the time. From this he deduced the same result for the discrete version. Shortly thereafter, E.C. Titchmarsh turned this around. He gave a direct proof of the boundedness of the discrete version and from it deduced the same for the continuous version. Further, he showed that the discrete and continuous versions have the same p-norms. The following year Titchmarsh pointed out that his argument for equality of the norms was incorrect. The problem of equality has been a conjecture since. In this talk we describe some tools from stochastic integration that lead to a proof of equality of the norms. The talk is based on work with Mateusz Kwaśnicki of Wroclaw University, Poland.

Weinstock inequality in higher dimensions

Dorin Bucur, Université de Savoie

Abstract

After an introduction to the question of the maximization of Steklov eigenvalues under a volume constraint in $\mathbb{R}^n$ (existence of optimal geometries, qualitative properties and numerical approximations) I will prove that the Weinstock inequality for the first nonzero Steklov eigenvalue holds in $\mathbb{R}^n$, for $n \geq 3$, in the class of convex sets with prescribed surface area. The key result is a sharp isoperimetric inequality involving simultaneously the surface area, the volume and the boundary momentum of convex sets.

The results are obtained in joint works with B. Bogosel and A. Giacomini, respectively E. Ferone, C. Nitsch and C. Trombetti.

A geometric stability result for Riesz-potentials

Almut Burchard, University of Toronto

Abstract

Riesz' rearrangement inequality implies that integral functionals (such as the Columb energy of a charge distribution) which are defined by a pair interaction potential (such as the Newton potential) which decreases with distance are maximized (under appropriate constraints) only by densities that are radially decreasing about some point. I will describe recent and ongoing work Greg Chambers on the stability of this inequality for the special case of the Riesz-potentials in n dimensions (given by the kernels $|x-y|^{-(n-\lambda)}$), for densities that are uniform on a set of given volume. For $1 < \lambda < n$, we bound the square of the symmetric difference of a set from a ball by the difference in energy of the corresponding uniform distributions.

Resonances under rank one perturbations

Rafael del Rio, UNAM Mexico

Abstract

This talk is about resonances generated by rank one perturbations of selfadjoint operators with eigenvalues embedded in the continuous spectrum. Instability of these eigenvalues is analyzed and almost exponential decay for the associated resonant states is exhibited. These results can be applied to Sturm-Liouville operators. Main tools are the Aronszajn-Donoghue theory for rank one perturbations, a reduction process of the resolvent based on Feshbach-Livsic formula, the Fermi golden rule and a careful analysis of the Fourier transform of quasi-Lorentzian functions. This is joint work with Olivier Bourget, Víctor H. Cortés and Claudio Fernández from the Pontificia Universidad Católica de Chile.

Spectral continuity for aperiodic quantum systems

Giuseppe De Nittis, Pontificia Universidad Católica de Chile

Abstract

How does the spectrum of a Schrödinger operator vary if the corresponding geometry and dynamics change? Is it possible to define approximations of the spectrum of such operators by defining approximations of the underlying structures? In this talk a positive answer is provided using the rather general setting of groupoid C^*-algebras. A characterization of the convergence of the spectra by the convergence of the underlying structures is proved. In order to do so, the concept of continuous field of groupoids is used. The approximation scheme is expressed through the tautological groupoid, which provides a sort of universal model for fields of groupoids. The use of the Hausdorff topology turns out to be fundamental in understanding why and how these approximations work. The construction presented during the talk is adapted to the case of Schrödinger operator with Delone potential (i.e. quasi-crystals).

The talk is based on a joint work with: S. Beckus and J. Bellissard.

Domains for Dirac operators and applications to min-max characterization of eigenvalues

Maria J. Esteban, CNRS & University Paris-Dauphine

Abstract

Several years ago, with J. Dolbeault and E. Séré, we proposed an abstract min-max method to characterize the eigenvalues of operators with gaps, and we applied it to the particular case of Dirac operators in relativistic quantum mechanics. Some important questioned remained open. Recently with M. Lewin and E. Séré we have studied the domains of critical Dirac-Coulomb operators, and proved a new density result that has allowed us to solve those open problems. In particular we know now the whole range of spaces where the min-max can be used, and this is very important for applications and for numerical computations.

On loops, cones, and stars: striving for the optimal shape

Pavel Exner, Czech Technical University

Abstract

One of the main contribution Rafael made to spectral geometry concerns the optimization problem, i.e. identifying the shape for which certain spectral quantities reach an extremal value. In this talk we address questions of that type for singular Schrödinger operators which can be formally written as $-\Delta - \alpha\delta(x-\Gamma)$. Various classes of the support Γ will be discussed: loops, manifolds homothetic to a sphere, cones and stars; we will consider also situations where the interaction is strongly singular, either of the δ' type or with the support of codimension two.

A Hardy-Lieb-Thirring inequality for fractional Pauli operators

Søren Fournais, Aarhus University

Abstract

In this talk we will discuss recent work on Hardy-Lieb-Thirring inequalities for the Pauli operator. The classical Lieb-Thirring inequality estimates the sum of the negative eigenvalues of a Schrödinger operator $-\Delta + V$ by an integral of a power of the potential. In 3-dimensions, this becomes

$$\operatorname{tr}(-\Delta + V)_- \leq C \int (V(x))_-^{5/2}\, dx$$

The classical Hardy inequality states that (also in $3D$),

$$-\Delta - \frac{1}{4|x|^2} \geq 0,$$

where the constant $\frac{1}{4}$ is the sharp constant for this bound.

It is well known, that these inequalities can be combined to yield "Hardy-Lieb-Thirring inequalities", i.e. the Lieb-Thirring inequality above still holds (possibly with a different constant) if V is replaced by $-\frac{1}{4|x|^2} + V$ on the left side.

In this talk we will discuss similar inequalities, where the non-relativistic kinetic energy operator $-\Delta$ is replaced by a magnetic Pauli operator. In particular, we will discuss a relativistic version, where the kinetic energy is the square root of a Pauli operator, and where $\frac{1}{4|x|^2}$ is replaced by $\frac{c_H}{|x|}$, with c_H being the critical Hardy constant for the relativistic problem.

This is joint work with Gonzalo Bley.

Spectral shift functions and Dirichlet-to-Neumann maps

Fritz Gesztesy, Baylor University, Waco, TX, USA

Abstract

The spectral shift function of a pair of self-adjoint operators is expressed via an abstract operator-valued Weyl-Titchmarsh m-function. This general result is then applied to different self-adjoint realizations of second-order elliptic partial differential operators on smooth domains with compact boundaries and Schrödinger operators with compactly supported potentials. In these applications the spectral shift function is determined in an explicit form with the help of (energy parameter dependent) Dirichlet-to-Neumann maps. These spectral shift functions are naturally related to differences of eigenvalue counting functions.

This is based on joint work with J. Behrndt and S. Nakamura.

Molecular propagation through conical intersections of electron energy levels

George A. Hagedorn, Virginia Tech

Abstract

We study propagation of molecular wave packets through the most important type of intersection of elecron energy levels, conical intersections.

Lower bounds on the fundamental spectral gap for single-well potentials

Evans M. Harrell II, Georgia Institute of Technology

Abstract

In a 1989 article Ashbaugh and Benguria proved an optimal lower bound to the fundamental spectral gap of a family of Sturm-Liouville equations on an interval, assumed to possess symmetric, single-well potentials. This work inspired many later results for other categories of potentials, with relaxed assumptions imposed on the "single-well" potentials or alternative constraints such as convexity, as in a notable paper of Lavine from 1994.

In recent work with Zakaria El Allali a direct variational approach is used to study lower bounds on spectral gaps for Sturm-Liouville problems, with which we obtain optimal lower bounds for a wide cagtegory of Sturm-Liouville operators. I shall also discuss prospects for extending these results to the case of quantum graphs.

Sturm's theorem on the zeros of sums of eigenfunctions: Gelfand's strategy implemented

Bernard Helffer, University of Nantes

Abstract

In the second section "Courant-Gelfand theorem" of his last published paper, V. Arnold recounts Gelfand's strategy to prove that the zeros of any linear combination of the n first eigenfunctions of the Sturm-Liouville problem

$$-y''(s) + q(x)y(x) = \lambda y(x) \text{ in }]0,1[, \text{ with } y(0) = y(1) = 0,$$

divide the interval into at most n connected components, and concludes that "the lack of a published formal text with a rigorous proof. . . is still distressing."

Inspired by quantum mechanics, Gelfand's strategy consists in replacing the analysis of linear combinations of the n first eigenfunctions by that of their Slater determinant which is the first eigenfunction of the associated n particle operator acting on Fermions.

Connected results by Haar, Kellogg, Gantmacher-Krein will also be discussed. In this talk, we give a complete proof of the above assertion. Refining Gelfand's strategy, we prove a stronger property taking the multiplicity of zeros into account, a result which actually goes back to Sturm (1836).

This is a work in collaboration with Pierre Bérard.

Multiplicities of eigenvalues, some results and open problems

Thomas Hoffmann-Ostenhof, University of Vienna

Abstract

We discuss some results on the multiplicities of eigenvalues for membranes and mention some related open problems. In addition we consider also multiplicities of eigenvalues for problems where the Hamiltonian is a direct sum of 1D Schrödinger operators.

Counting bound states in quantum mechanics with maximal Fourier-multiplier estimates

Dirk Hundertmark, Karlsruhe Institute of Technology

Abstract

We show how one can use bounds for maximal Fourier-multipliers, well-known in harmonic analysis, to drastically simplify and improve Cwikel's proof of the famous Cwikel-Lieb-Rozenblum bound. It turns out, that the constant one gets in this way are "echt nicht schlecht" (i.e., "really not bad," in german).

From Neumann to Steklov and beyond, via Robin

Richard Laugesen, University of Illinois at Urbana-Champaign

Abstract

Weinberger showed the first nontrivial Neumann eigenvalue of the Laplacian is maximal for the ball, among domains of fixed volume. Brock did the same for the first nontrivial Steklov eigenvalue. We connect these two results by generalizing to the second Robin eigenvalue of the Laplacian. The Robin parameter here is negative, lying in the range between the Neumann and Steklov eigenvalues and going even somewhat beyond the Steklov regime.

On the semiclassical spectrum of the Dirichlet-Pauli operator

Loïc Le Treust, Aix - Marseille University

Abstract

This talk is devoted to semiclassical estimates of the eigenvalues of the Pauli operator on a bounded open set whose boundary carries Dirichlet conditions. Assuming that the magnetic field is positive and a few generic conditions, we establish the simplicity of the eigenvalues and provide accurate asymptotic estimates involving Bergman-Hardy spaces associated with the magnetic field.

Breaking of symmetries in Hartree-Fock Jellium

Mathieu Lewin, CNRS & Paris-Dauphine University

Abstract

Jellium, where electrons are placed in a uniform positive background, is a fundamental system in quantum physics and chemistry, used in particular for the construction of effective functionals in Density Functional Theory. In the talk I will present recent results obtained jointly with David Gontier (Paris-Dauphine) and Christian Hainzl (Tübingen), on the Hartree-Fock (HF) approximation of Jellium.

If HF Jellium were a fluid, one can prove that the ground state would be the free Fermi gas, hence the HF energy per unit volume would exactly be $c_{\rm TF}\rho^{5/3} - c_{\rm D}\rho^{4/3}$ where $c_{\rm TF}$ and $c_{\rm D}$ are respectively the Thomas-Fermi and Dirac constants. With spin, the Coulomb interaction induces a first-order transition with the ferromagnetic free Fermi gas at low densities and the paramagnetic free Fermi gas at high densities. The fluid HF phase diagram is more involved at positive temperatures, as we will illustrate on some numerical computations.

However, Overhauser has shown in the 60s that Hartree-Fock Jellium is never a fluid at zero temperature. Recent numerical simulations indeed suggest that HF Jellium is crystallized at all densities. I will then explain that the energy gain due to the breaking of translation-invariance is exponentially small at large densities, namely the true HF energy per unit volume is $c_{\rm TF}\rho^{5/3} - c_{\rm D}\rho^{4/3} + O(e^{-a\rho^{1/6}})$ for some $a > 0$. The proof reduces to the study of the first eigenvalue of a Schrödinger operator with dispersion relation degenerating on a sphere.

A dual form of the sharp Nash inequality

Elliott Lieb, Princeton University

Abstract

Nash's inequality, $\|u\|_2^{1+2/n} \leq C_n \|\nabla u\|_2 \, \|u\|_1^{2/n}$ is a weakened version of Sobolev's inequality $\|u\|_{2n/(n-2)} \leq C_n' \|\nabla u\|_2$, and can be obtained from it by using Hölder's inequality. Nevertheless, it is extremely important for obtaining smoothing estimates for contraction semigroups, such as the heat equation semigroup $e^{t\Delta}$. The sharp constant C_n is, therefore, important and was found by Carlen and Loss in a beautiful paper in 1993.

The well known duality between Sobolev's inequality and the Hardy-Littlewood-Sobolev inequality $\|(-\Delta)^{-1/2} u\|_2 \leq C_n'' \, \|u\|_{2n/(n+2)}$ suggests that the Nash inequality should also have an interesting dual form. We provide one here. This dual inequality relates the L^2 norm to the infimal convolution of the L^∞ and H^{-1} norms. Like the HLS inequality, it turns out to be an interesting statement in potential theory.

(with Eric Carlen) *A Dual Form of the Sharp Nash Inequality and its Weighted Generalization,* Bull. London Math. Soc. DOI:110.1112/blms.12220. arxiv:1703,09325.

Some results for functionals of Aharanov-Bohm type

Michael Loss, Georgia Tech

Abstract

In this talk I present some variational problems of Aharanov-Bohm type, i.e., they include a magnetic flux that is entirely concentrated at a point. This is maybe the simplest example of a variational problems for systems, the wave function being necessarily complex. The functional is rotationally invariant and the issue to be discussed is whether the optimizer have this symmetry or whether it is broken.

Another look at a rearrangement idea of Szegő for membranes and plates

Rajesh Mahadevan, Universidad de Concepción

Abstract

In connection with an ongoing work with Ashbaugh and Benguria on eigenvalue inequalities for clamped plates with lower order terms, we shall review briefly the ideas of rearrangement proposed by G. Szegő in his papers on membranes and plates [2, 3] from the 50's and look for connections with Talenti's comparison theorems [4, 5, 6] since the 70's.

[1] M. Ashbaugh, R. Benguria and R. Mahadevan, Minimization of the lowest eigenvalue of the vibrating clamped plate under compression, manuscript

[2] G. Szegő, On membranes and plates, *Proc. Nat. Acad. Sci.* **36** (1950), 210-216

[3] G. Szegő, Note to my paper "On membranes and plates", *Proc. Nat. Acad. Sci.* **44** (1958), 314-316

[4] G. Talenti, Elliptic equations and rearrangements, *Ann. Scuo. Norm. Sup. Pisa* (Ser. 4) **3** (1976), no. 2, 697-718

[5] G. Talenti, On the first eigenvalue of the clamped plate, *Ann. Mat. Pura Appl.* (Ser. 4) **129** (1981), 265-280

[6] An inequality between u^* and $|\mathrm{grad}\, u|^*$, General inequalities, 6 (Oberwolfach, 1990), *International Series of Numer. Math.* **103**, 175-182, Birkhäuser, Basel, 1992

Stability of the superselection sectors of two-dimensional quantum lattice models

Bruno Nachtergaele, University of California, Davis

Abstract

Kitaev's quantum double models provide a rich class of examples of two-dimensional lattice systems with topological order in the ground states and a spectrum described by anyonic elementary excitations. The infinite volume ground states of the abelian quantum double models come in a number of equivalence classes called superselection sectors. We prove that the superselection structure remains unchanged under uniformly small perturbations of the Hamiltonians. (joint work with Matthew Cha and Pieter Naaijkens)

Local eigenvalue asymototics of the perturbed Krein Laplacian

Georgi Raikov, Pontificia Universidad Católica de Chile

Abstract

I will consider the Krein Laplacian on a regular bounded domain, perturbed by a real-valued multiplier V vanishing on the boundary. Assuming that V has a definite sign, I will discuss the asymptotics of the eigenvalue sequence which converges to the origin. In particular, I will show that the effective Hamiltonian that governs the main asymptotic term of this sequence, is the harmonic Toeplitz operator with symbol V, unitarily equivalent to a pseudodifferential operator on the boundary. This is a joint work with Vincent Bruneau (Bordeaux, France).

The financial support of the Chilean Science Foundation Fondecyt under grant 1170816 is gratefully acknowledged.

Absence of transport in quasicrystals

Constanza Rojas-Molina, University of Düsseldorf

Abstract

In this talk we review some recent results on Delone operators, which are used to model the electronic transport in aperiodic systems, like quasicrystals. We show how random Schrödinger operators appear naturally in this setting, as auxiliary models to study absence of transport. This connection allows us to apply the well-developed theory for random operators to obtain dynamical localization in the aperiodic setting. We discuss the ingredients needed in the proof of localization in this setting and its consequences on the spectral type of Delone operators near the bottom of the spectrum. This is based on joint work with Peter Müller (Munich).

Quantum many-body systems with point interactions

Robert Seiringer, IST Austria

Abstract

We investigate the stability of quantum many-body systems with point interactions. In particular, we present a proof that a system of N fermions interacting with an additional particle via point interactions is stable if the ratio of the mass of the additional particle to the one of the fermions is larger than some critical value. For this impurity problem, we also show that the ground state energy of the system at given non-zero mean density differs from the one of the ideal gas by a term depending only on the density and the scattering length of the interactions, independently of N. While the general problem with more than one impurity remains open, we can show stability for the simplest such system, the one consisting of two fermions interacting with two (fermionic) impurities. (Joint work with T. Moser)

The density of heavy atoms close to the nucleus: proof of the strong Scott conjecture for the atomic Chandrasekhar operator

Heinz Siedentop, Ludwig-Maximilians-Universität München

Abstract

The innermost electrons of heavy atoms, namely those on the scale $1/Z$ where Z is the atomic number, move with considerable speed. An realistic analysis of these electrons requires therefore to take relativity into account. For simplicity with start the presentation for a Hamiltonian with a kinetic energy used already by Chandrasekhar in his analysis of stability of stars. We show that the density of electrons in the scale $1/Z$ converges towards the sum of the squares of the normalized eigenfunctions of $\sqrt{-\Delta+1}-\kappa/|x|$ in the limit of large Z and large velocity of light c with $\kappa := Z/c$ fixed.

Time permitting we will indicate how this result can also be shown for the no-pair operator in the Furry picture, a Hamiltonian that is known to yield energies within chemical accuracy.

The talk is based on joined work with Rupert Frank, Konstantin Merz, and Barry Simon.

Seeing through space time

Gunther Uhlmann, University of Washington

Abstract

We consider the inverse problem of determining the structure of the Universe both from active and passive measurements.

Sign changing solutions of Poisson's equation

Michiel van den Berg, University of Bristol

Abstract

Let Ω be an open, possibly unbounded, set in Euclidean space $\mathbb{R}^m$ with boundary $\partial\Omega$, let A be a measurable subset of Ω with measure $|A|$, and let $\gamma \in (0,1)$. We investigate whether the solution $v_{\Omega,A,\gamma}$ of $-\Delta v = \gamma \mathbf{1}_{\Omega-A} - (1-\gamma)\mathbf{1}_A$ with $v = 0$ on $\partial\Omega$ changes sign. Bounds are obtained for $|A|$ in terms of geometric characteristics of Ω (bottom of the spectrum of the Dirichlet Laplacian, torsion, measure, or R-smoothness of the boundary) such that $v_{\Omega,A,\gamma}$ is either non-negative or is sign changing. Joint work with Dorin Bucur, Université de Savoie.

Spectral theory for systems of ordinary differential equations with distributional coefficients

Rudi Weikard, University of Alabama at Birmingham

Abstract

We discuss the spectral theory of the first-order system $Ju' + qu = wf$ of differential equations on the real interval (a, b) when J is a constant, invertible skew-Hermitian matrix and q and w are matrices whose entries are distributions of order zero with q Hermitian and w non-negative. We do not require the definiteness condition customarily made on the coefficients of the equation. Specifically, we construct associated minimal and maximal relations, and study self-adjoint restrictions of the maximal relation. For these we construct Green's function and prove the existence of a spectral (or generalized Fourier) transformation.

Random matrices beyond mean-field models

Horng-Tzer Yau, Harvard University

Abstract

In this talk, we'll review the current status of random matrix theory beyond Wigner ensembles. Two main ensembles to be discussed in this lecture are band matrices and heavy tail random matrices.